Massachusetts Institute of Technology
Center for Advanced Engineering Study

MIT Video Course

Video Course Manual

Electronic Feedback Systems

James K. Roberge Professor of Electrical Engineering, MIT

Reorder No. 74-2100

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Preface

Feedback control is an important technique that is used in many modern electronic and electromechanical systems. The successful inclusion of this technique improves performance, reliability, and cost effectiveness of many designs.

Many of us have had an introductory feedback control subject during our student days. In most cases this introduction was a typically academic one with little emphasis on how theoretical concepts could be applied to actual physical hardware.

In this series of lectures we reintroduce the analytical concepts that underlie classical feedback system design. The application of these concepts is illustrated by a variety of experiments and demonstration systems. The diversity of the demonstration systems reinforces the value of the analytic methods previously introduced and provides the motivation for future lectures.

The lectures incorporate the material in this area that I have found to be most important in my own research and consulting experience. In fact, most of the demonstration systems are closely related to actual systems that I have designed. The lectures also reflect the important comments made by many of the students who have taken a similar subject at MIT.

Each lesson consists of a taped lecture, a reading assignment in the text, and problems, generally also in the text. The suggested sequence is to first view the lecture, then read the suggested sections of the text, and finally solve the problems.

In many cases, the material in the text is more detailed than that provided in the taped lectures. The material learned from the lectures should expedite the self-study of the extensions and amplifications of the text.

It is vitally important that participants in this course solve the problems that are included in the assignments. Simply viewing the tapes, even when combined with the suggested reading assignments, provides a superficial knowledge of the subject matter at best. True mastery requires the in-depth exposure that only comes from wrestling with the concepts through problem solving.

The problem solutions presented in this manual should be consulted only after diligent effort has been made by the participant. We intend the solutions to illustrate what we feel is a good way to solve the problems, rather than to serve as a crutch. Similarly, discussion of the problems with fellow course participants, after having solved them individually, will improve your understanding of the material.

The text frequently includes problems related to the material presented in a lecture that are not specifically assigned. In view of the importance of the first-hand experience gained by problem solving, participants should also solve these additional problems as their schedules permit.

I'd like to thank two of my students for their contributions to the video tapes and the manual. Mike Johnson set up all of the experiments and demonstrations in the studio. He was responsible for the real time magic that replaced one demonstration with another as I lectured away from the demonstration area. Dave Trumper provided all of the problem solutions presented in this manual. The explanations and insights he provides reflect his own excellent understanding of the material.

Most of the viewgraphs used to illustrate the lectures were copied from figures in the text. John Wiley & Sons kindly allowed us to use this material.

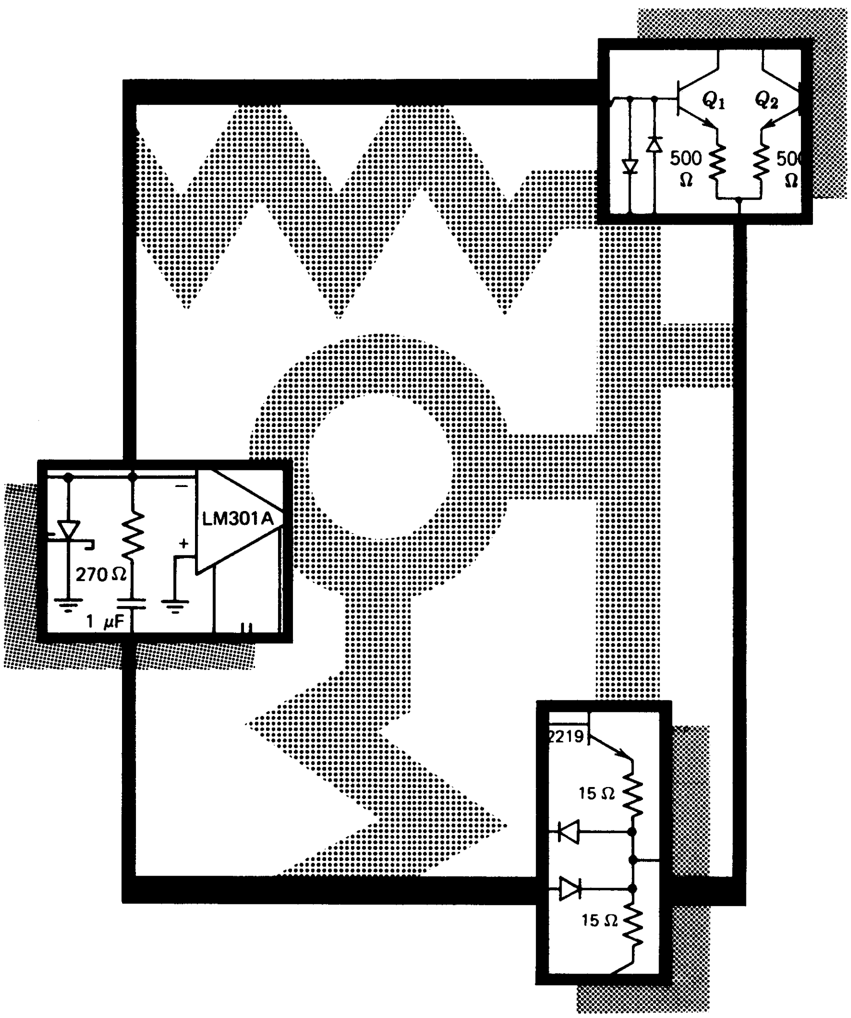
Finally, I'd like to thank the many staff members of the MIT Center for Advanced Engineering Study who made the production of the tapes, the manual, and the promotional material possible. Their tolerance of my procrastination was also much appreciated!

James K. Roberge
Lexington, Massachusetts
January, 1986

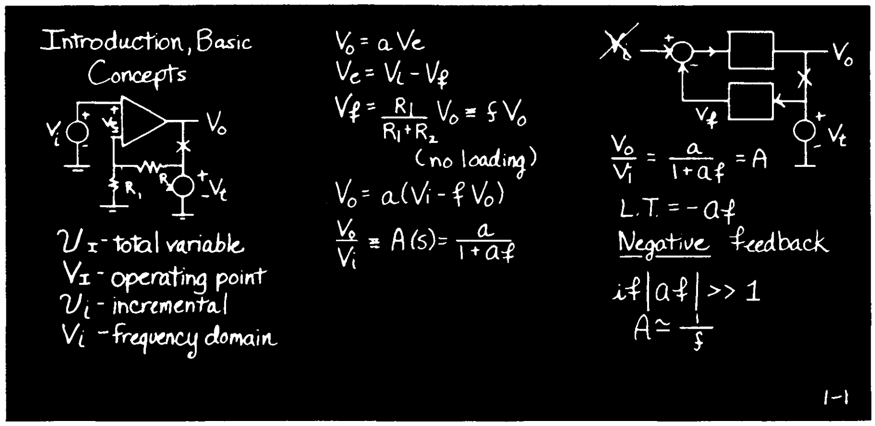
Lectures

VIDEO COURSE MANUAL

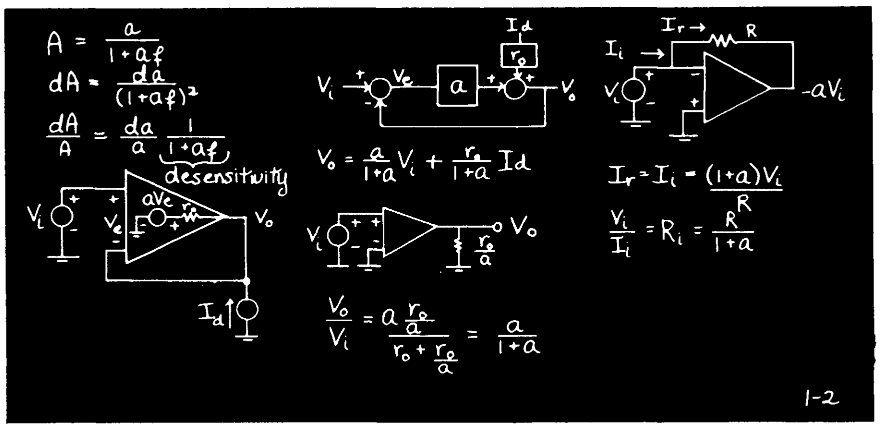
Introduction and Basic Concepts 1



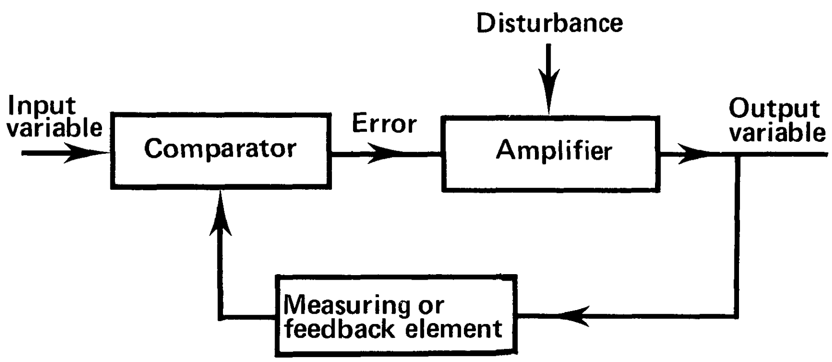
Blackboard 1.1



Blackboard 1.2



Viewgraph 1.1



A typical feedback system

Comments

This lecture serves to introduce the course material and establish the notation that we will use. Important properties of feedback systems are discussed, and the impact of loop transmission on these properties is described. The examples used for illustration are familiar operational amplifier connections.

Reading

Textbook: Skim Chapter 1 if you have not worked with operational amplifier circuits recently.

Chapter 2: Sections 2.3.2 and 2.3.3 may be postponed until Lecture 2, or may be read now for continuity. We will not use the material in Section 2.4.3 directly. However, you will find that this method can save you considerable time in the analysis of complex systems.

Problems

Problem numbers followed by numbers in parentheses (see, for example, P1.6 below) refer to the textbook that accompanies these lectures: James K. Roberge, *Operational Amplifiers: Theory and Practice* (New York: John Wiley & Sons, 1975). When a problem number in parentheses is given without any accompanying text, the problem is to be followed as it appears in the textbook. For some problems referring to the textbook, we have added comments or modifications, which appear in this manual beneath the problem it refers to (see, for example, P2.1 below).

Problem 1.1: Design a circuit using a single operational amplifier that provides an ideal input–output relationship:

$$V_o = -V_{i1} - 2V_{i2} - 3V_{i3}$$

Keep the value of all resistors between 10 kΩ and 100 kΩ.

Draw a block diagram for your amplifier connection, assuming that the operational amplifier has infinite input resistance and zero output resistance but finite open-loop gain *a*. Use your block diagram to determine the loop transmission of your connection. How large must the open-loop gain of the operational amplifier be so that errors from finite gain are no more than 0.01%?

Problem 1.2 (P1.6)

Problem 1.3 (P2.1): Solve the problem as stated in the textbook, and also reduce the block diagram to a form that has one forward element and one feedback element between input and output and determine the gains of these elements.

Problem 1.4 (P2.11)

Problem 1.5 (P2.9)

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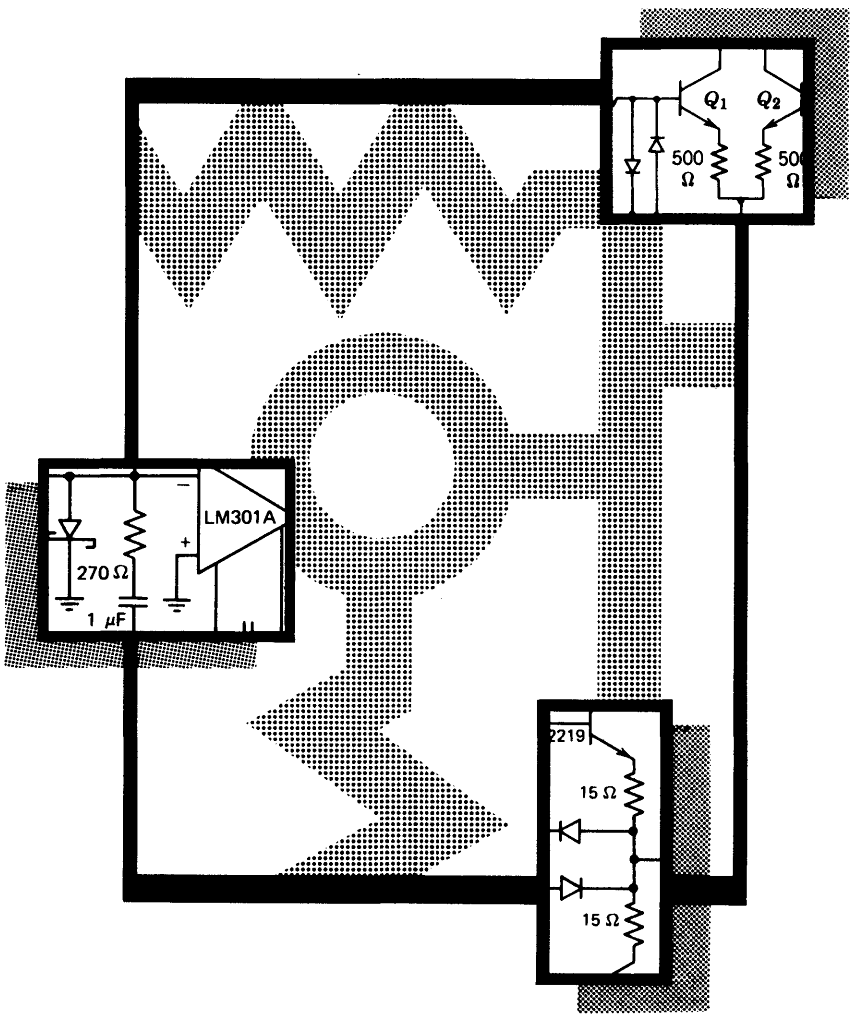
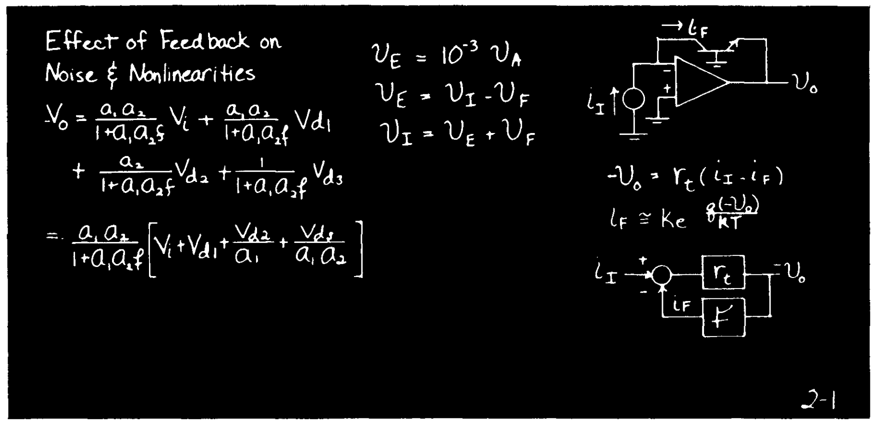
Effects of Feedback on Noise and Nonlinearities

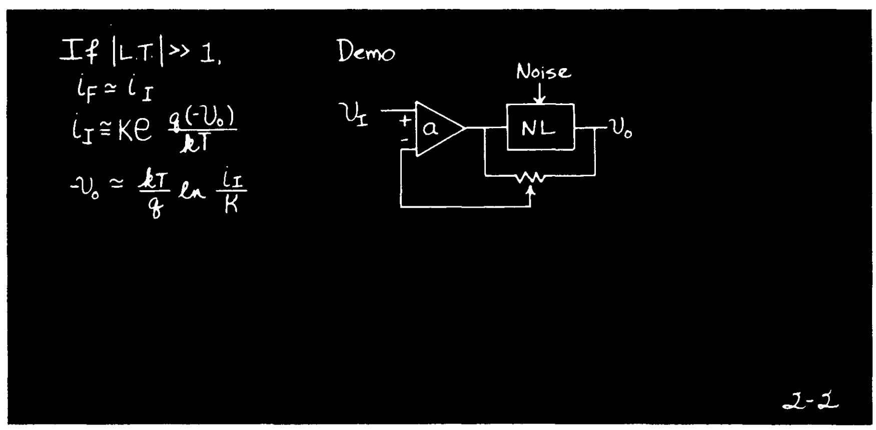
2



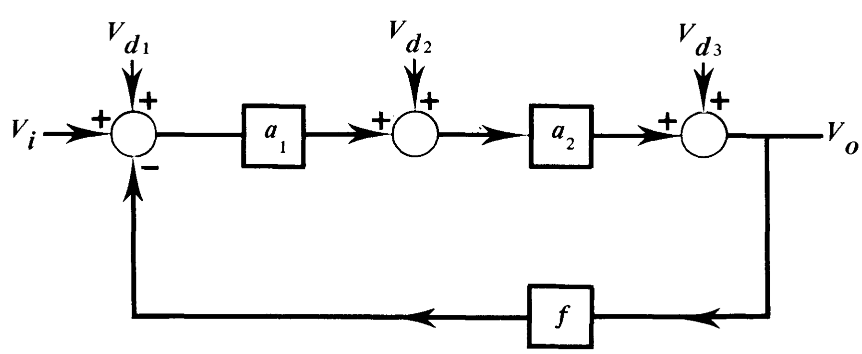
Blackboard 2.1



Blackboard 2.2

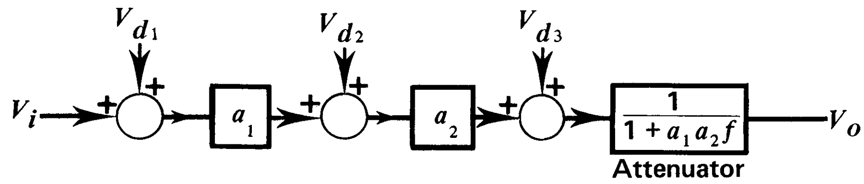


Viewgraph 2.1



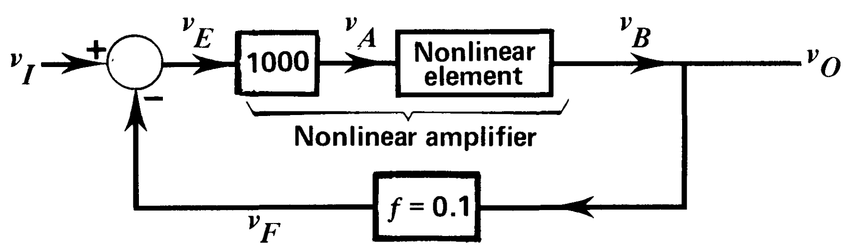
Feedback system illustrating effects of disturbances

Viewgraph 2.2



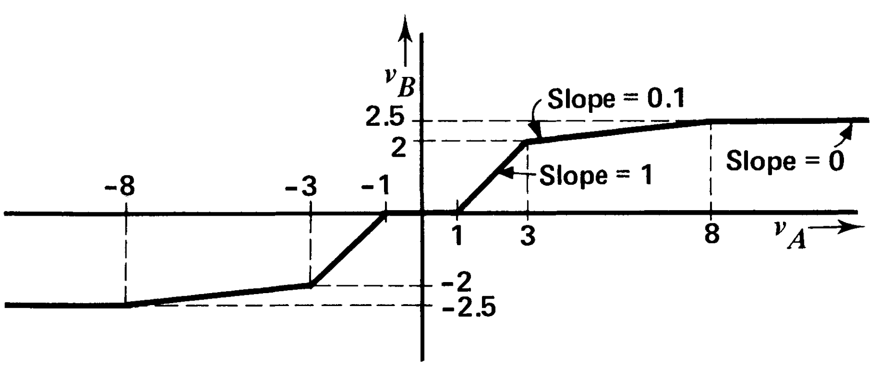
Open-loop system illustrating
effects of disturbances

Viewgraph 2.3



System

Viewgraph 2.4



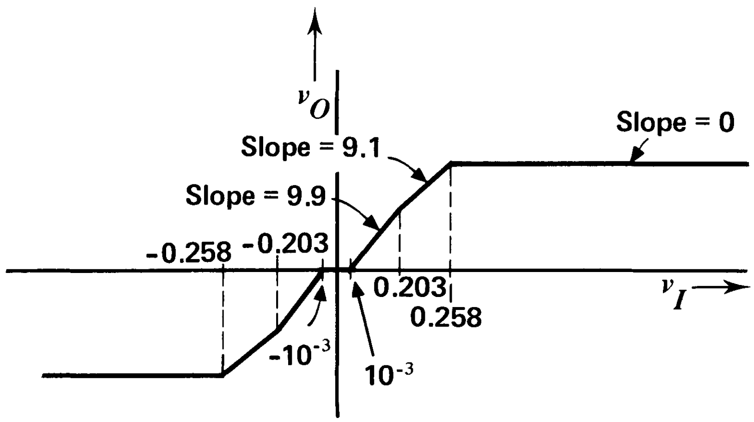
Transfer characteristics of the
nonlinear element.

Viewgraph 2.5

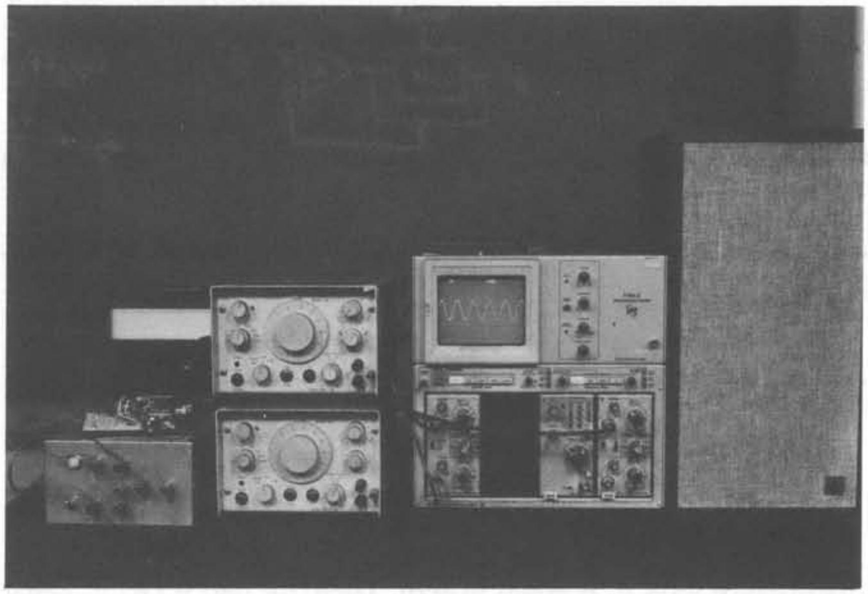
Values of Variables at Breakpoints

v_I	$v_E = v_I - v_F$	$v_A = 10^3 v_E$	$v_B = v_O$	$v_F = 0.1 v_O$
< -0.258	$v_I + 0.250$	$10^3 v_I + 250$	-2.5	-0.25
-0.258	-0.008	-8	-2.5	-0.25
-0.203	-0.003	-3	-2	-0.2
-10^{-3}	-10^{-3}	-1	0	0
10^{-3}	10^{-3}	1	0	0
0.203	0.003	3	2	0.2
0.258	0.008	8	2.5	0.25
> 0.258	$v_I - 0.250$	$10^3 v_I - 250$	2.5	0.25

Viewgraph 2.6



System transfer characteristics
(closed loop) (Not to scale)



Demonstration Photograph 2.1
Nonlinear amplifier demonstration

Much of our effort in this subject involves the properties of linear feedback control systems. However, feedback is often used to either provide controlled nonlinearities or to moderate the effects of the nonlinearities associated with virtually all physical components.

The economic impact of improving the performance of a nonlinear power handling element by adding gain at a low signal power level of the system can be substantial.

Similarly, feedback often provides a convenient and economical means for reducing the sensitivity of systems to externally applied disturbances.

Comments

Reading

Textbook: The material in this lecture parallels that in Sections 2.3.2 and 2.3.4. Please read this material if you have not done so in connection with Lecture 1.

Problems

Problem 2.1 (P2.3)

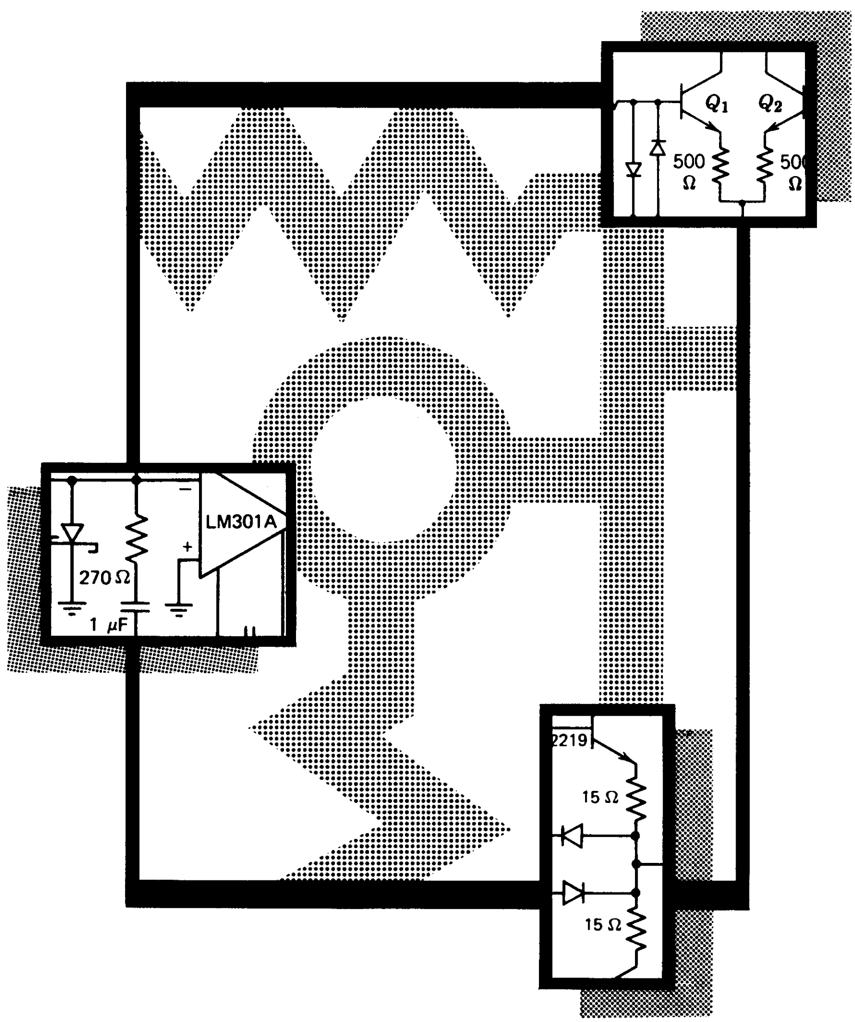
Problem 2.2 (P2.4)

Problem 2.3 (P2.5): For parts (a) and (b) assume that the sinusoidal disturbance term v_N is equal to zero. Then, for part (c) let v_N equal the indicated value of *sin* 377*t*.

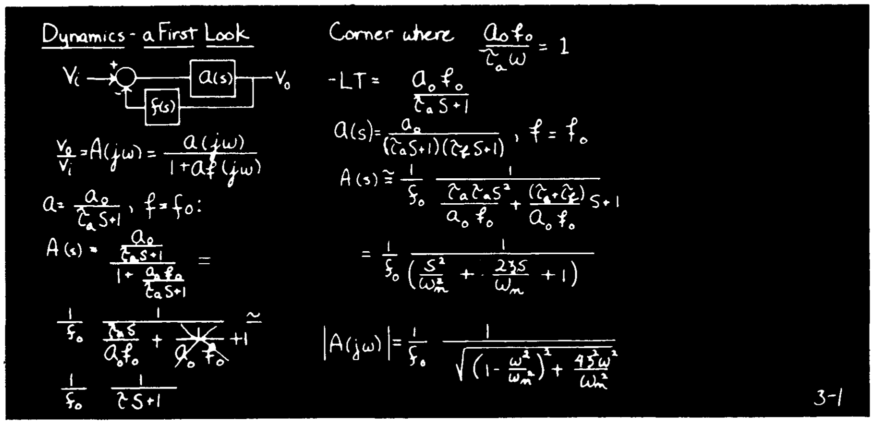
VIDEO COURSE MANUAL

Introduction to Systems with Dynamics

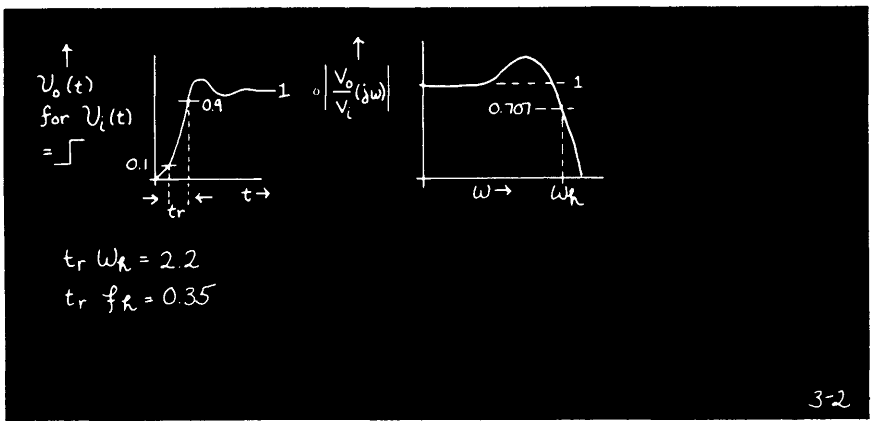
3



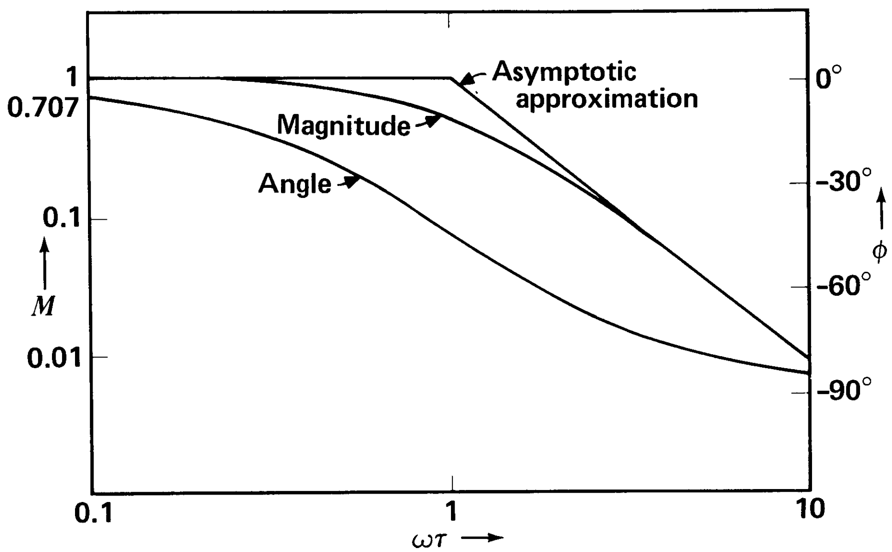
Blackboard 3.1



Blackboard 3.2

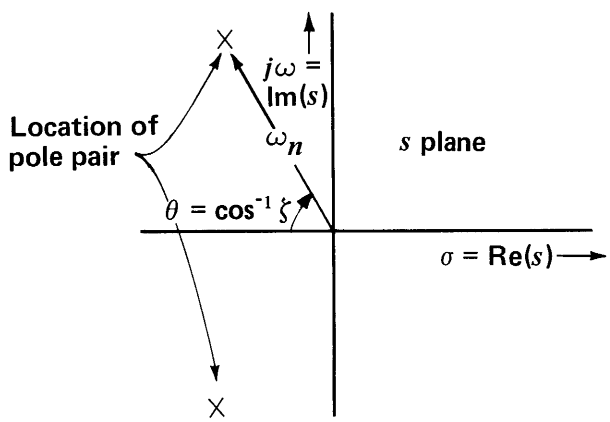


Viewgraph 3.1



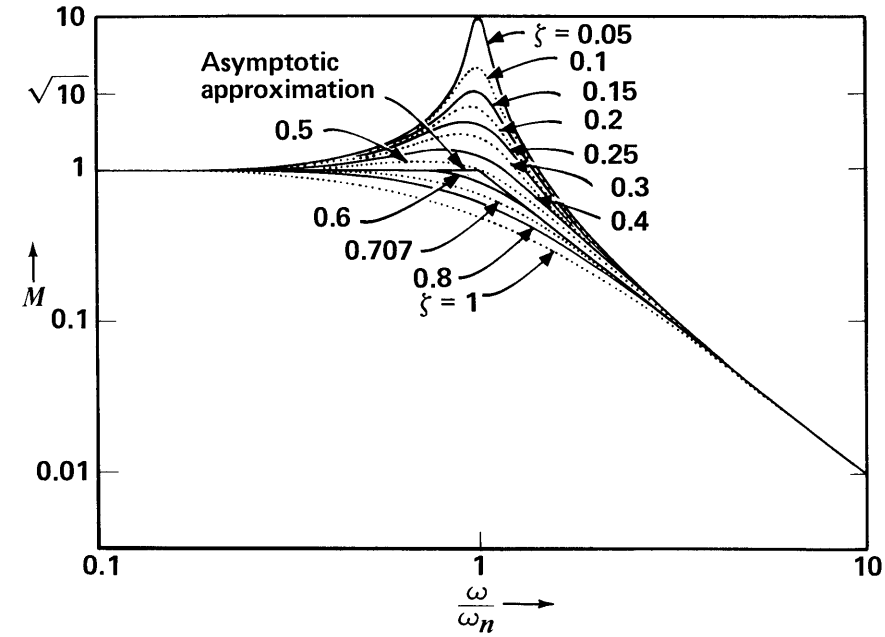
Frequency response of first-order system.

Viewgraph 3.2



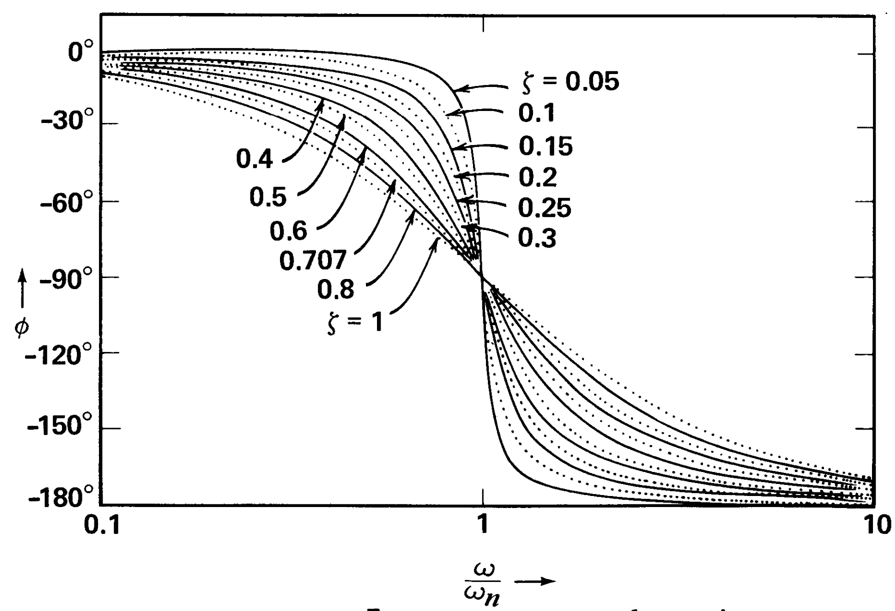
s-plane plot of complex pole pair

Viewgraph 3.3



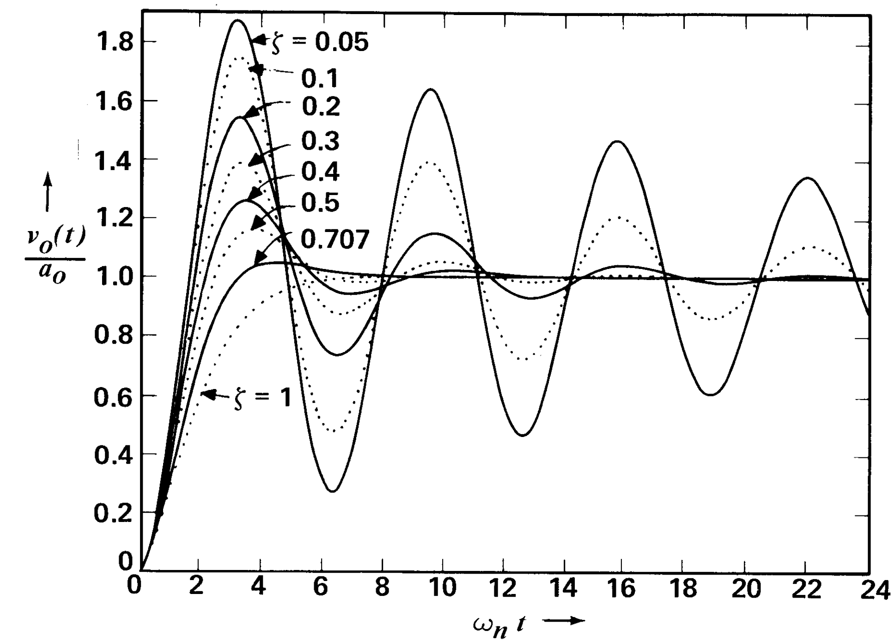
Frequency response of second-order system. (a) Magnitude

Viewgraph 3.4



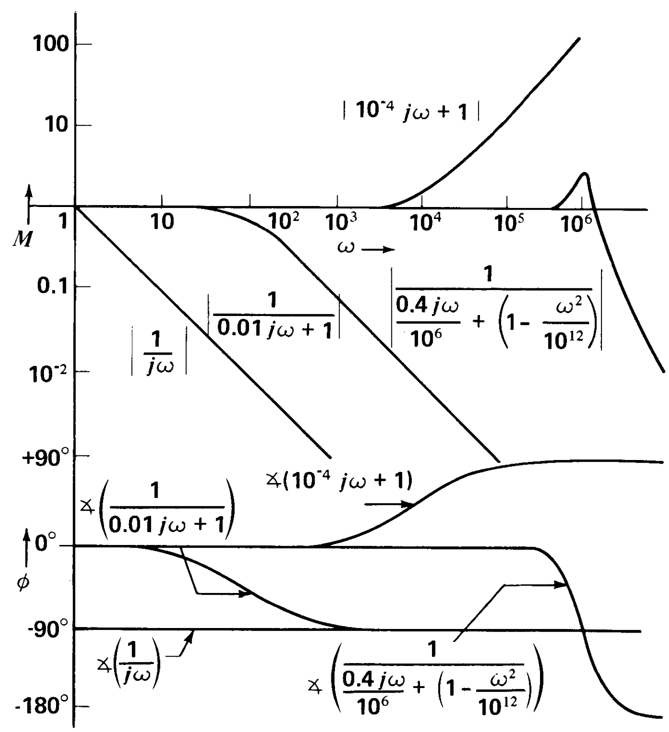
Frequency response of second-order system. (b) Angle.

Viewgraph 3.5



Step responses of second-order system.

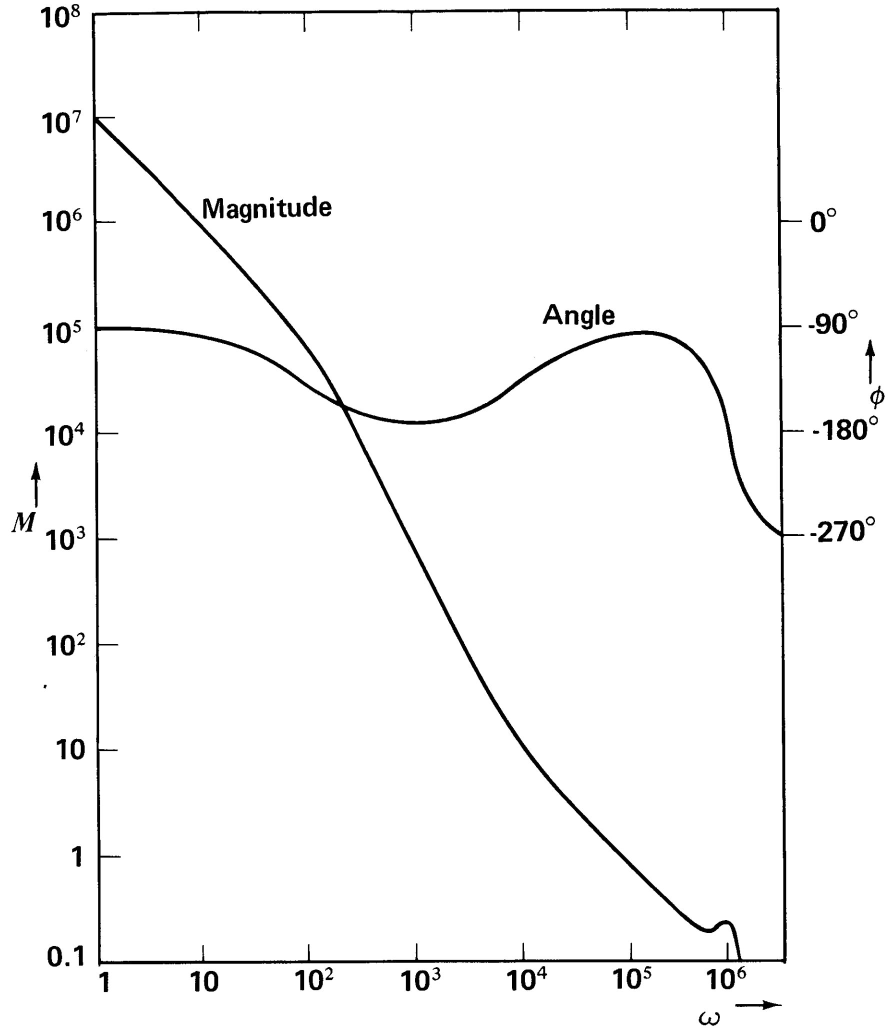
Viewgraph 3.6



(a) Individual factors

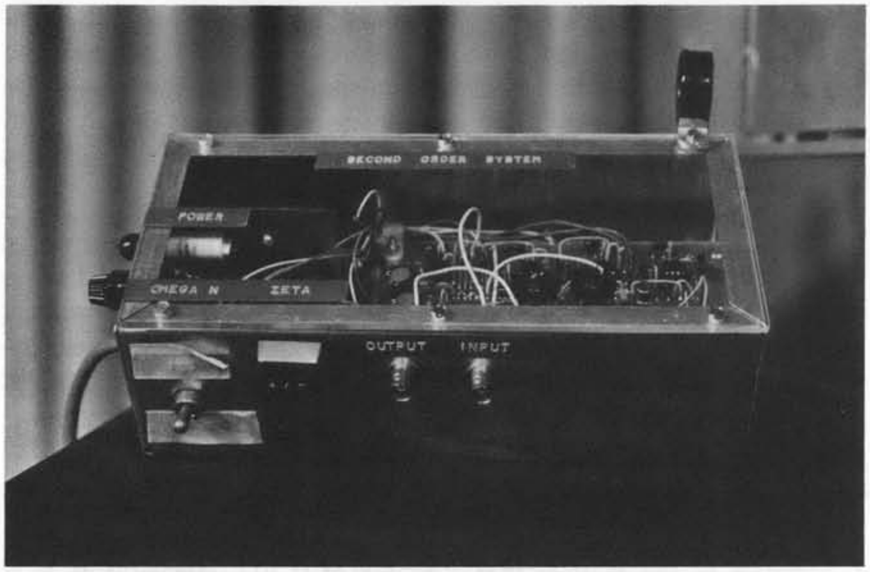
Bode plot of $\dfrac{10^7(10^{-4}s + 1)}{s(0.01s + 1)(s^2/10^{12} + 2(0.2)s/10^6 + 1)}$

Viewgraph 3.7

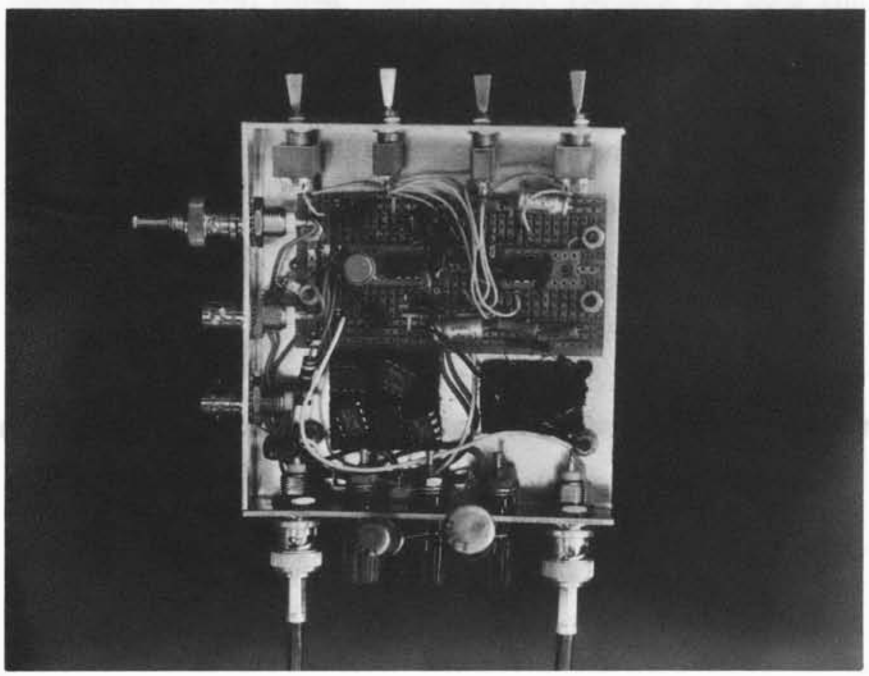


(b) Bode plot

Bode plot of $\dfrac{10^7(10^{-4}s+1)}{s(0.01s+1)(s^2/10^{12}+2(0.2)s/10^6+1)}$



Demonstration Photograph 3.1
Second-order system



Demonstration Photograph 3.2
Operational-amplifier for comparison with second-order response

Comments

This lecture serves as an introduction to the dynamics of feedback systems. Aspects of this topic form the basis for more than half the material covered here. If the dynamics of systems could be adjusted at will, it would be possible to achieve arbitrarily high desensitivities and to modify electrical or mechanical impedances in any required way.

We will never solve for the exact closed-loop transient response of a high-order system, preferring instead to estimate important properties by considering lower-order systems that accurately approximate the actual behavior. A demonstration indicating a specific example of this type of approximation is included.

Additional Discussion

I mention in the lecture that a factor of 0.707 corresponds to a -3 dB change on a decibel scale. This reflects the convention usually used for feedback systems where gains (even dimensioned ones) are converted to dB as $20 \log_{10}$ (gain).

Note that in viewgraphs 3.1, 3.3, 3.4, and 3.5, the horizontal axis is normalized so that the resultant curves can be easily scaled for any particular bandwidth system. Thus the horizontal axis in 3.1 is presented as a multiple of $\frac{1}{\tau}$, in 3.3 and 3.4 as a multiple of ω_n, and in 3.5 as a multiple of $\frac{1}{\omega_n}$.

Reading

Textbook: Sections 3.1, 3.3, 3.4, and 3.5. While we will not use the material in Section 3.2 directly, you may want to review it if you have not worked with Laplace transforms recently.

Problems

Problem 3.1 (P3.1)

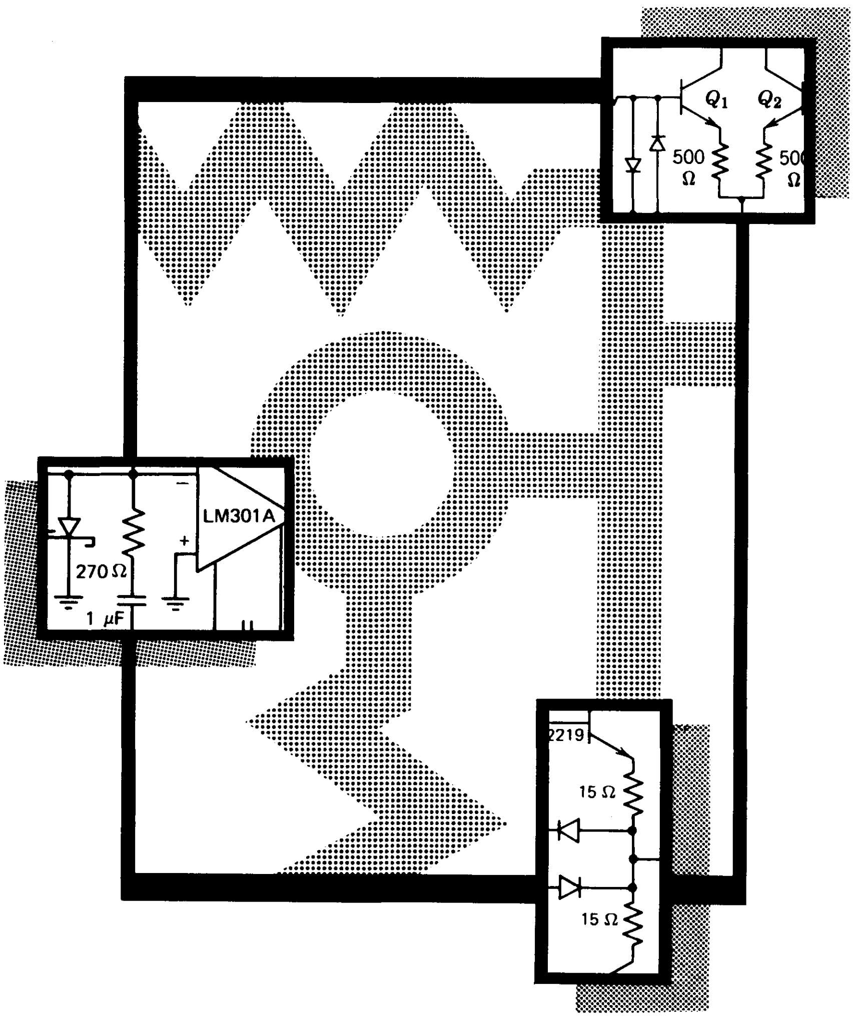
Problem 3.2 (P3.2)

Problem 3.3 (P3.5)

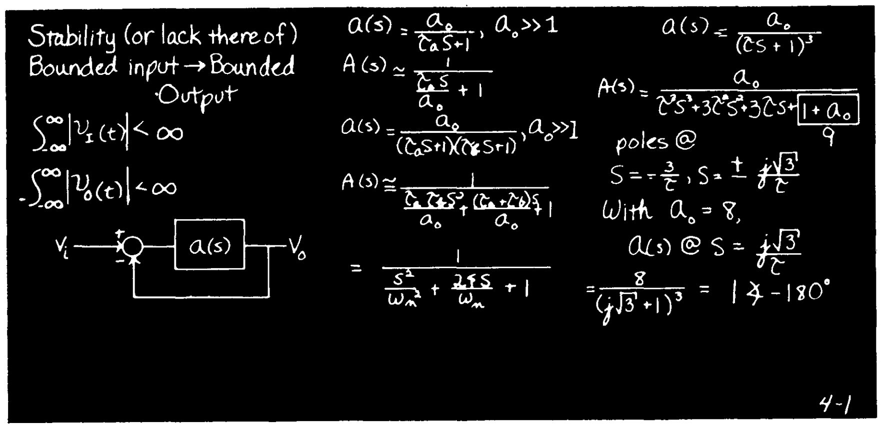
Problem 3.4 (P3.7)

Problem 3.5 (P3.8)

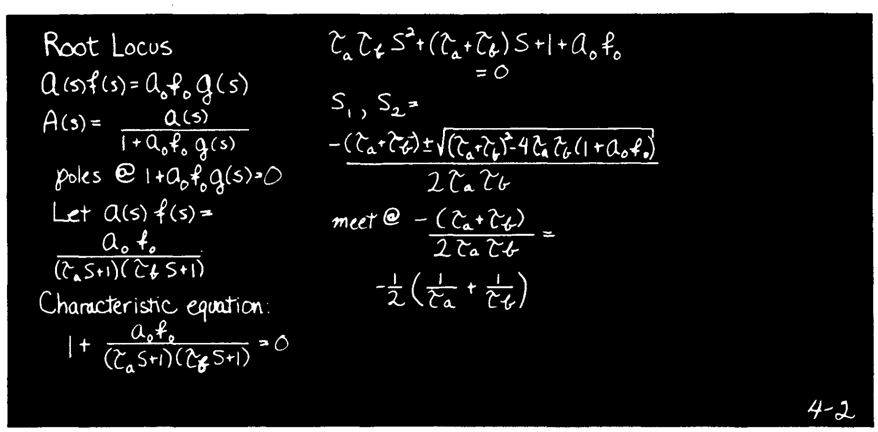
Stability 4



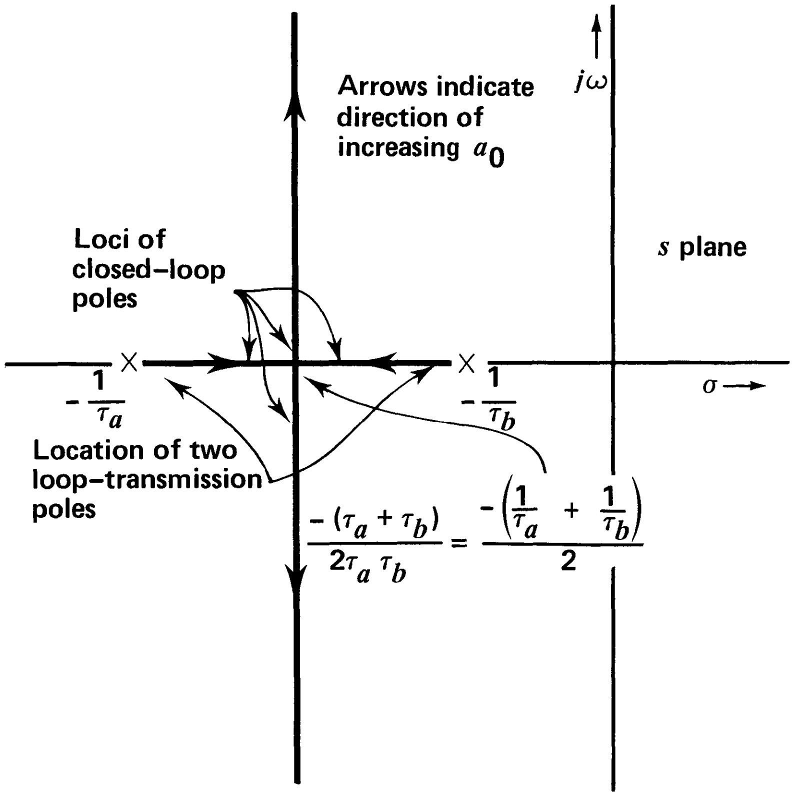
Blackboard 4.1



Blackboard 4.2



Viewgraph 4.1



Root-locus diagram for second-order system.

Comments

This lecture provides our introduction to the stability of feedback systems. We will see again and again that the effective design of feedback systems hinges on the successful resolution of the compromise between desensitivity and speed of response on one hand and stability on the other.

Examples of first-, second-, and third-order systems illustrate how the difficulty of achieving a given degree of stability increases dramatically as the order of the system increases.

Corrections

I left out dt in both of the integrals used in the definition of stability on Blackboard 4.1. These integrals should read

$$\int_{-\infty}^{\infty} |v_I(t)|\,dt < \infty \qquad \text{and} \qquad \int_{-\infty}^{\infty} |v_O(t)|\,dt < \infty$$

I also left out a t in the test generator signal on the same blackboard. The relationship should be

$$K \sin \frac{\sqrt{3}}{\tau} t = V_t$$

I may have left the wrong impression concerning evaluation of stability by means of loop-transmission frequency response. If the loop transmission is exactly $+1$ at some frequency, there is certainly a closed-loop pair of poles on the imaginary axis at that frequency. The point is that closed-loop pole locations (and particularly the important question concerning the number of closed-loop poles in the right-half of the s plane) can generally not be resolved with loop-transmission information at isolated frequencies. This quantity must be known at all frequencies to answer the stability question.

Reading

Textbook: Chapter 4 through page 120.

Problems

Problem 4.1 (P4.1)

Problem 4.2 (P4.2)

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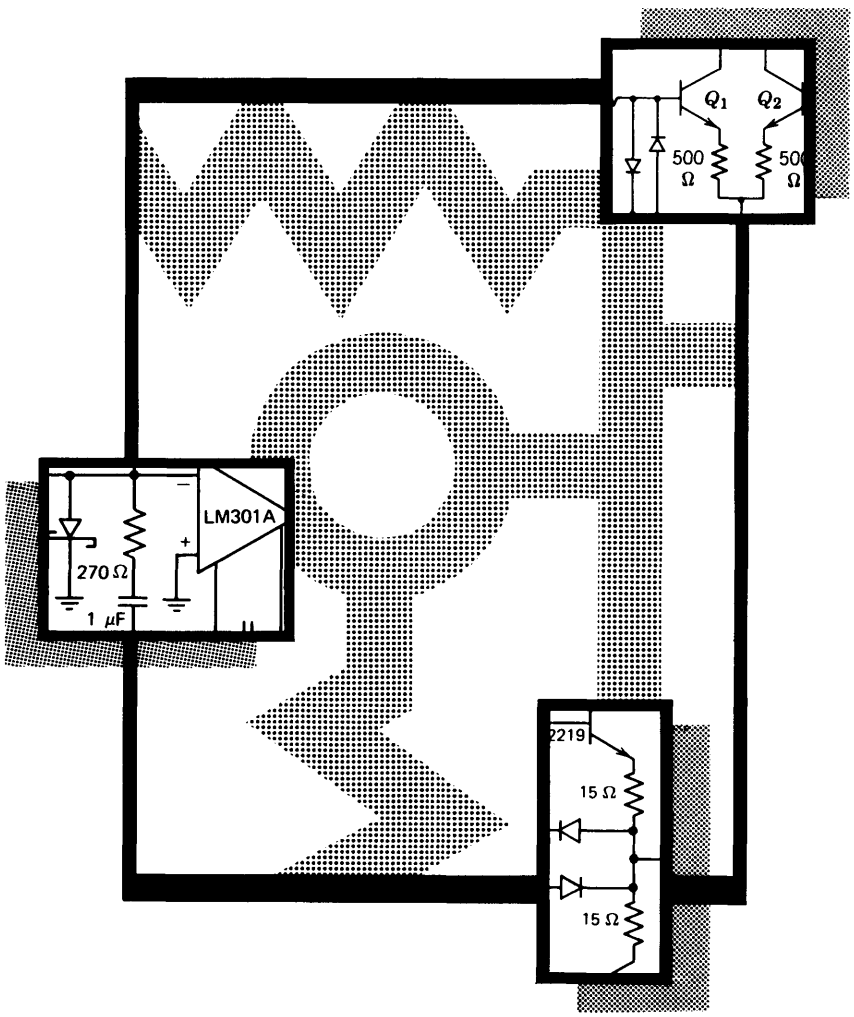
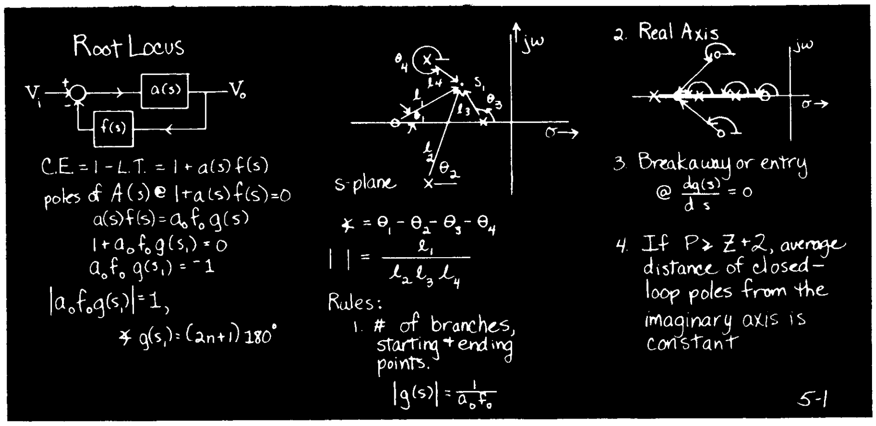
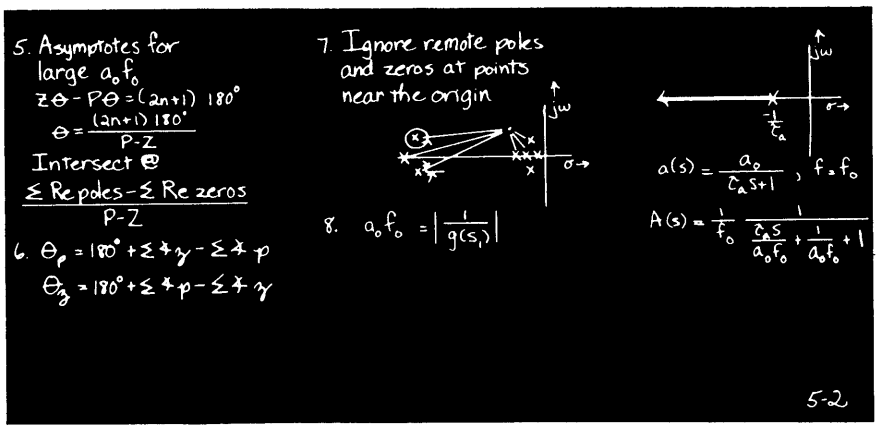
Root Locus 5



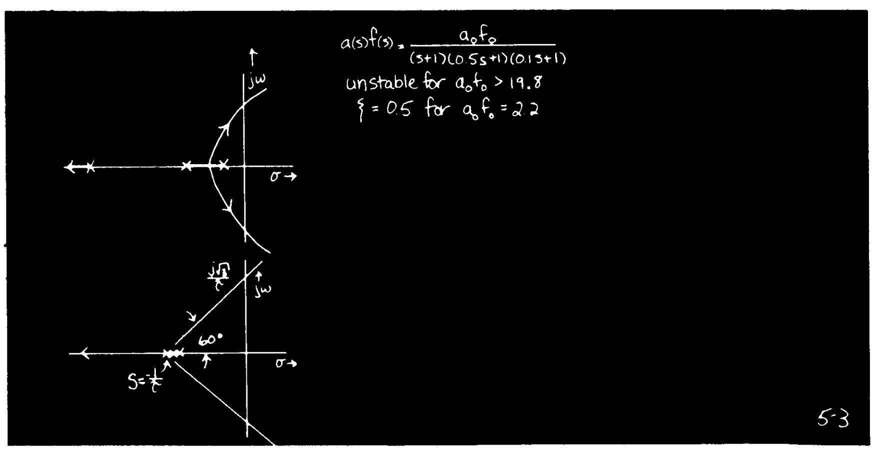
Blackboard 5.1



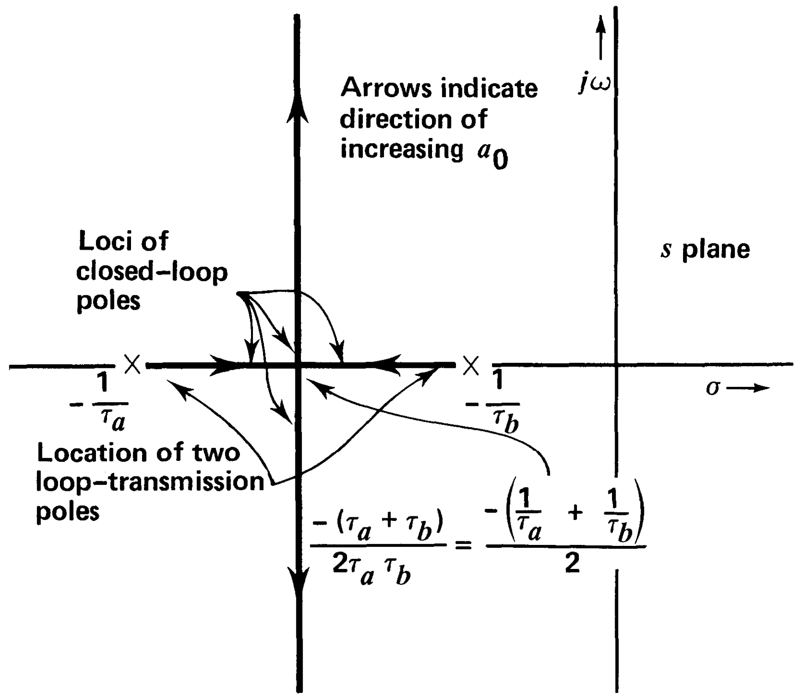
Blackboard 5.2



Blackboard 5.3

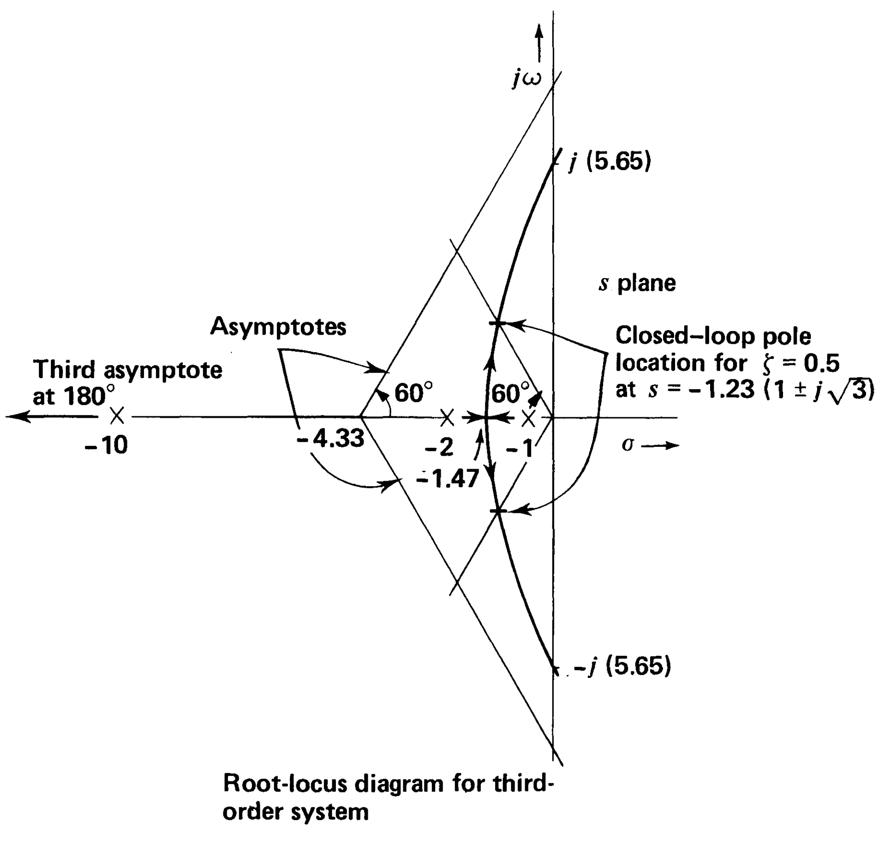


Viewgraph 5.1



Root-locus diagram for second order system.

Viewgraph 5.2



Root-locus diagram for third-order system

Comments

In Lecture 4 we introduced the concept of a root-locus diagram by directly factoring the characteristic equation of the two-pole system used for illustration. This method is tedious for higher-order systems.

The material in this lecture shows how the fact that the $a(s)f(s)$ product must equal -1 at a closed-loop pole location can be exploited to determine rapidly important features of the root-locus diagram. We also see that simple numerical methods can provide certain quantitative results when required.

Corrections

Note that there is a mistake in the videotape on blackboard 5-1 where it states root-locus Rule 4. The blackboard says that the average distance from the real axis is constant. This is identically satisfied for all physically realizable systems. The corrected blackboard in the Video Course Manual states that the average distance from the *imaginary* axis is constant under the conditions of Rule 4. See page 123 of the textbook for clarification.

Reading

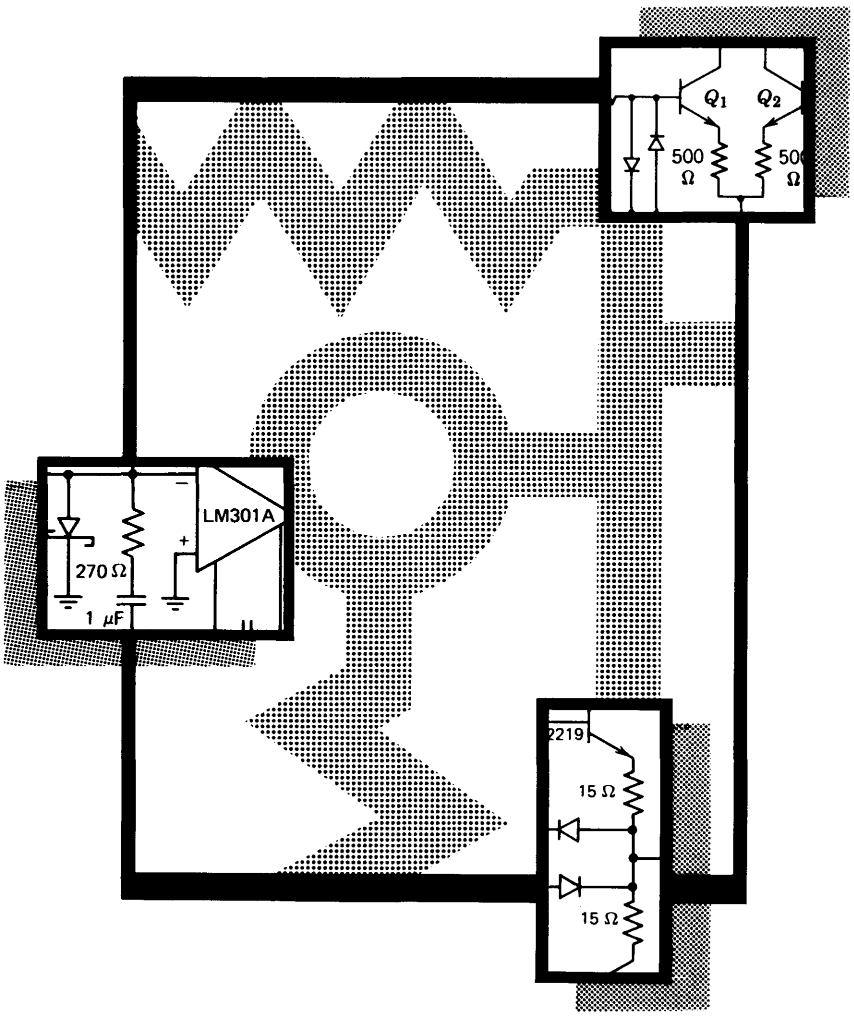
Textbook: Sections 4.3.1 and 4.3.2.

Problems

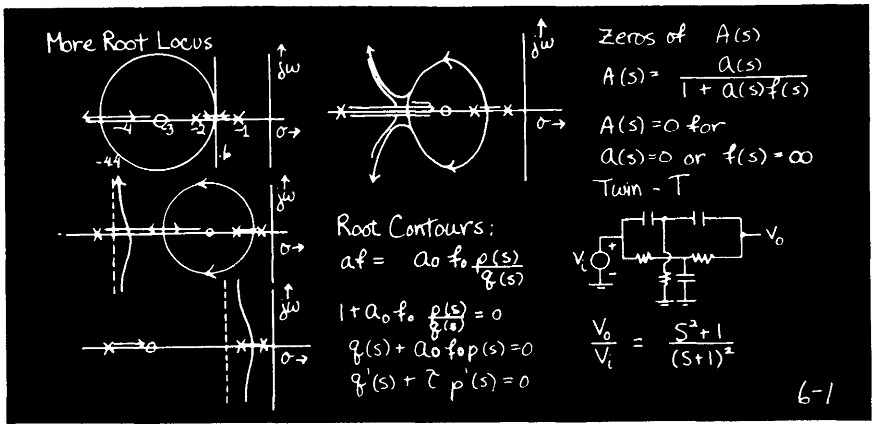
Problem 5.1 (P4.5)

Problem 5.2 (P4.7)

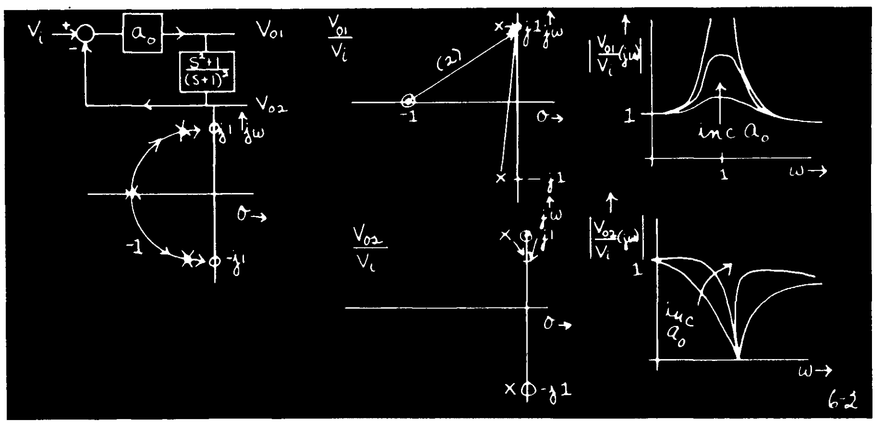
More Root Locus 6

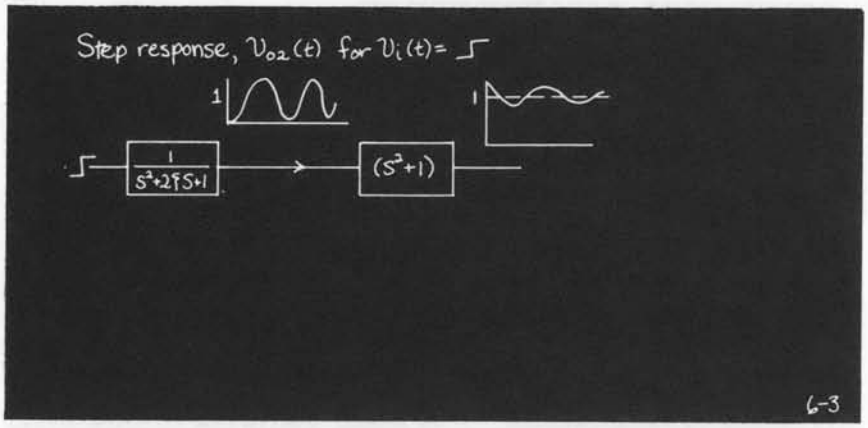


Blackboard 6.1

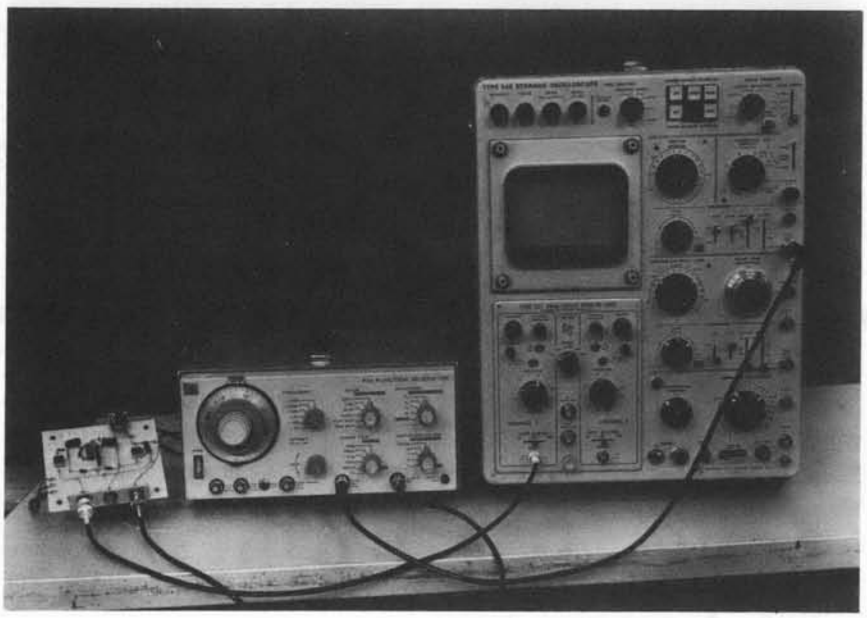


Blackboard 6.2



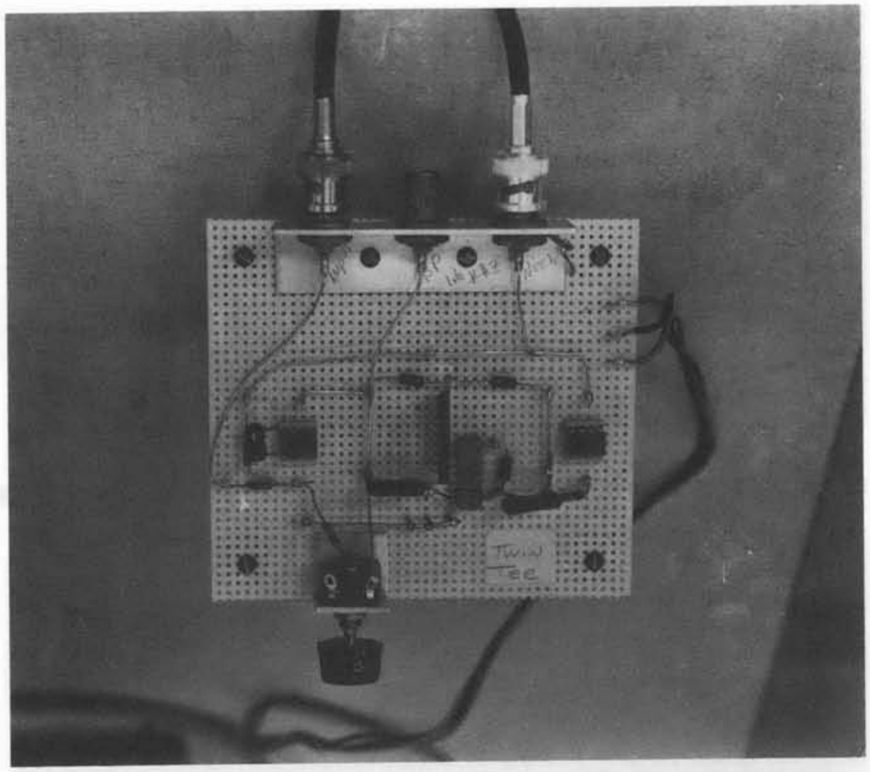


Blackboard 6.3



Demonstration Photograph 6.1
Frequency-selective amplifier demonstration

Demonstration Photograph 6.2
Close-up of frequency-selective amplifier



Comments

This lecture extends our understanding of root-locus techniques. We see that the addition of a zero to a multiple-pole loop transmission increases the amount of desensitivity that can be achieved for a given damping ratio. Future material on compensation will show how this technique can be used to improve system performance.

We also saw how the root-locus method can be generalized to determine how system poles are related to the value of a single parameter other than d-c loop transmission magnitude.

Finally we determined the location of zeros of the closed-loop transfer function and saw the profound effect they can have on the performance of certain systems.

Correction

The phase of the signal associated with the rejection amplifier is wrong in my blackboard drawing. As I mentioned, the initial value theorem can be used to show that the initial value of the step response will be unity. The waveform displayed in the demonstration correctly shows the phase for a small damping ratio as approximately that of $-\sin \omega t$.

Reading

Textbook: Sections 4.3.3 through 4.3.5.

Problems

Problem 6.1 (P4.6)

Problem 6.2 (P4.8)

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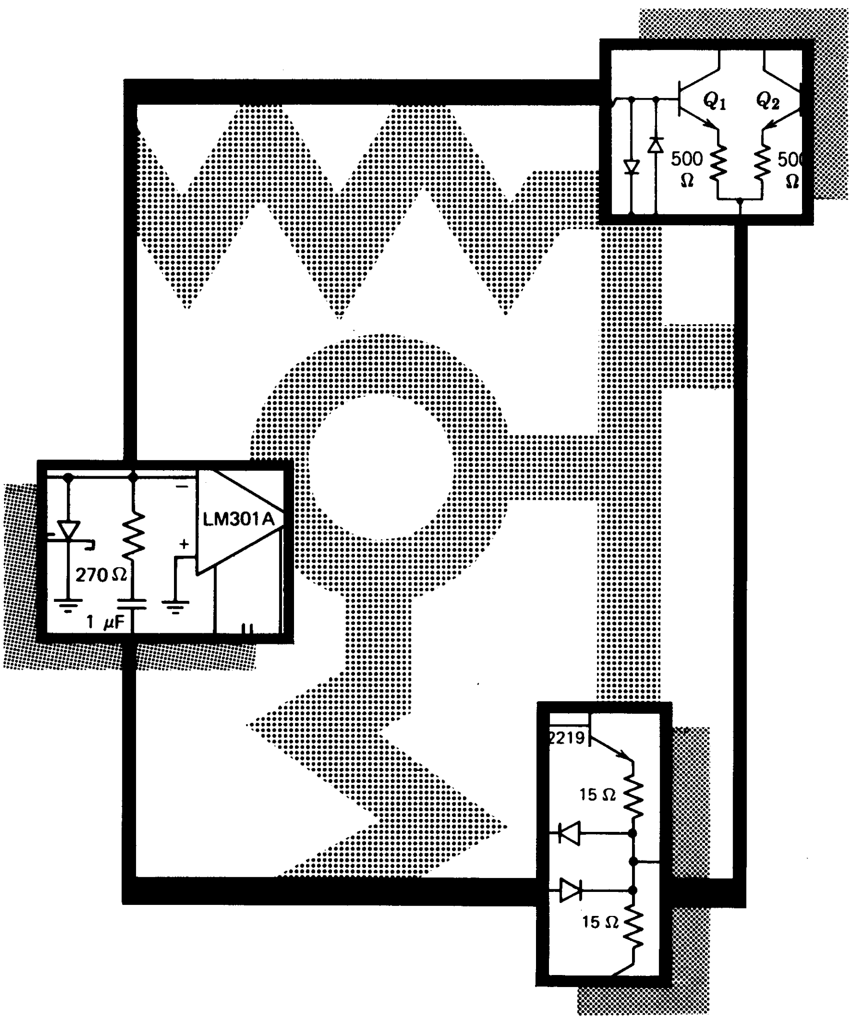
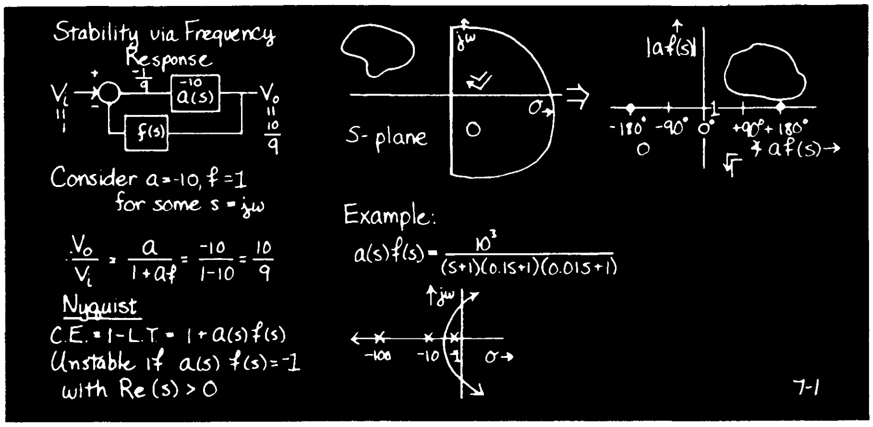
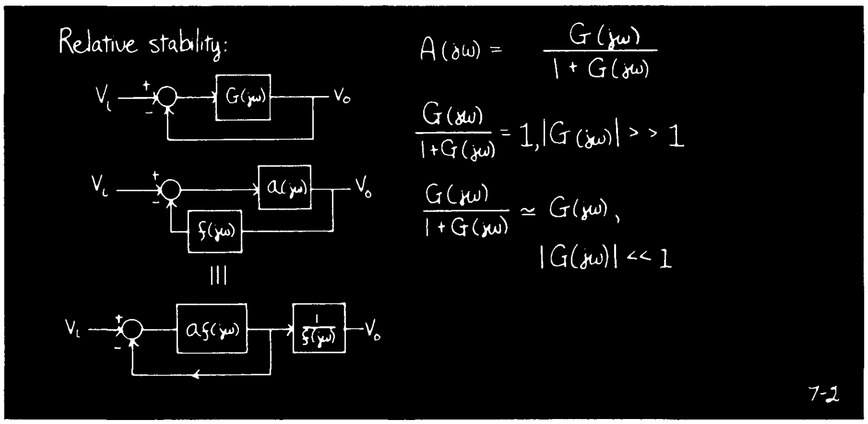
Stability via Frequency Response 7



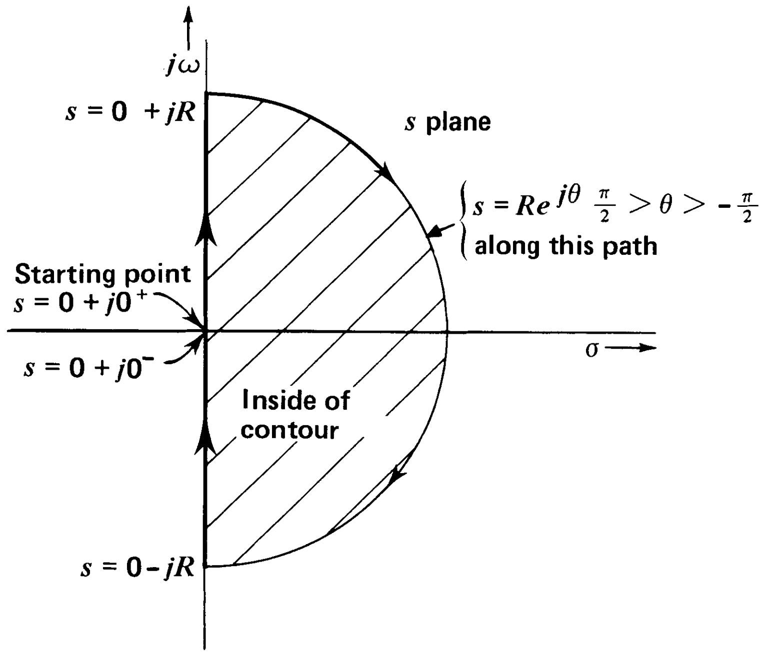
Blackboard 7.1



Blackboard 7.2

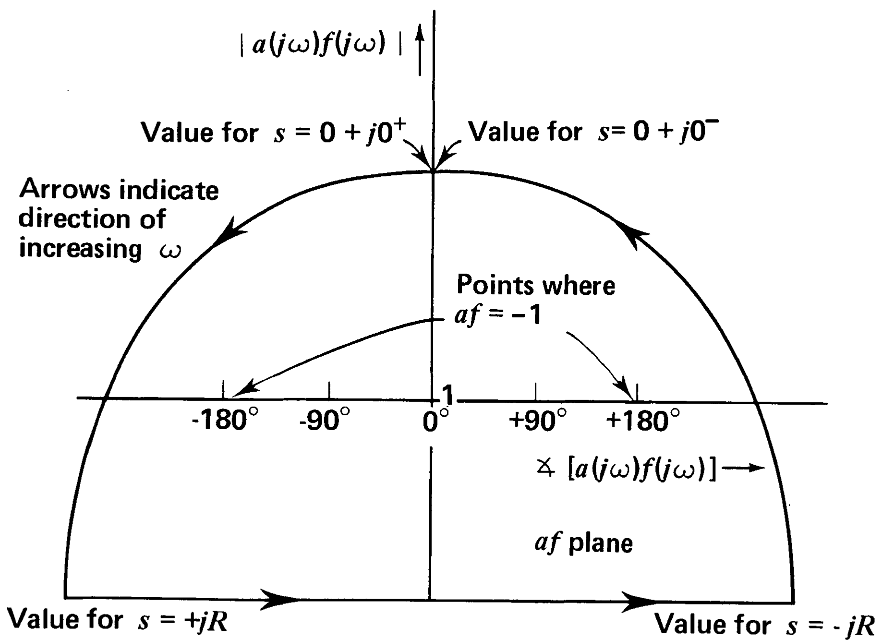


Viewgraph 7.1



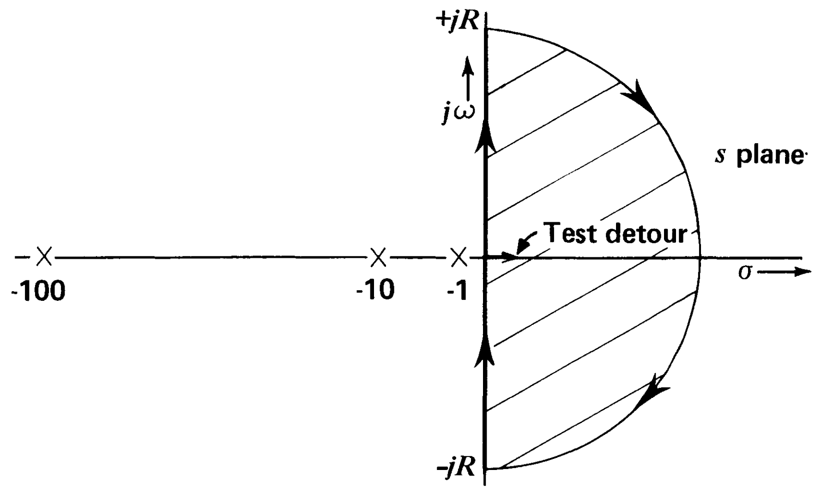
Contour Used to evaluate $a(s)f(s)$.

Viewgraph 7.2



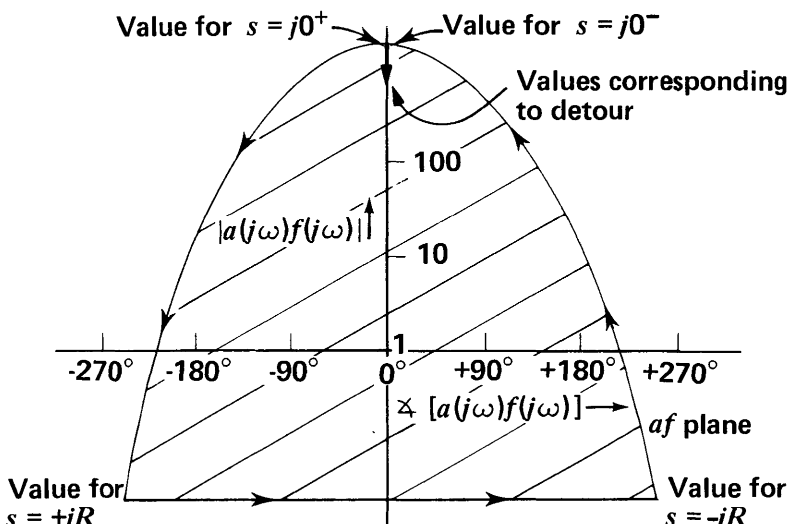
Plot of $a(s)f(s)$ as s varies along contour of previous figure.

Viewgraph 7.3



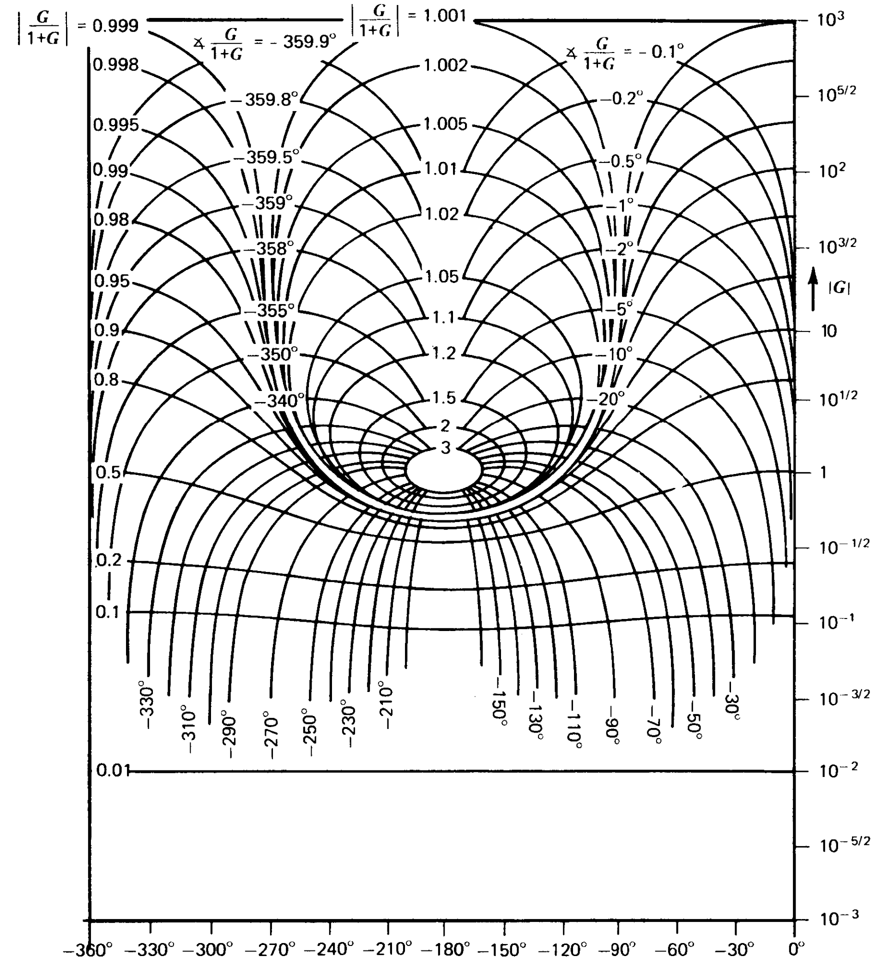
Nyquist test for $a(s)f(s) =$
$10^3/[(s+1)(0.1s+1)(0.01s+1)]$.
(a) s-plane plot.

Viewgraph 7.4



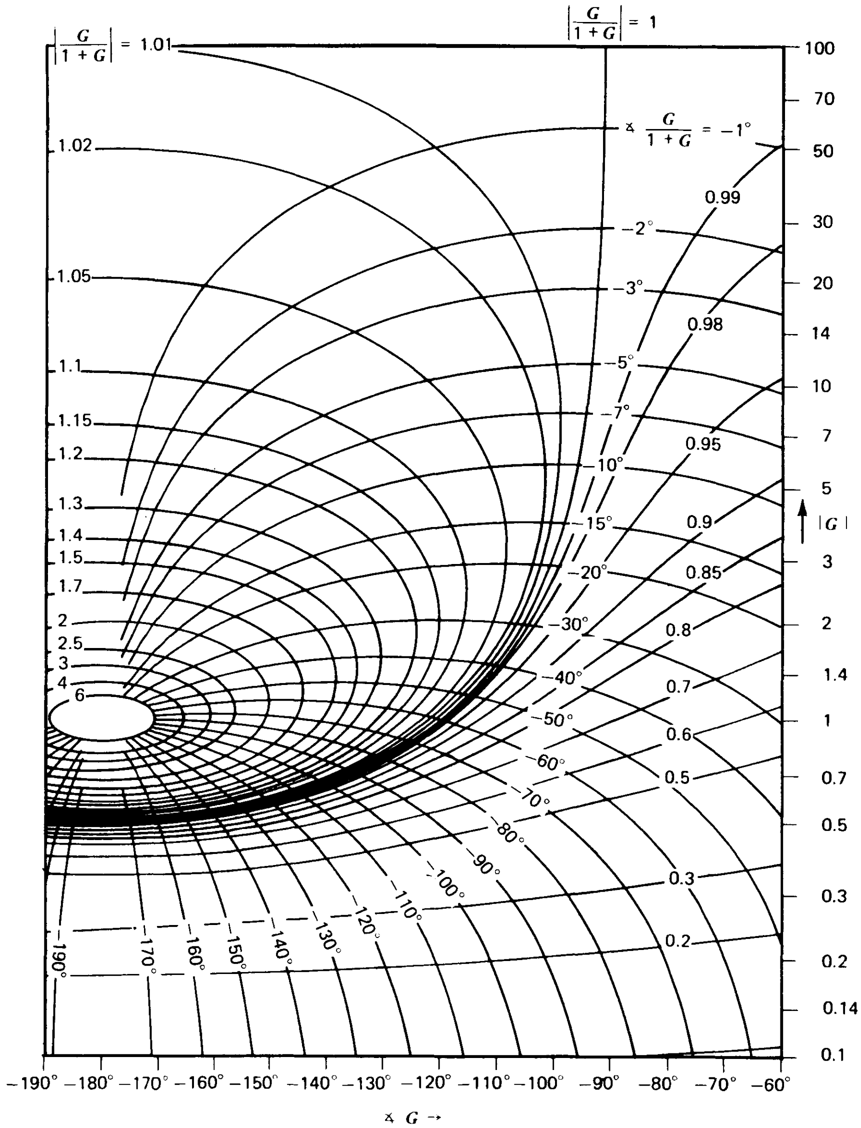
Nyquist test for $a(s)f(s) =$
$10^3/[(s+1)(0.1s+1)(0.01s+1)]$.
(b) af-plane plot.

Viewgraph 7.5

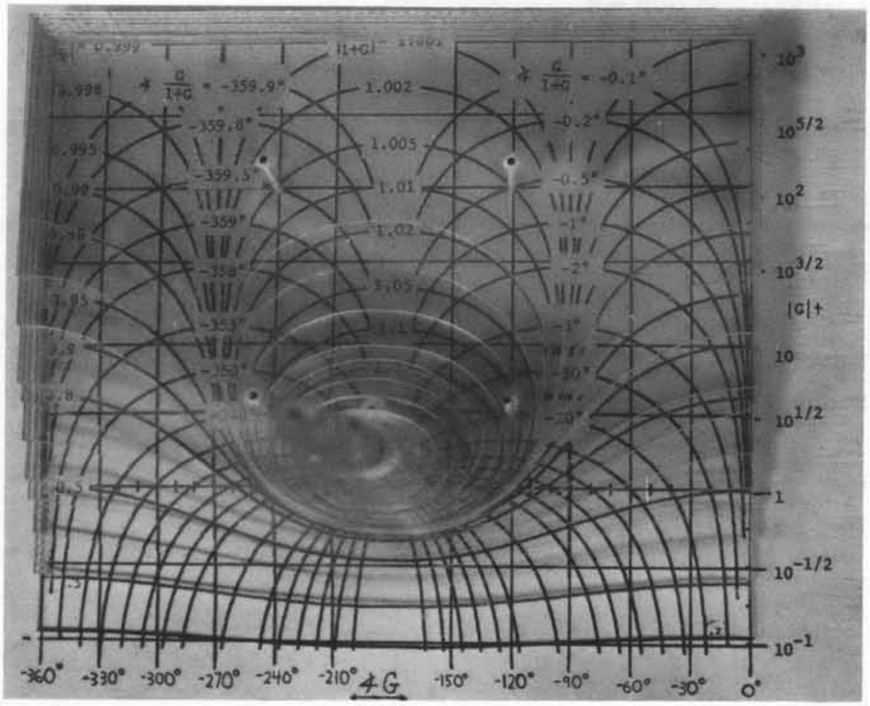


Nichols chart.

Viewgraph 7.6



Expanded Nichols chart.



Demonstration Photograph 7.1
3-dimensional Nichol's chart

Comments

The root-locus method introduced earlier provides a good insight into the behavior of many feedback systems, but it has its limitations. For example, experimental measurements made on an open-loop system may be difficult to convert to the required forms. Furthermore, quantitative results can only be obtained via possibly involved algebraic manipulations.

An alternative that is useful in many cases involves frequency domain manipulations, where the evaluation of relative stability is based on the resonant peak of the closed-loop transfer function. The conversion from open-loop to closed-loop quantities is achieved via the Nichol's chart.

Reading

Textbook: Section 4.4.

Problems

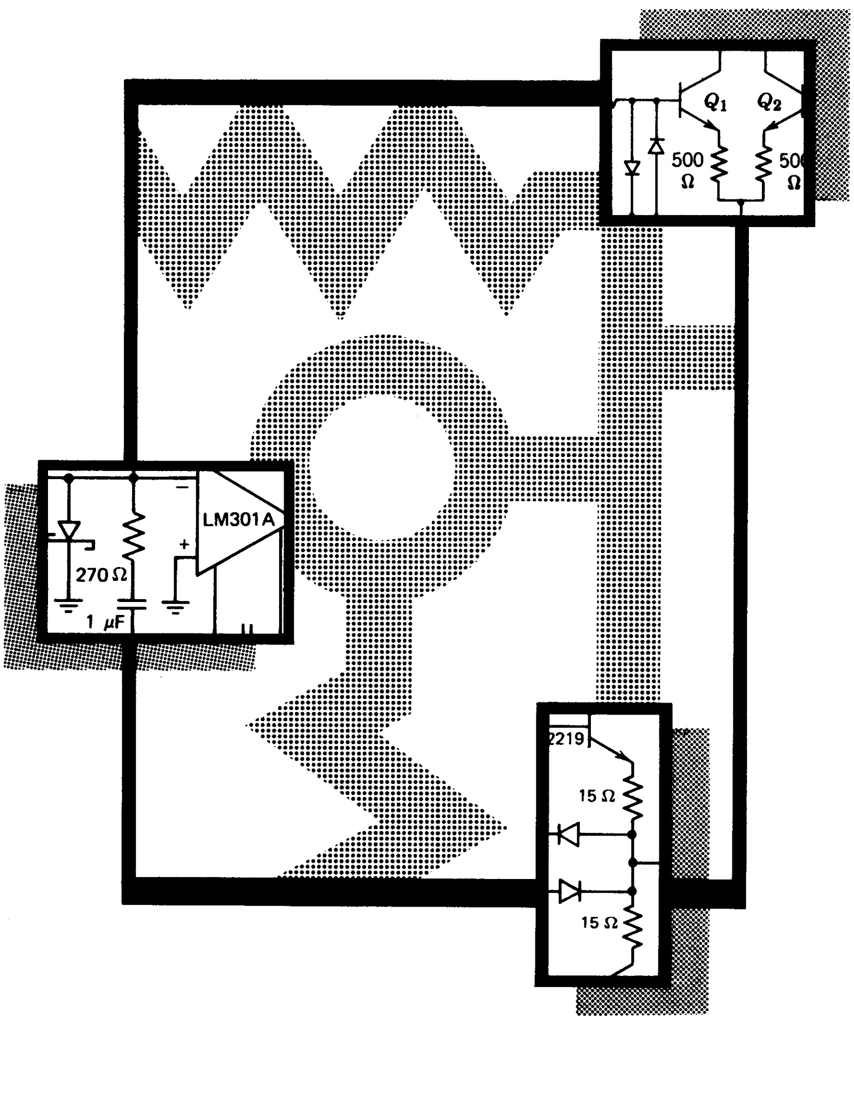
Problem 7.1 (P4.9)

Problem 7.2 (P4.10)

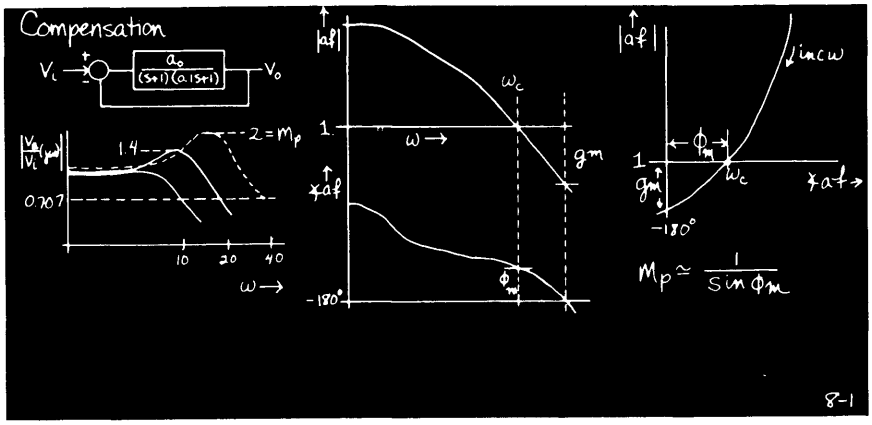
Problem 7.3 (P4.11)

VIDEO COURSE MANUAL

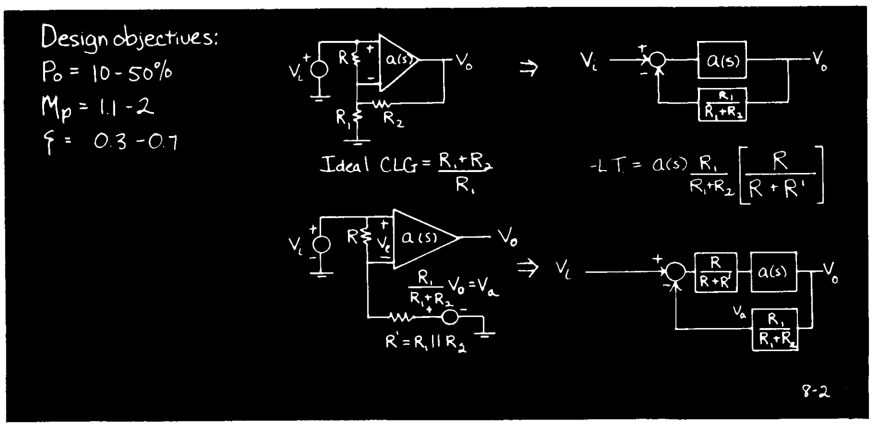
Compensation 8



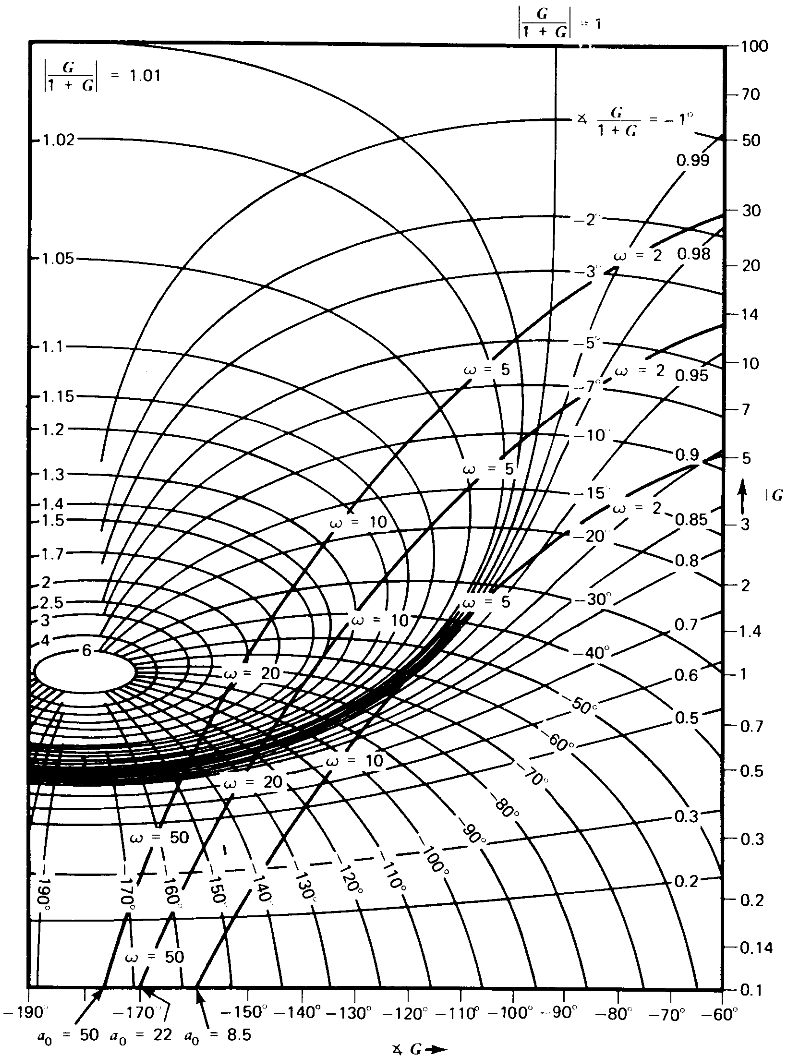
Blackboard 8.1



Blackboard 8.2

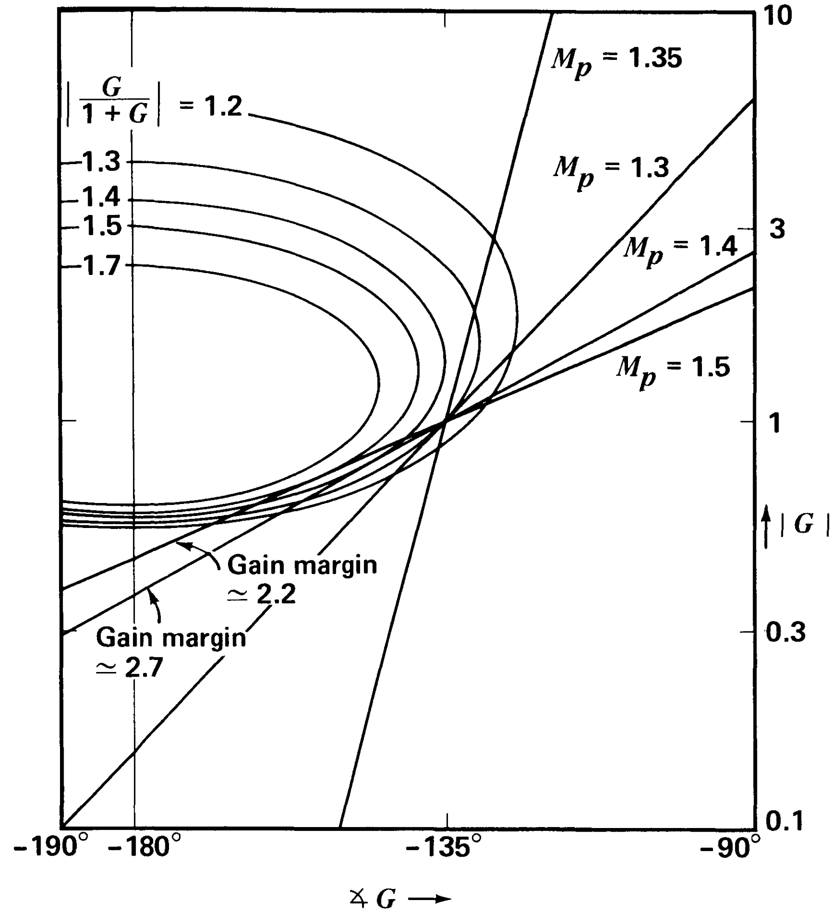


Viewgraph 8.1



Determination of closed-loop transfer function for $a(s) = a_0/[(s+1)(0.1s+1)]$, $f = 1$.

Viewgraph 8.2



M_p for several systems with 45° of phase margin.

Comments

In this lecture we define phase margin and show that it is a valuable indicator of the relative stability of a feedback system. Because of the ease with which they are obtained and the accuracy of estimates based on them, frequency-domain measures are generally used for the quantitative design of feedback systems.

Our discussion of compensation is initiated in this lecture by showing how changes in the $a_o f_o$ product influence stability for typical systems.

Reading

Textbook: Review material in Sections 4.4.2 and 4.4.3. Chapter 5 through Section 5.2.1.

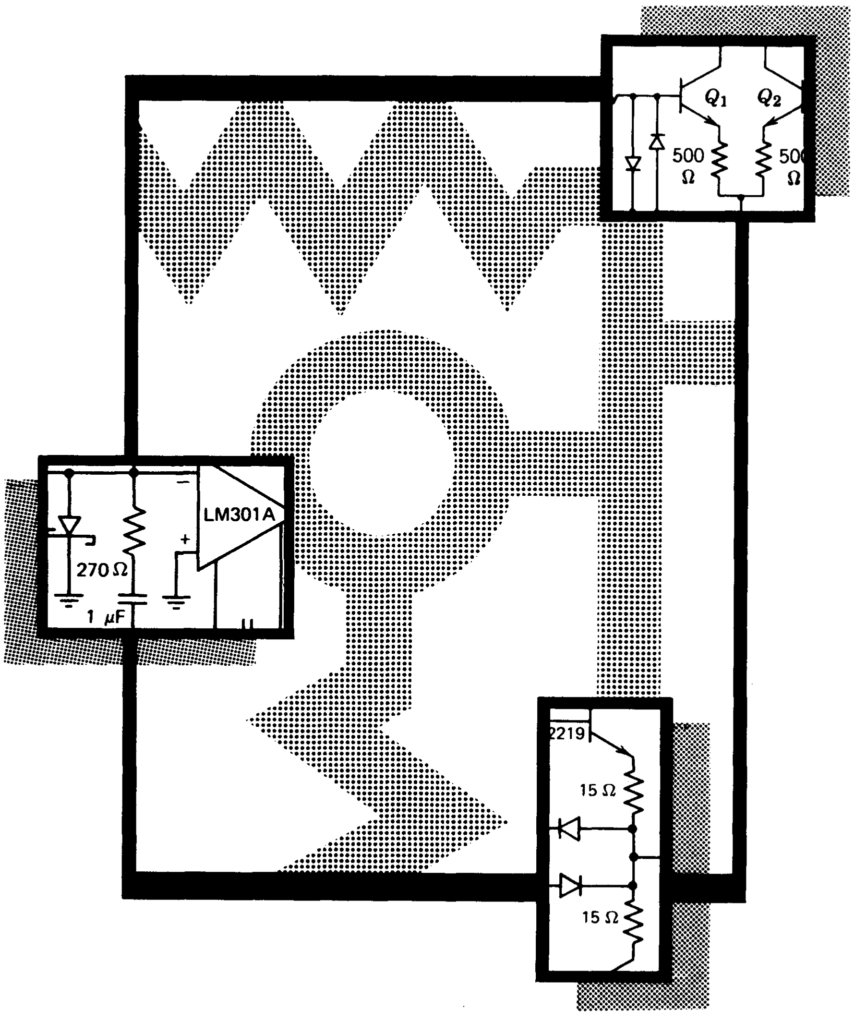
Problems

Problem 8.1 (P4.13)

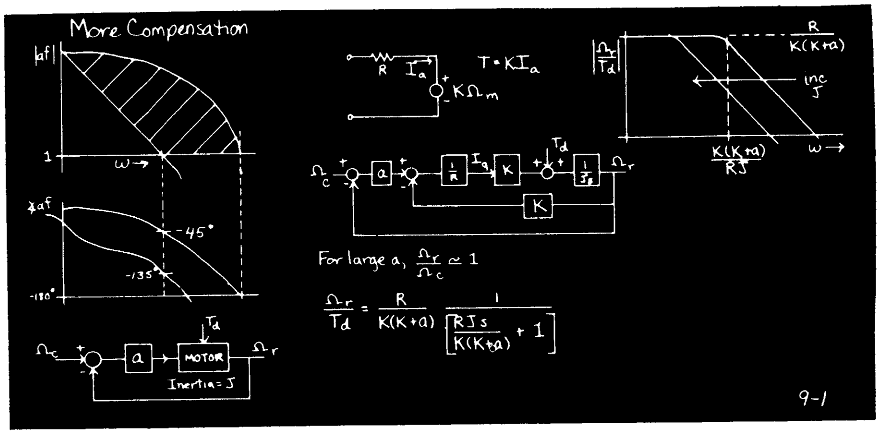
Problem 8.2 (P5.1)

Problem 8.3 (P5.2)

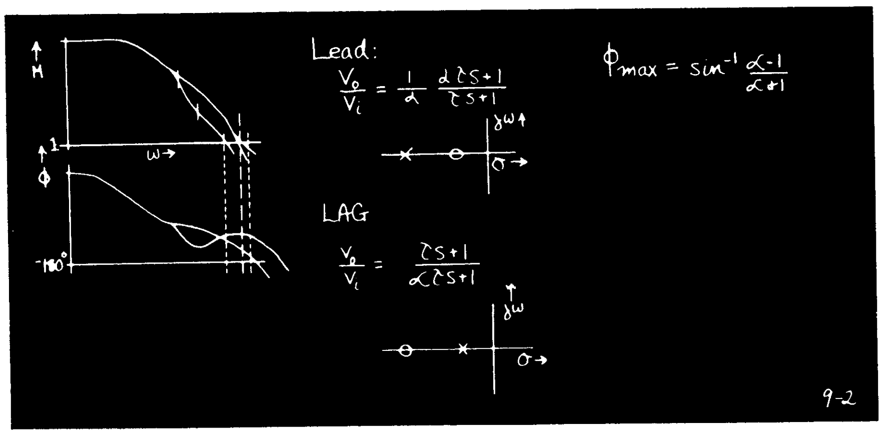
More Compensation 9



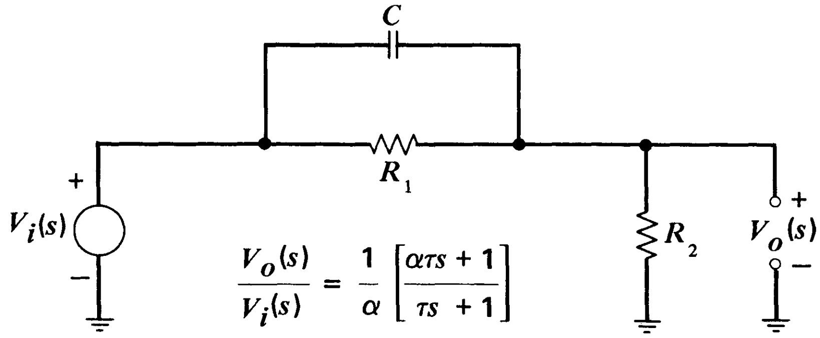
Blackboard 9.1



Blackboard 9.2



Viewgraph 9.1



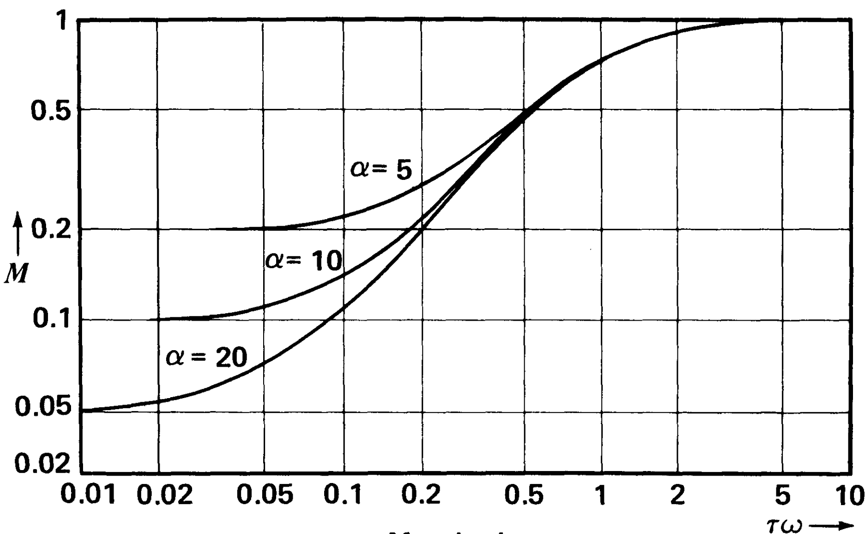
$$\frac{V_o(s)}{V_i(s)} = \frac{1}{\alpha}\left[\frac{\alpha\tau s + 1}{\tau s + 1}\right]$$

where

$$\alpha = \frac{R_1 + R_2}{R_2}$$

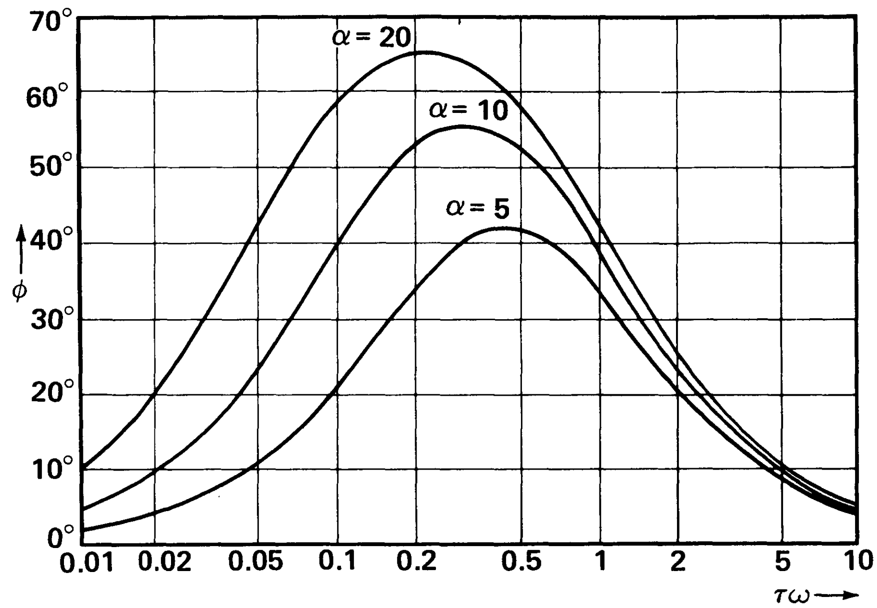
$$\tau = (R_1 \parallel R_2)C$$

Viewgraph 9.2



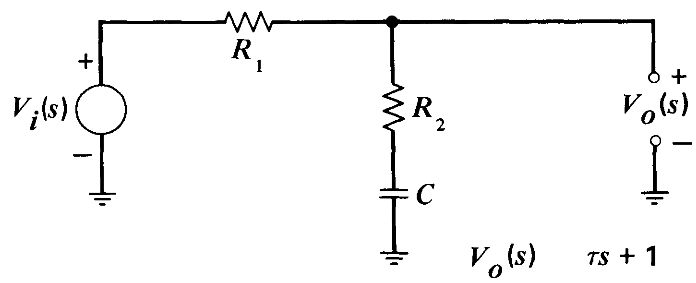
Magnitude
Lead network characteristics for
$V_o(s)/V_i(s) = (1/\alpha)[(\alpha\tau s + 1)/(\tau s + 1)]$.

Viewgraph 9.3



Angle
Lead network characteristics for $V_o(s)/V_i(s) = (1/\alpha)[(\alpha\tau s + 1)/(\tau s + 1)]$.

Viewgraph 9.4



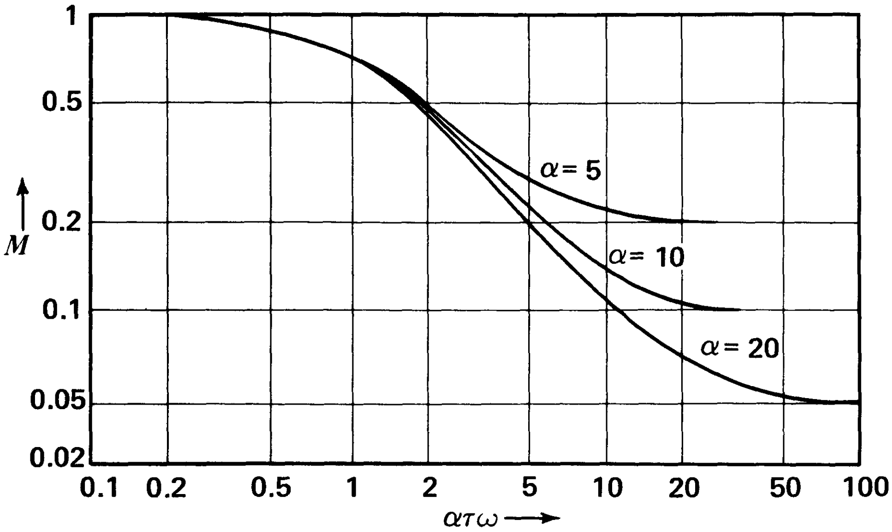
$$\frac{V_o(s)}{V_i(s)} = \frac{\tau s + 1}{\alpha\tau s + 1}$$

where

$$\alpha = \frac{R_1 + R_2}{R_2}$$

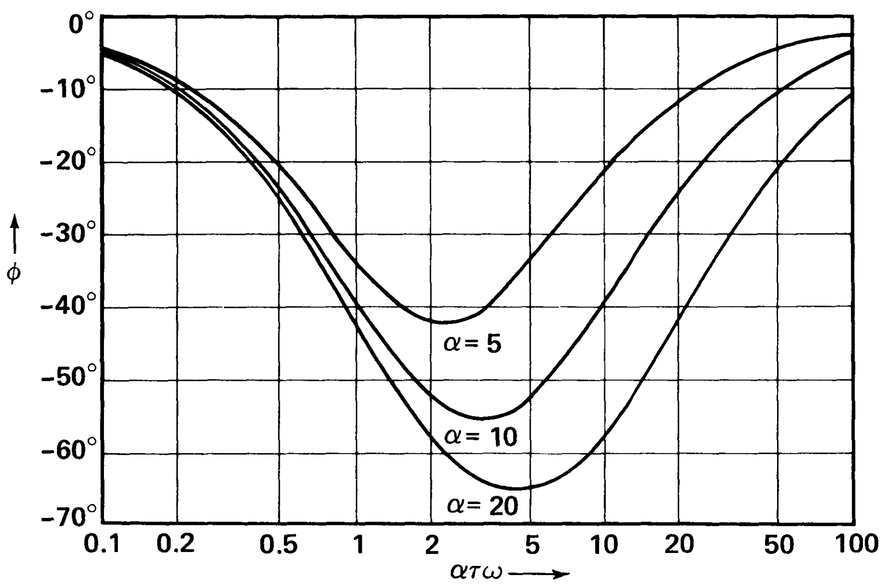
$$\tau = R_2 C$$

Viewgraph 9.5



Magnitude
Lag network characteristics for
$V_o(s)/V_i(s) = (\tau s + 1)/(\alpha\tau s + 1)$.

Viewgraph 9.6



Angle
Lag network characteristics for
$V_o(s)/V_i(s) = (\tau s + 1)/(\alpha\tau s + 1)$.

Comments

In this lecture we indicate the general types of changes that may be made to the dynamics of the loop transmission in order to improve performance. The exact way in which these techniques are applied is strongly dependent on the physical details of the system, and consequently the discussion is quite general. Specifics will be illustrated with examples in future lectures.

Reading

Textbook: Sections 5.2.2 and 5.2.3.

Problems

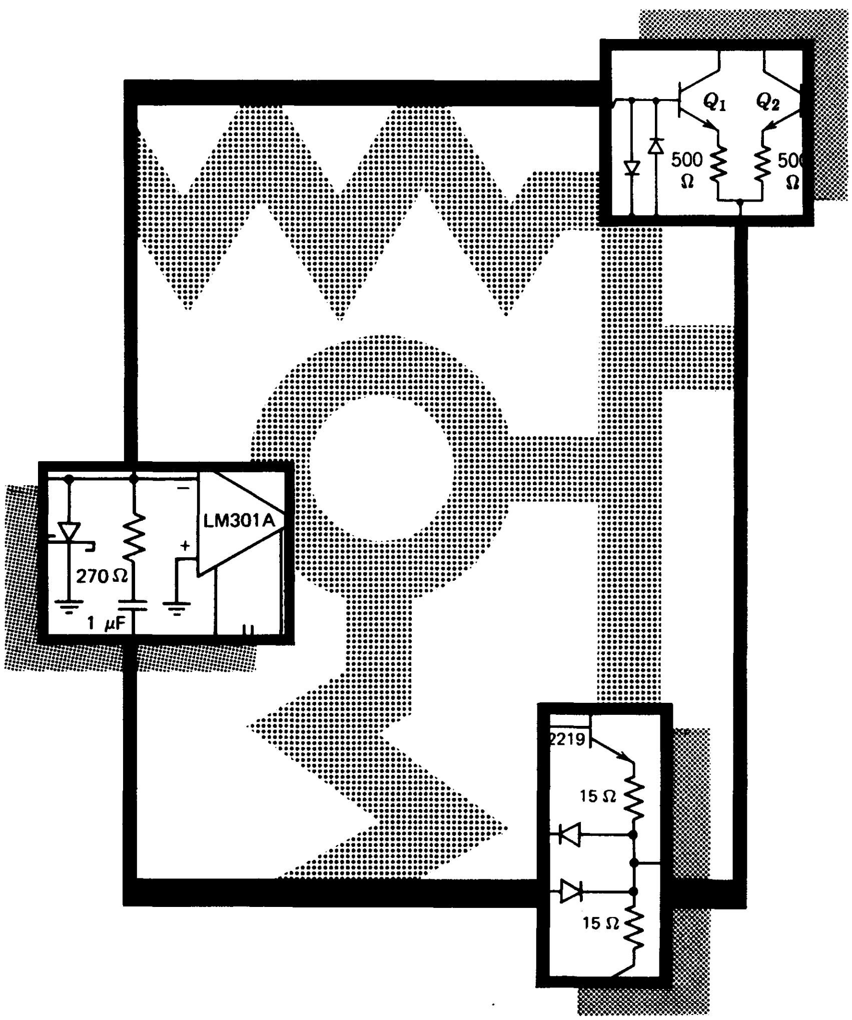
Problem 9.1 (P5.3)

Problem 9.2 (P5.4): Do not do part d.

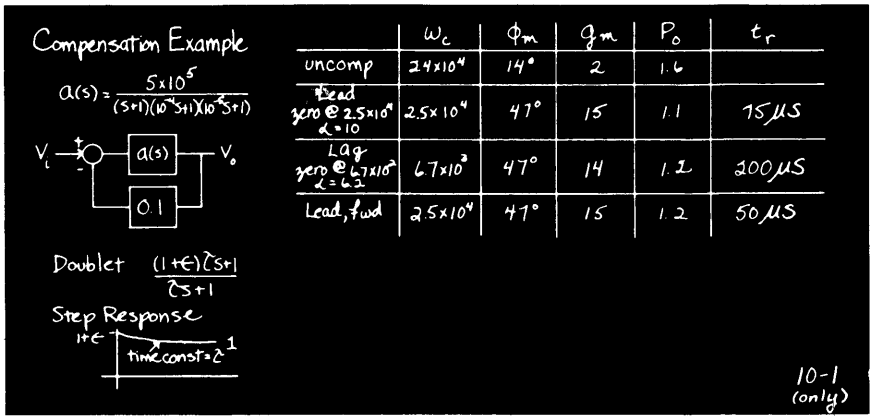
Problem 9.3 (P5.5)

Problem 9.4 (P5.6)

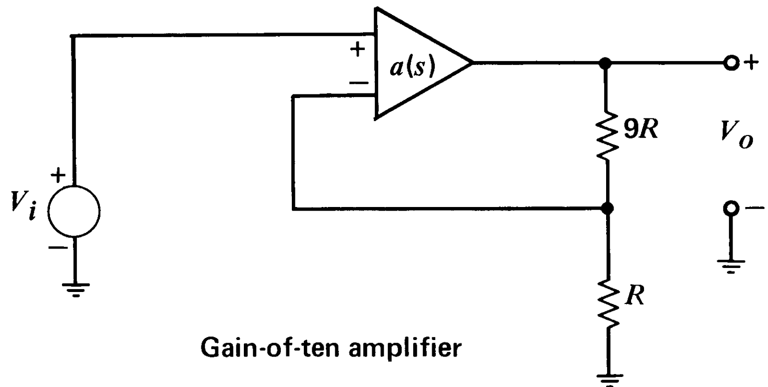
Compensation Example 10



Blackboard 10.1

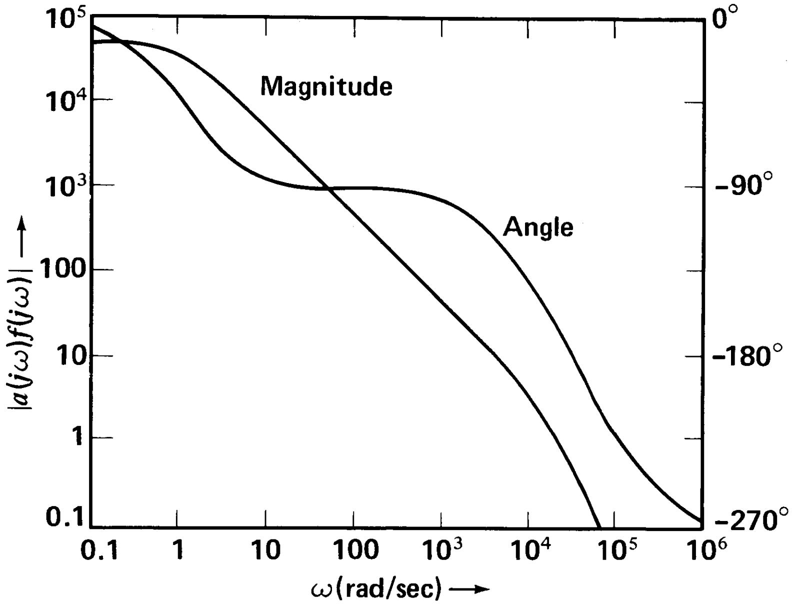


Viewgraph 10.1



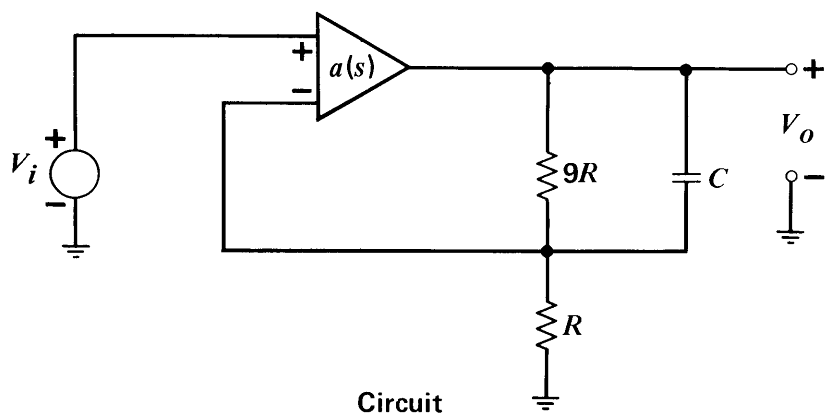
Gain-of-ten amplifier

Viewgraph 10.2



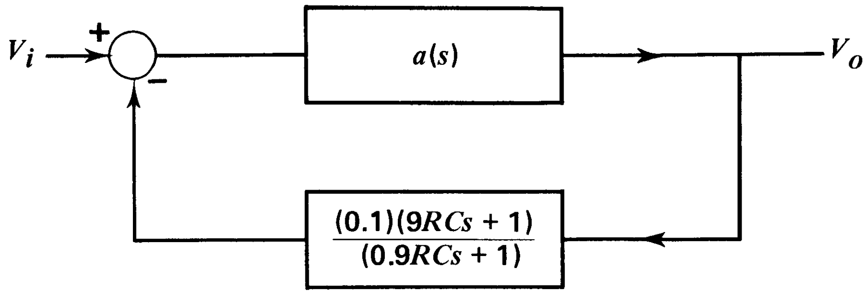
Bode plot for uncompensated gain-of-ten amplifier
$af = 5 \times 10^4/[(s+1)(10^{-4}s+1)(10^{-5}s+1)]$.

Viewgraph 10.3



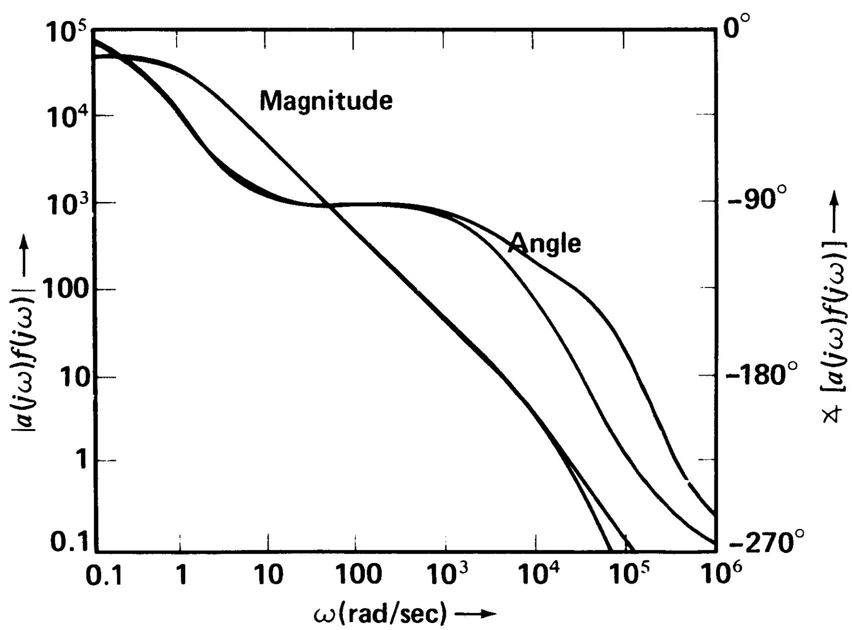
Circuit
Gain-of-ten amplifier with lead network in feedback path.

Viewgraph 10.4



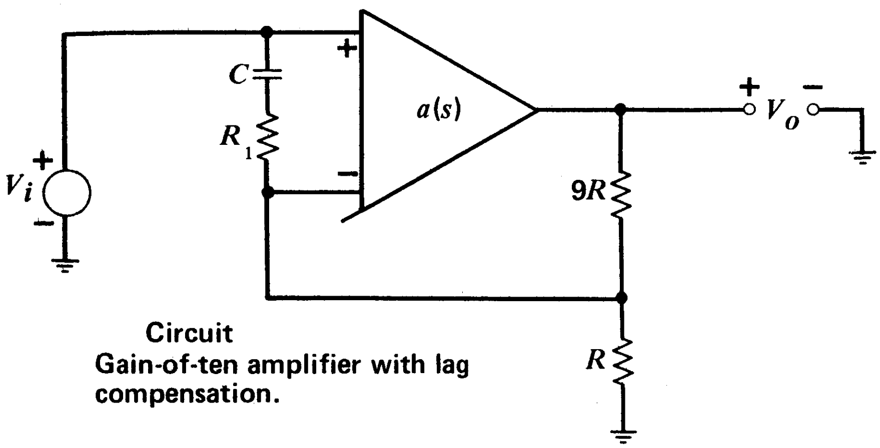
Block diagram
Gain-of-ten amplifier with lead network in feedback path.

Viewgraph 10.5



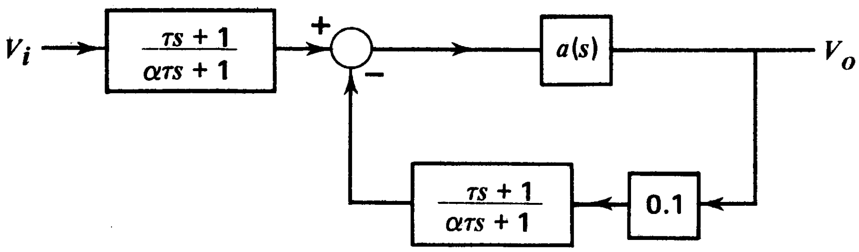
Bode plot for uncompensated gain-of-ten amplifier
$af = 5 \times 10^4/[(s + 1)(10^{-4}s + 1)(10^{-5}s + 1)]$.

Viewgraph 10.6



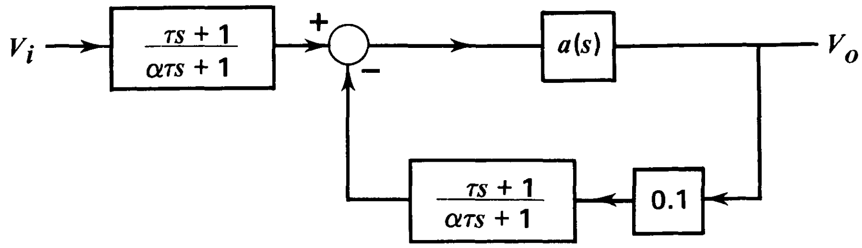
Circuit
Gain-of-ten amplifier with lag compensation.

Viewgraph 10.7



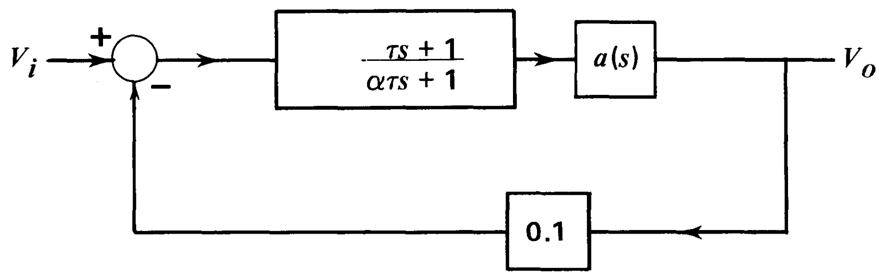
Block diagram
Gain-of-ten amplifier with lag compensation.

Viewgraph 10.8



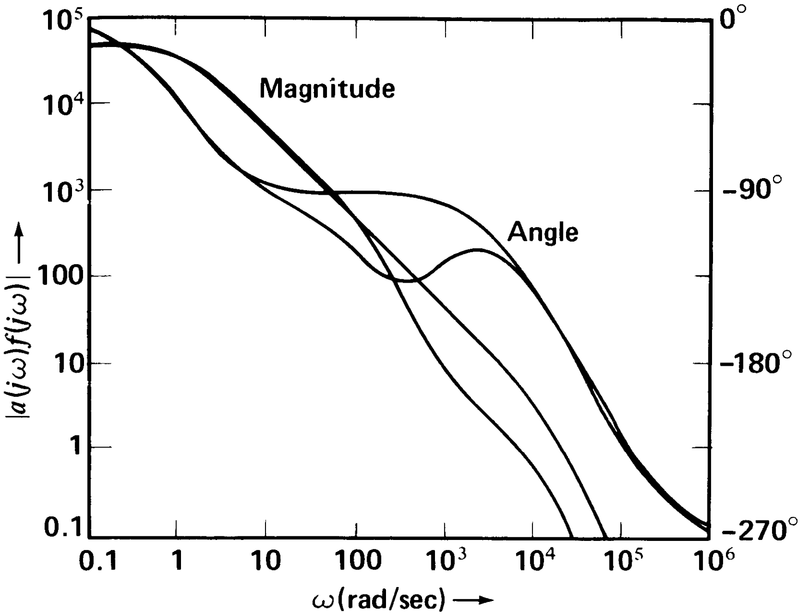
Block diagram
Gain-of-ten amplifier with lag compensation.

Viewgraph 10.9



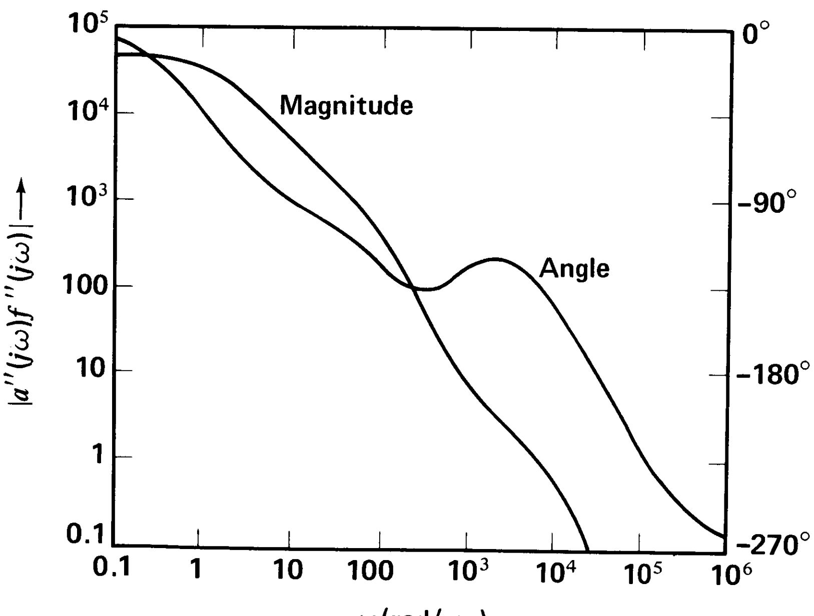
Block diagram following rearrangement.
Gain-of-ten amplifier with lag compensation.

Viewgraph 10.10



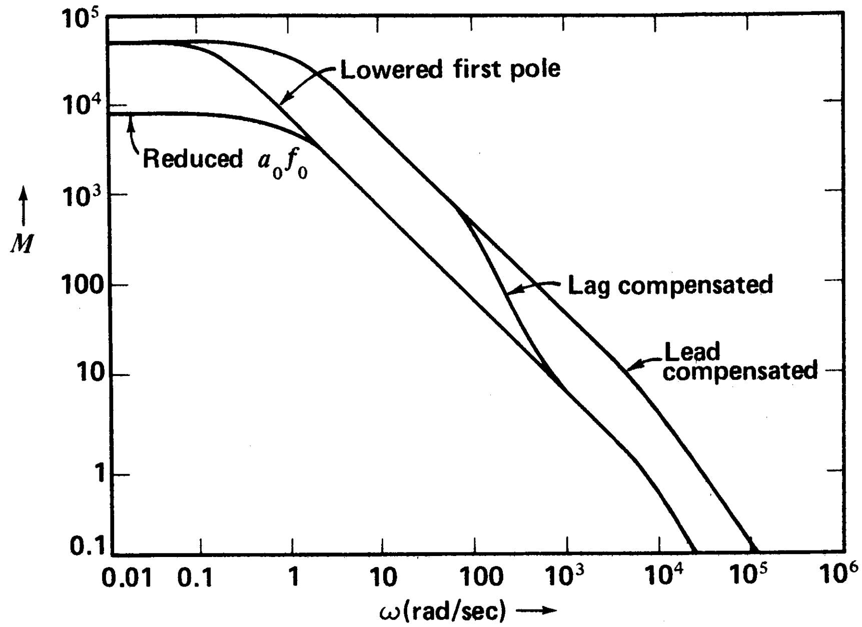
Bode plot for uncompensated
gain-of-ten amplifier
$af = 5 \times 10^4/[(s+1)(10^{-4}s+1)(10^{-5}s+1)]$.

Viewgraph 10.11



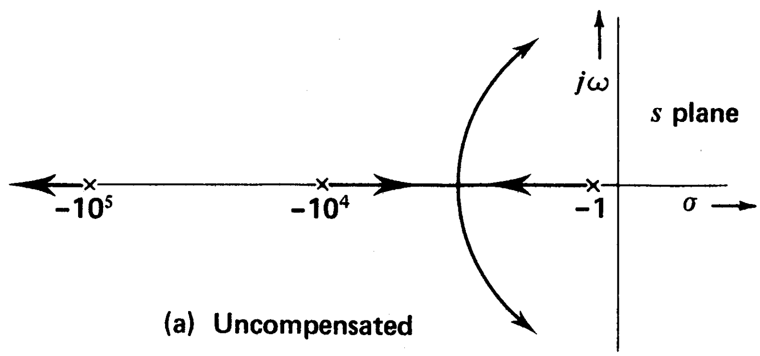
Bode plot for lag compensated
gain-of-ten amplifier.
$a''f'' = 5 \times 10^4(1.5 \times 10^{-3}s+1)/$
$[(s+1)(10^{-4}s+1)(10^{-5}s+1)(9.3 \times 10^{-3}s+1)]$.

Viewgraph 10.12



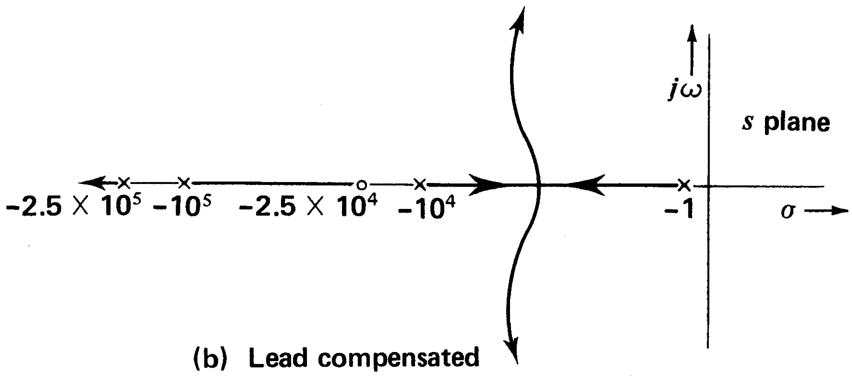
Effects of various types of compensation on loop-transmission magnitude.

Viewgraph 10.13



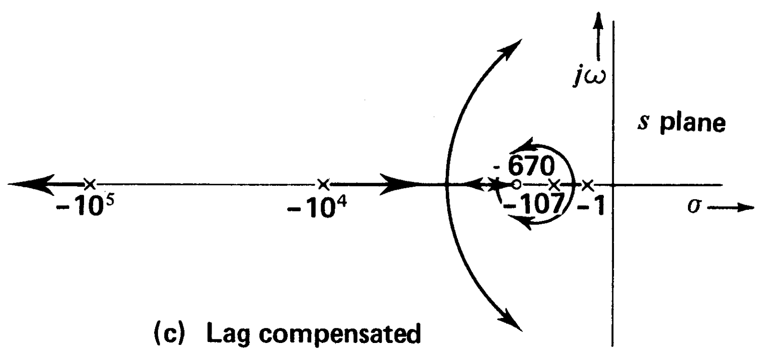
(a) Uncompensated

Viewgraph 10.14



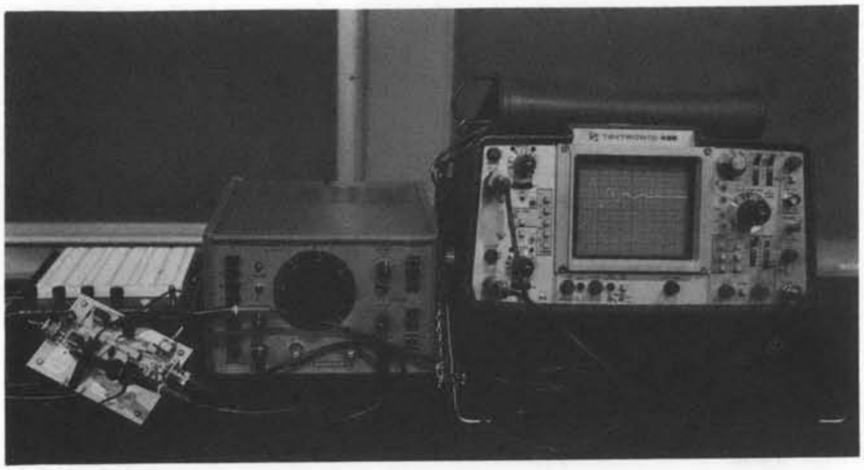
(b) Lead compensated

Viewgraph 10.15

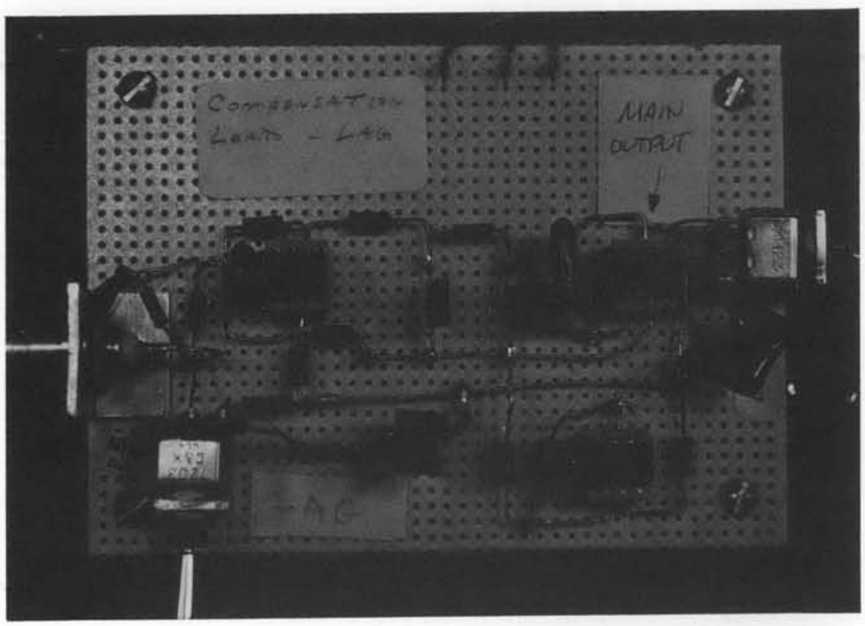


(c) Lag compensated

Demonstration Photograph
10.1 Gain-of-ten amplifier demonstration



Demonstration Photograph
10.2 Close-up of gain-of-ten amplifier



Comments

This lecture's analysis and the associated demonstrations show how the compensation methods developed earlier can be applied to a physical system. The demonstration system is, admittedly, somewhat contrived so that the required manipulations can be easily performed with accuracy. In many actual systems, exact system parameters are less certain. However, analysis parallel to that presented in the lecture generally provides an excellent first cut at the appropriate compensator, which may then be refined based on test results.

It is important that participants in this course have first-hand experience with the kind of analysis and measurements required for compensation. Toward this end, the problem (P5.15) suggested below, which includes substantial laboratory effort, is a very necessary part of the course.

Clarification

I suggest aiming for 47° of phase margin in the lag-compensated system without much justification for the design objective. The reason is effectively an educational one. We choose the lead compensation to provide the maximum possible phase margin given the constraints of the topology. This approach is reasonable, because the maximum achievable value of 47° is adequate, but certainly not overly conservative.

In the case of lag compensation, larger values of phase margin could be obtained by using larger values of α. The value of 47° was chosen so that direct stability and speed of response comparisons could be made with the lead-compensated amplifier.

Reading

Textbook: Sections 5.2.4 through 5.2.6.

Problems

Problem 10.1 (P5.8)

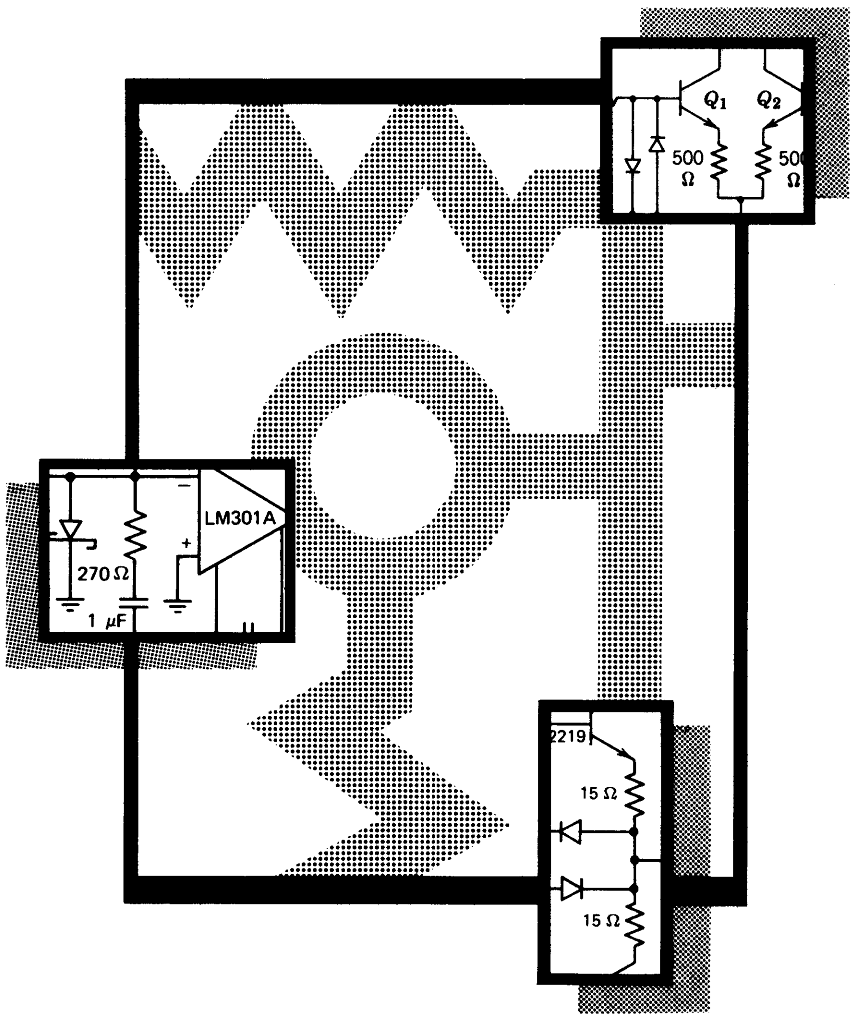
Problem 10.2 (P5.12)

Problem 10.3 (P5.13): In calculating the settling time for the lag-compensated gain-of-ten amplifier, make the simplifying assumption that the loop transmission is second order. That is, the poles of Equation 5.15 at 10^4 and 10^5 rad/sec may be ignored, and $a''(s)$ approximated by

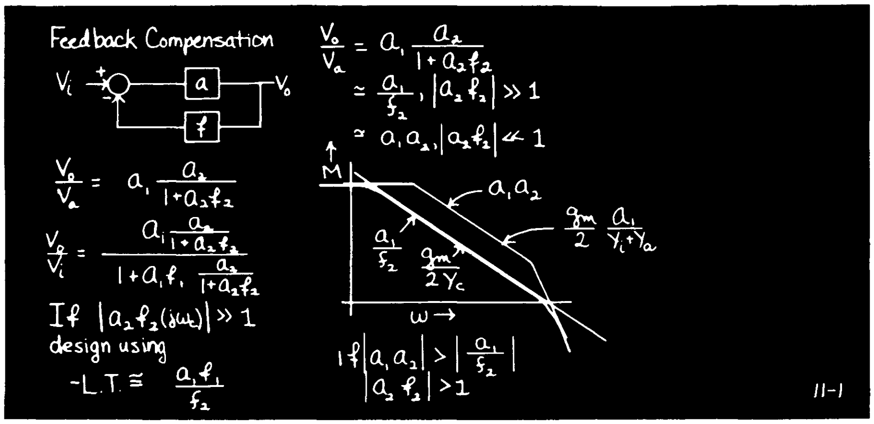
$$a''(s) \simeq \frac{5 \times 10^5(1.5 \times 10^{-3}s + 1)}{(s + 1)(9.3 \times 10^{-3}s + 1)}$$

Problem 10.4 (P5.15)

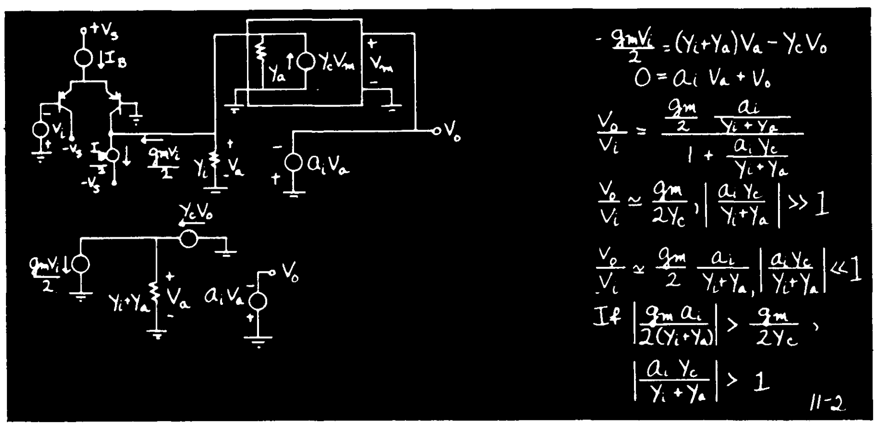
Feedback Compensation 11



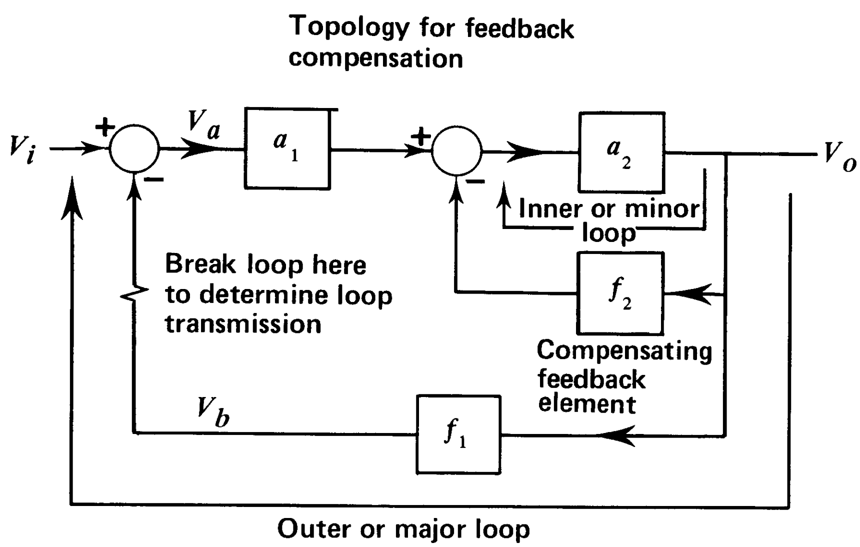
Blackboard 11.1



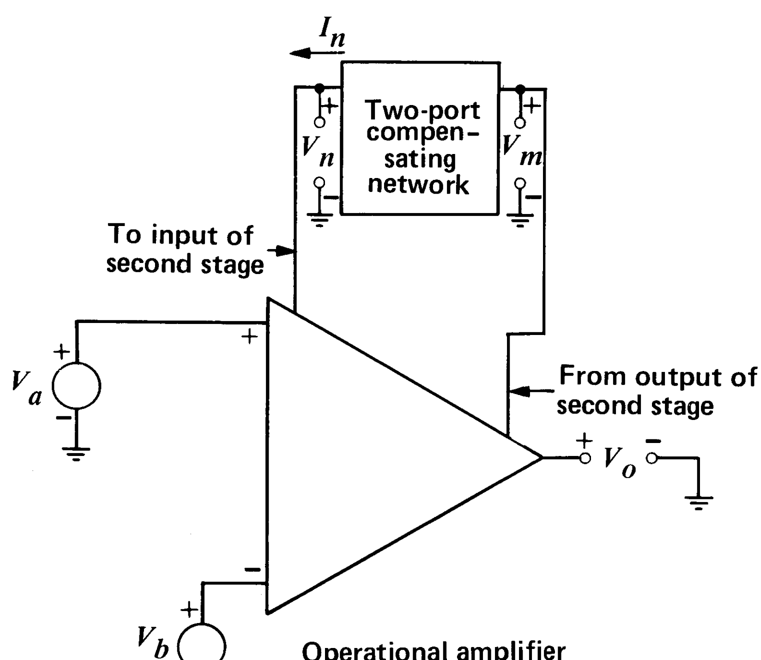
Blackboard 11.2



Viewgraph 11.1



Viewgraph 11.2



Operational amplifier compensated with a two-port network.

Comments

Minor-loop compensation provides a preferable alternative to cascade compensation for many physical systems. Examples include servomechanisms using tachometric feedback and a number of available integrated-circuit operational amplifiers.

The appropriate compensation for a particular application is generally determined by assuming that feedback controls the behavior of the minor loop at the major-loop crossover frequency. The possibility is realistic because the relatively fewer elements included in the minor loop permit it to have a higher crossover frequency.

Reading

Textbook: Sections 5.3 and 13.3.1.

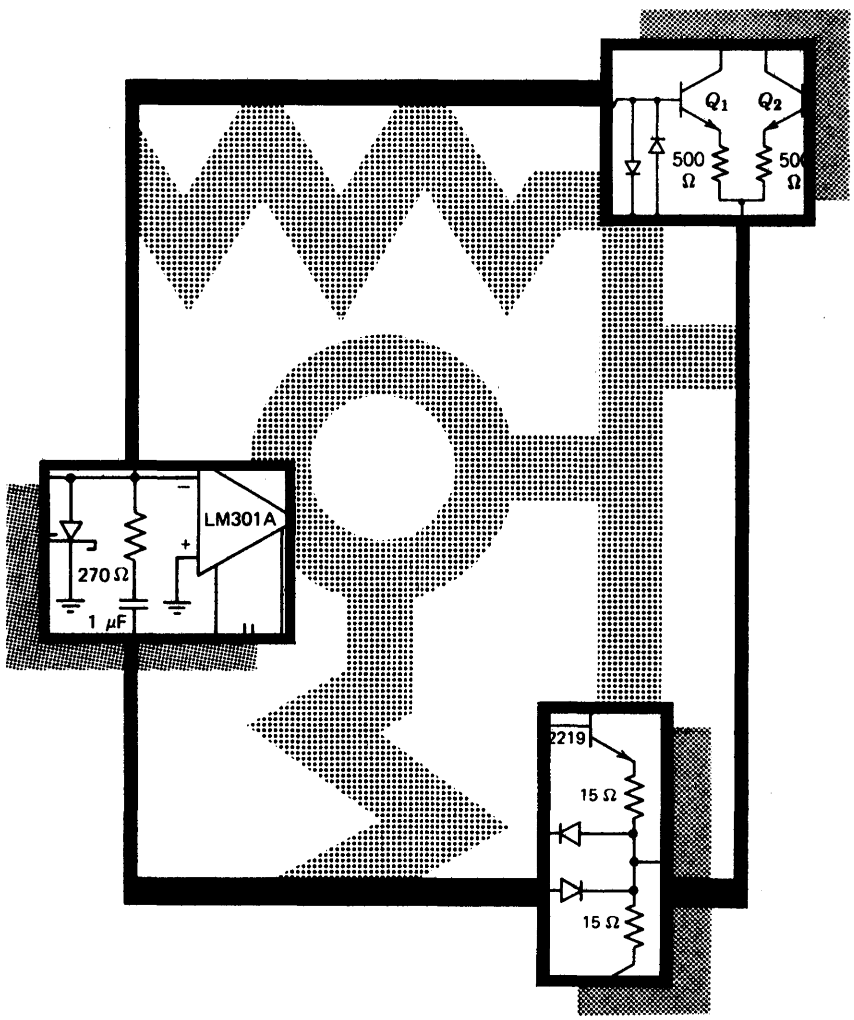
Problem

Problem 11.1 (P5.14)

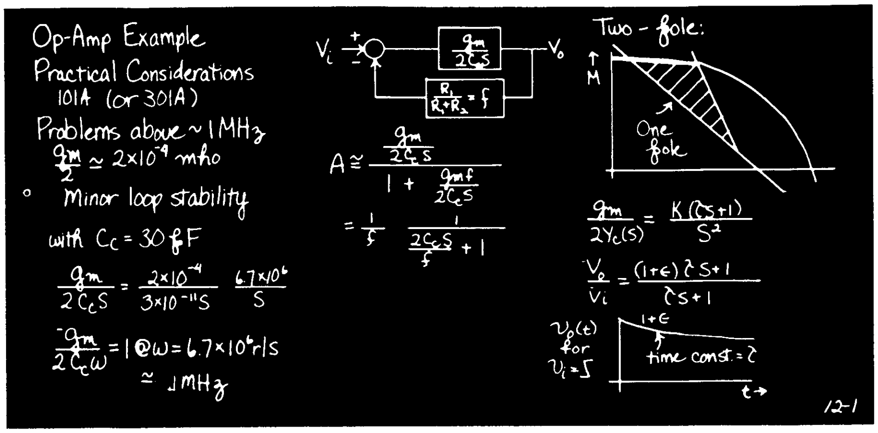
V I D E O   C O U R S E   M A N U A L

Feedback Compensation of an Operational Amplifier

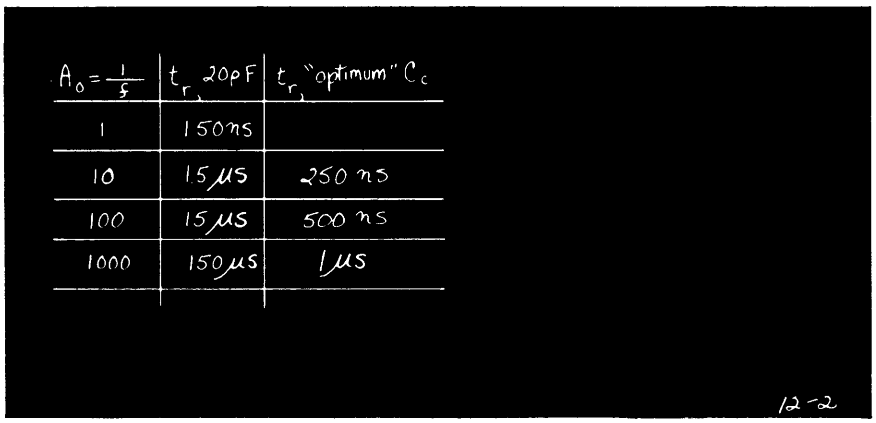
12



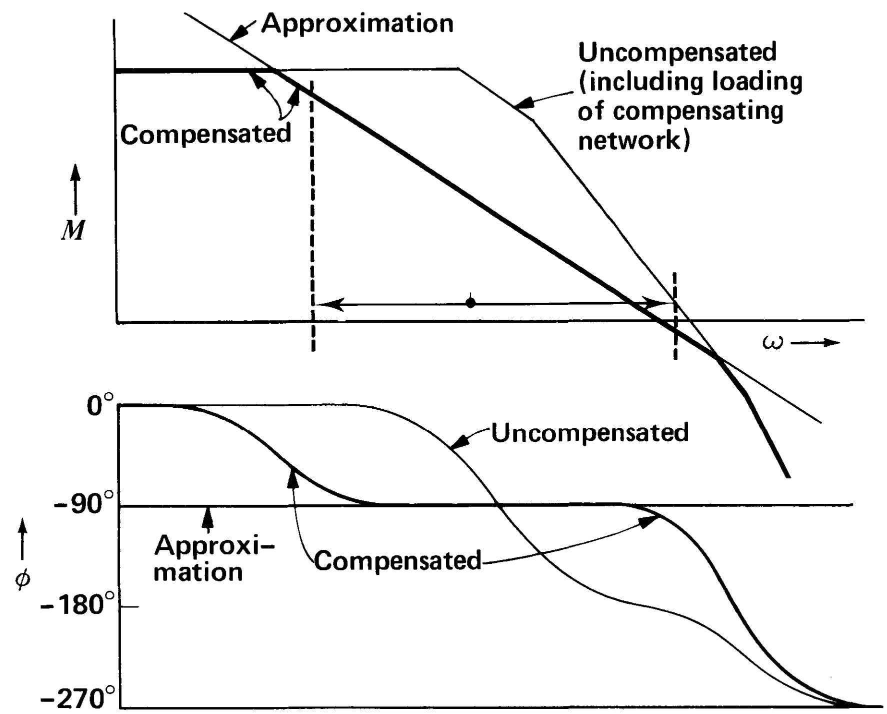
Blackboard 12.1



Blackboard 12.2

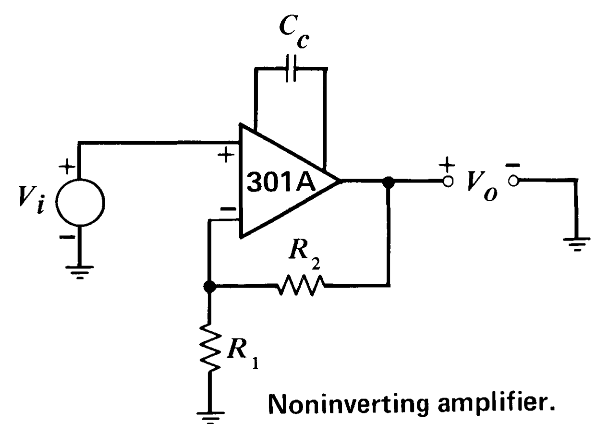


Viewgraph 12.1



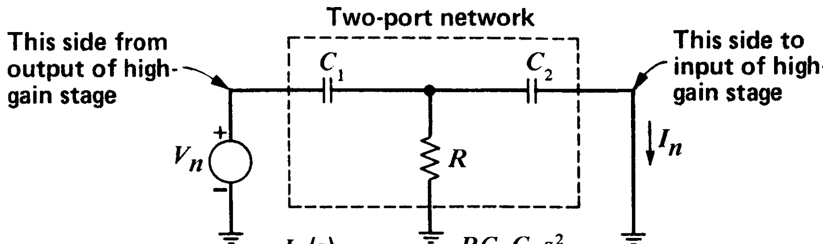
Open-loop transfer function for one-pole compensation.

Viewgraph 12.2



Noninverting amplifier.

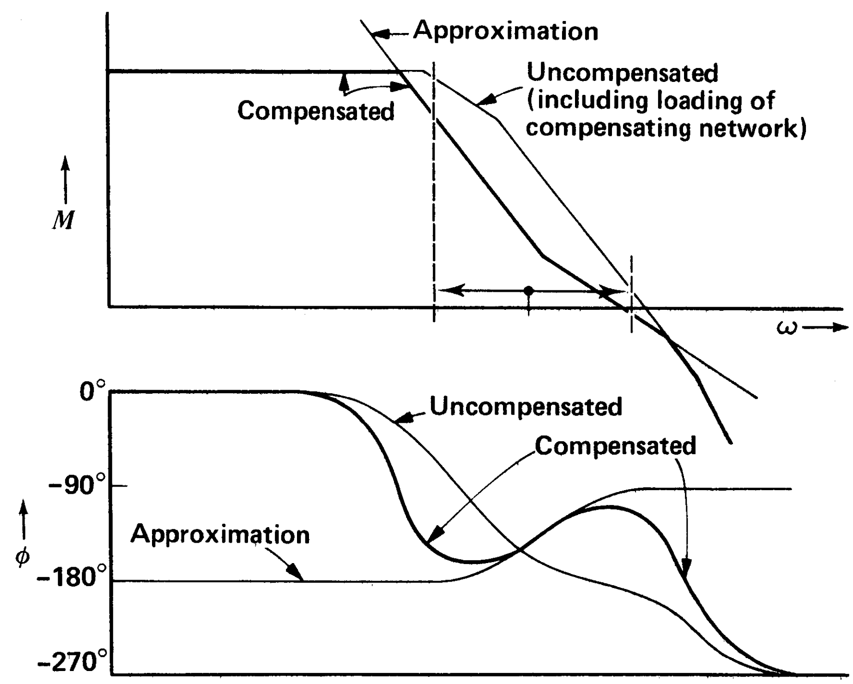
Viewgraph 12.3



$$\frac{I_n(s)}{V_n(s)} = \frac{RC_1C_2s^2}{R(C_1 + C_2)s + 1}$$

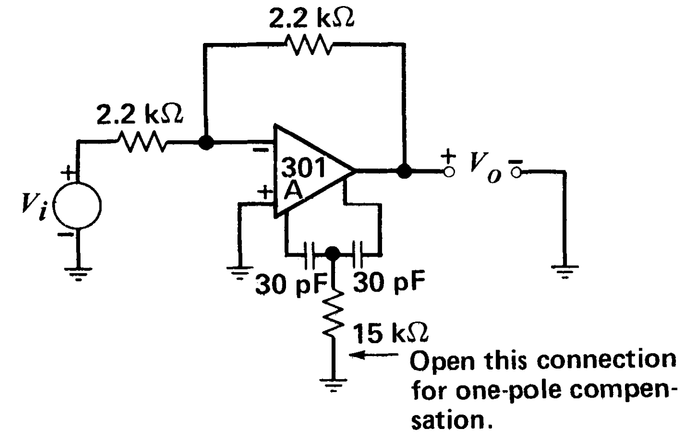
Network for two-pole compensation

Viewgraph 12.4



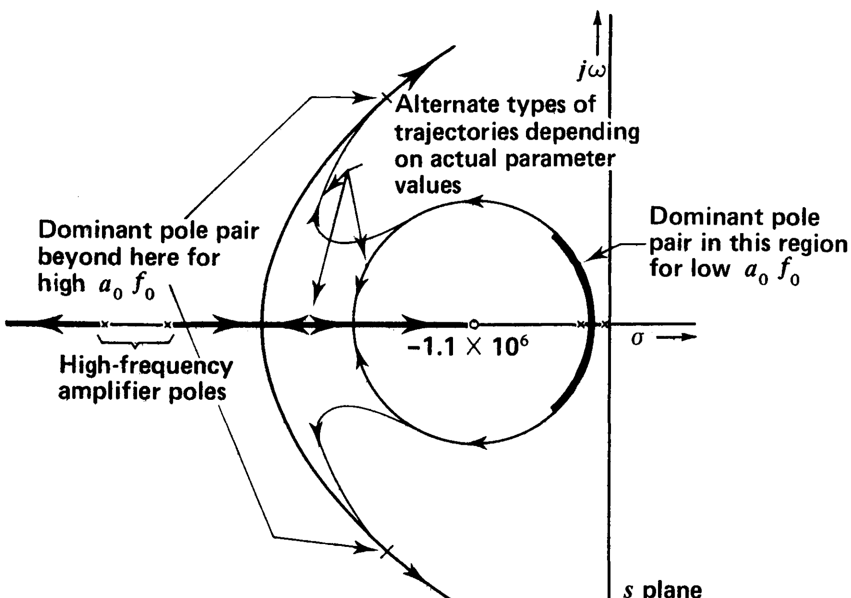
Open-loop transfer function for two-pole compensation.

Viewgraph 12.5



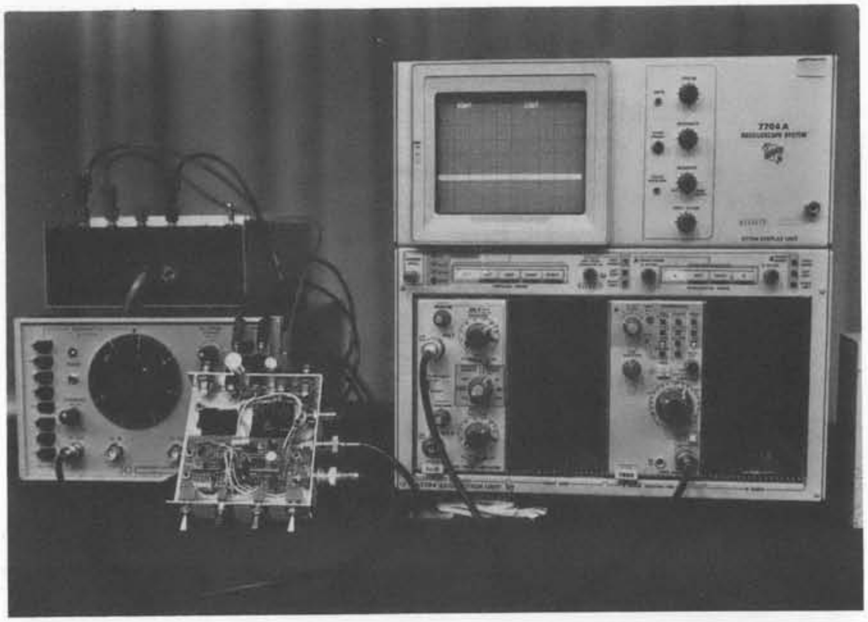
Unity-gain inverter with two-pole compensation.

Viewgraph 12.6

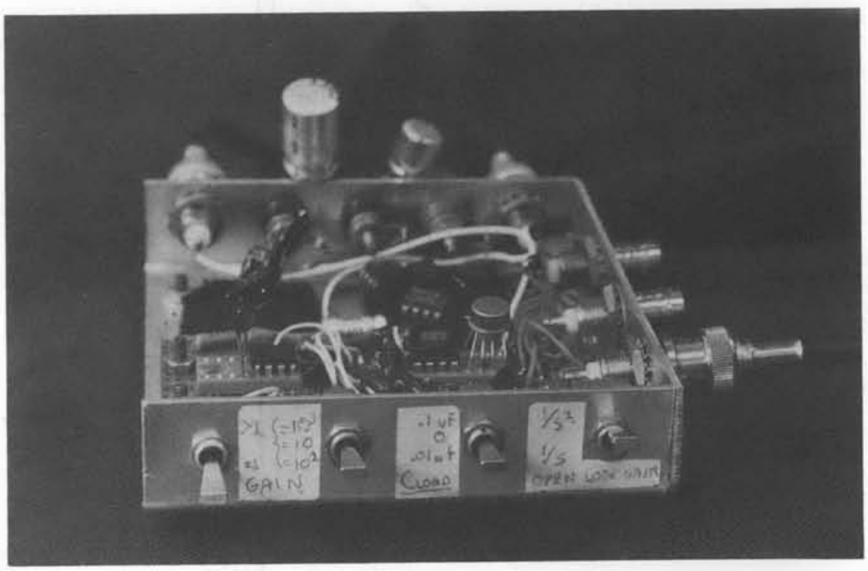


Root-locus diagram for inverter with two-pole compensation.

Demonstration Photograph
12.1 Operational-amplifier compensation demonstration



Demonstration Photograph
12.2 Close-up of operational-amplifier for compensation circuit



Comments

In this session we show how minor-loop compensation is used to control the dynamics of an available integrated-circuit operational amplifier. We find that in certain applications, dramatic performance improvements are possible compared with a similar amplifier that uses fixed compensation that is selected for unity-gain stability.

While specific values for the compensating components are selected based on parameters of the amplifier type used, the general methods are applicable to any amplifier that allows the choice of the components used for minor-loop compensation.

Correction

In about the middle of the first blackboard I give the expression for the closed-loop gain as:

$$A \simeq \frac{\dfrac{g_m}{2C_c s}}{1 + \dfrac{g_m f}{2C_c s}} = \frac{1}{f} \frac{1}{\dfrac{2C_c s}{f} + 1}$$

The final expression should read:

$$\frac{1}{f} \frac{1}{\dfrac{2C_c s}{g_m f} + 1}$$

(There is a g_m missing in the equation on the blackboard.)

Reading

Textbook: Sections 13.3.2 and 13.3.3.

Problems

Problem 12.1 (P13.5)

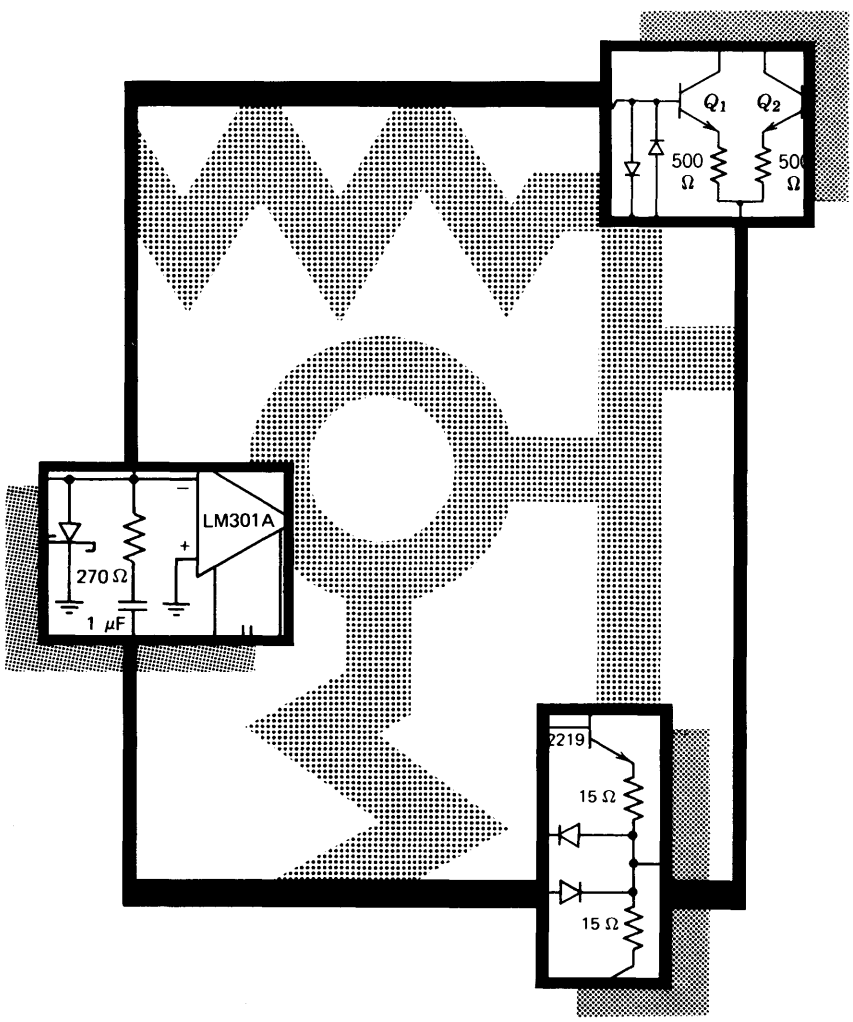
Problem 12.2 (P13.6)

Problem 12.3 (P13.7)

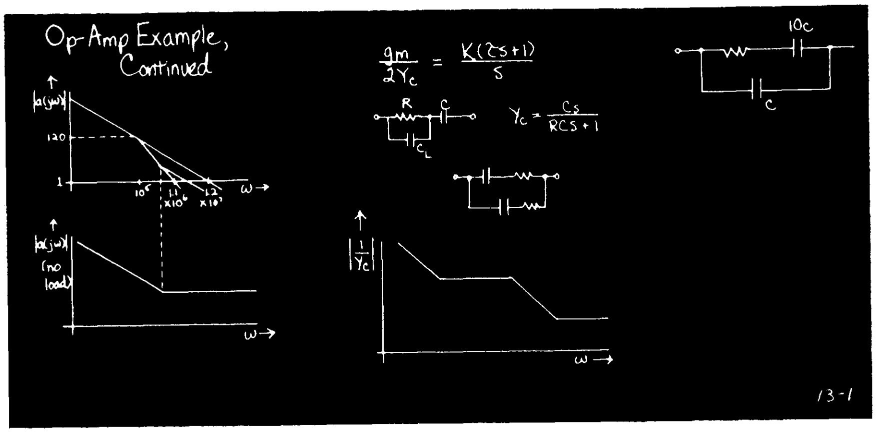
Problem 12.4 (P13.8)

Operational Amplifier Compensation (continued)

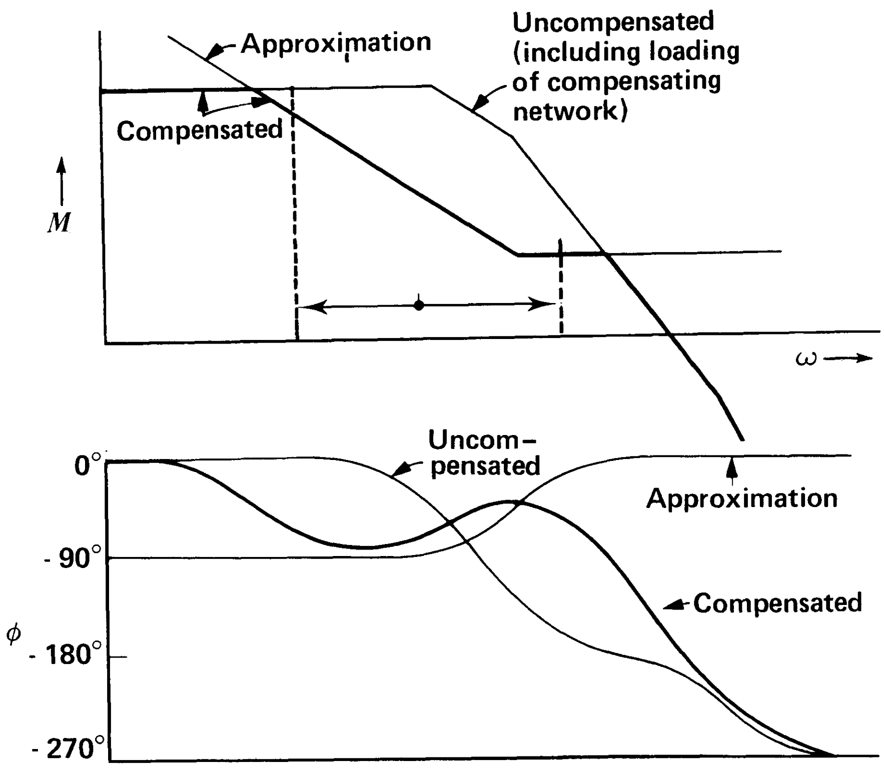
13



Blackboard 13.1

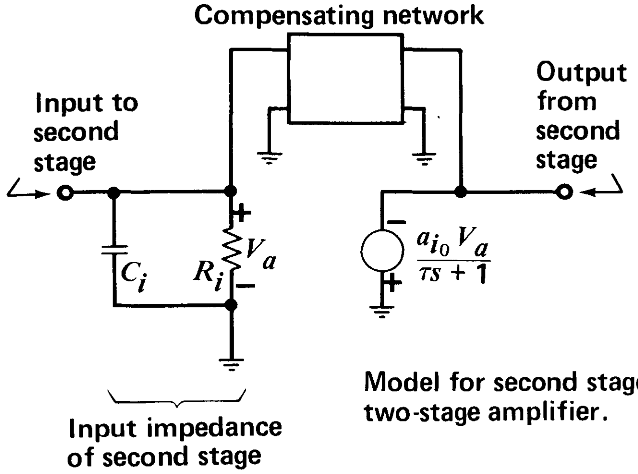


Viewgraph 13.1



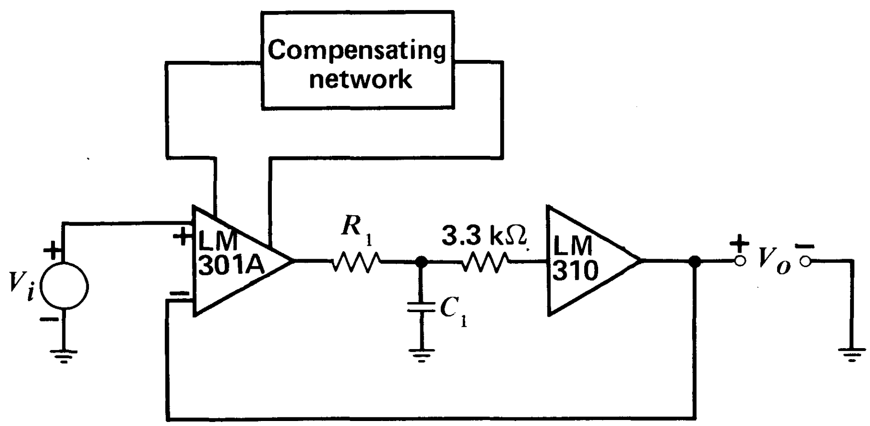
Unloaded open-loop transfer function for compensation that includes a zero.

Viewgraph 13.2



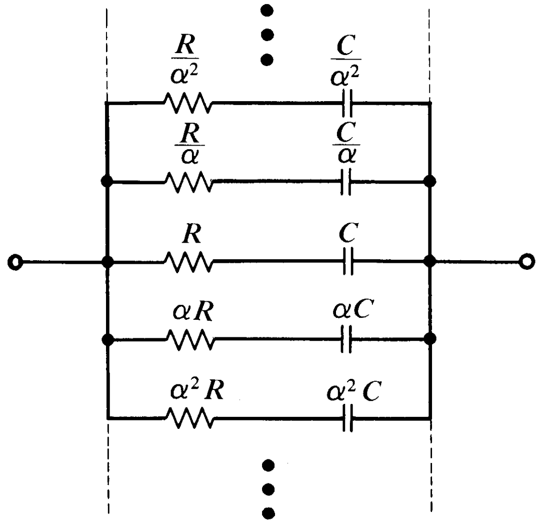
Model for second stage of a two-stage amplifier.

Viewgraph 13.3



Circuit used to evaluate slow-rolloff compensation.

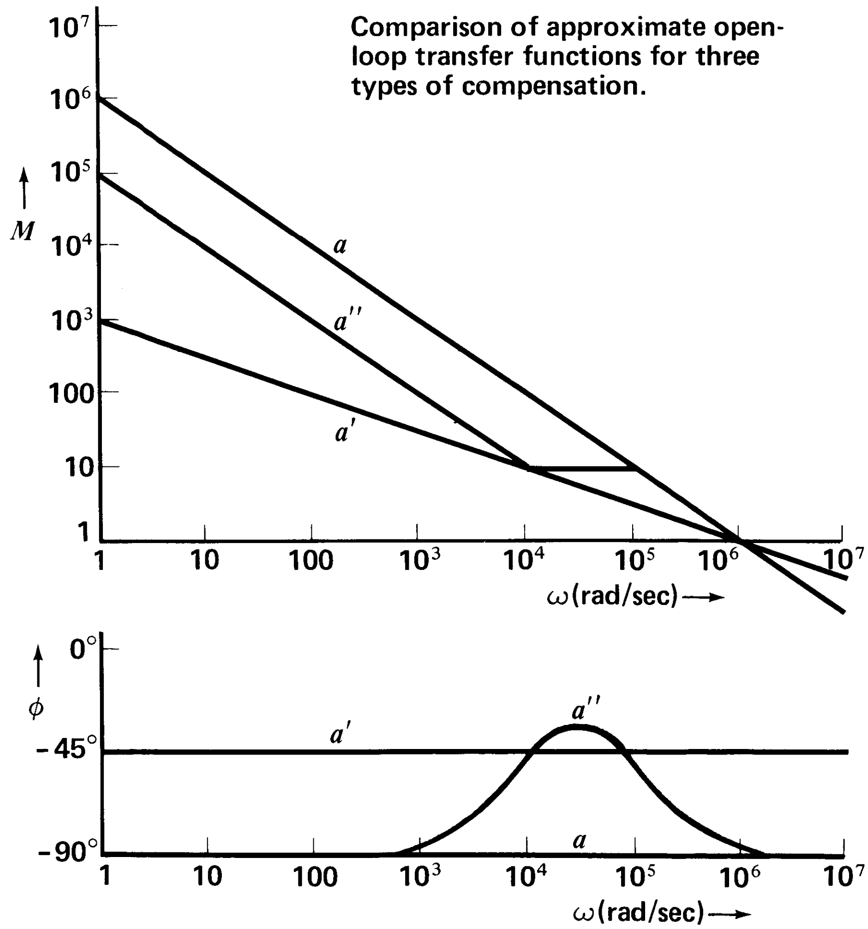
Viewgraph 13.4

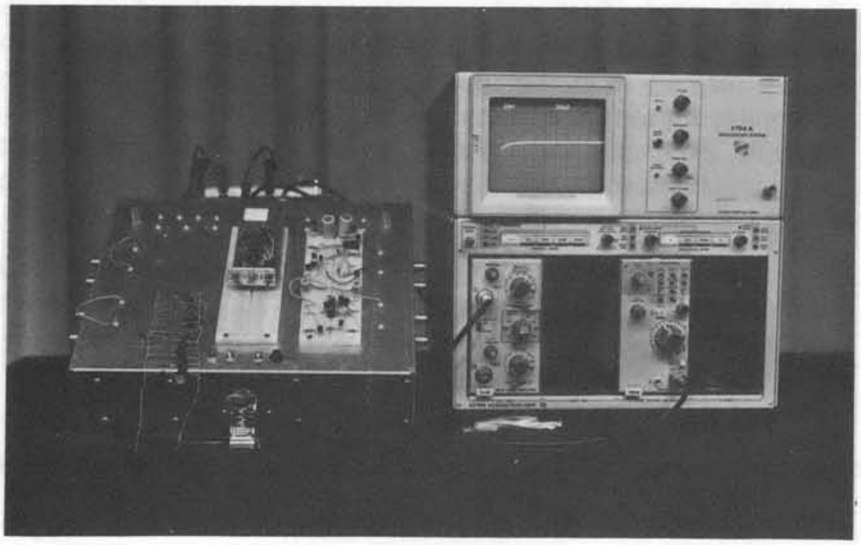


Network used to approximate an admittance proportional to $\sqrt{s}$

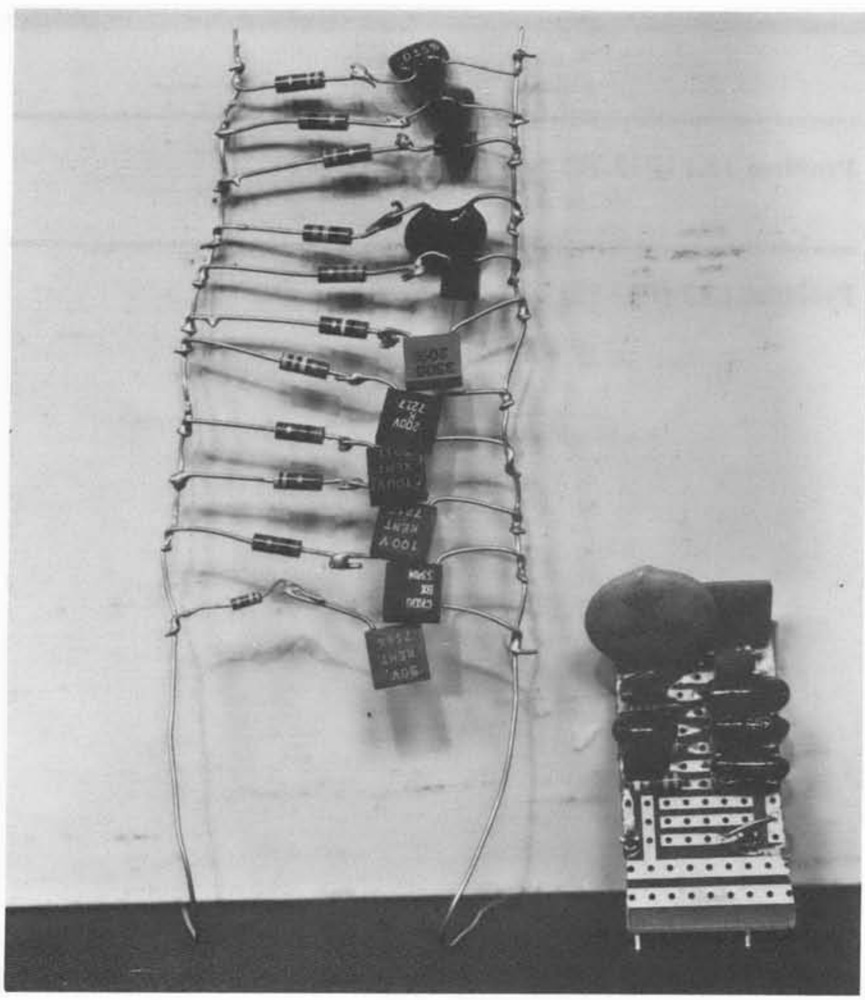
Viewgraph 13.5

Comparison of approximate open-loop transfer functions for three types of compensation.





Demonstration Photograph 13.1 Slow-roll-off compensation demonstration



Demonstration Photograph 13.2 Slow-roll-off network

Comments

Our discussion of minor-loop compensation tailored to specific applications is continued. We find that if a zero is added to the open- (major-) loop transmission at an appropriate frequency, acceptable stability can be maintained when an additional pole (for example, from capacitive loading) occurs. This type of compensation requires specific information concerning the location of the additional pole.

Conversely, compensation that rolls off more slowly than $1/s$ is advantageous when it is expected that the additional pole will be located over a range of frequencies.

Reading

Textbook: Sections 13.3.4 and 13.3.5.

Problems

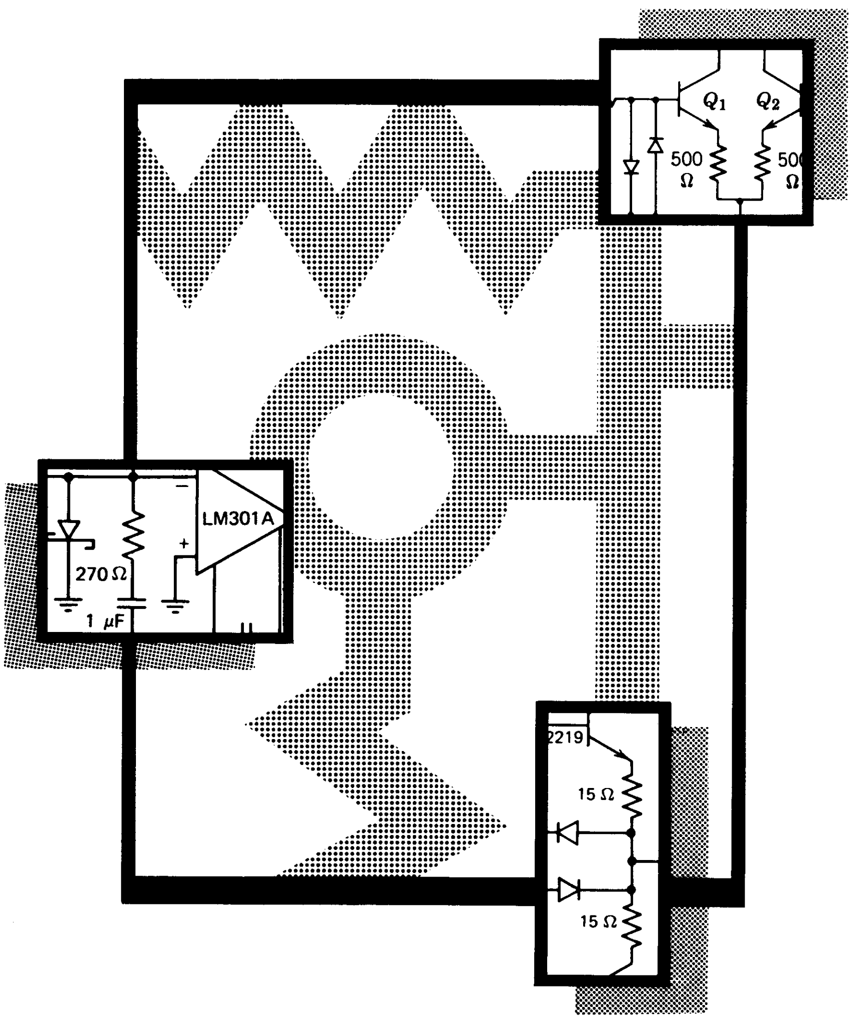
Problem 13.1 (P13.10)

Problem 13.2 (P13.11)

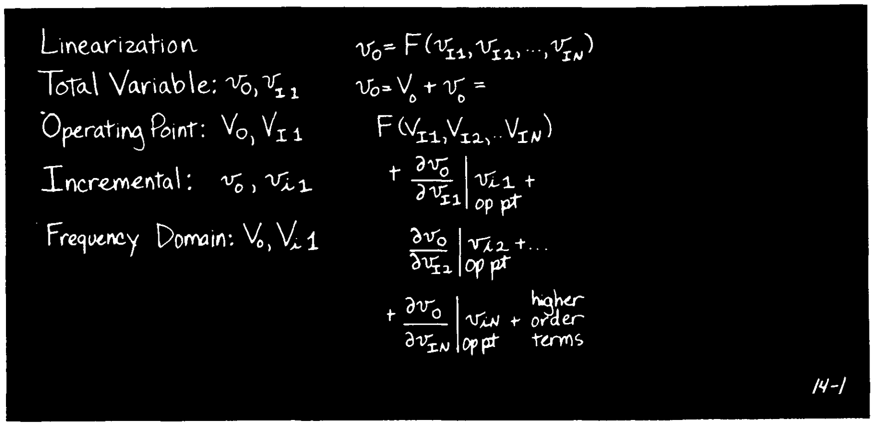
V I D E O C O U R S E M A N U A L

Linearized Analysis of Nonlinear Systems

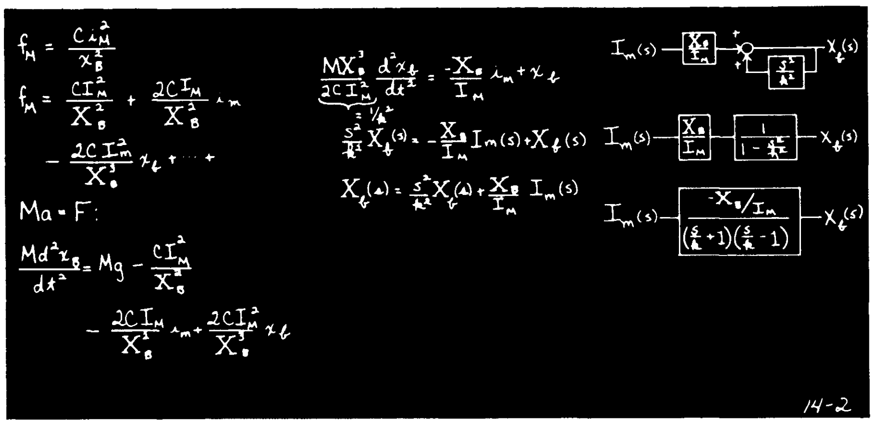
14



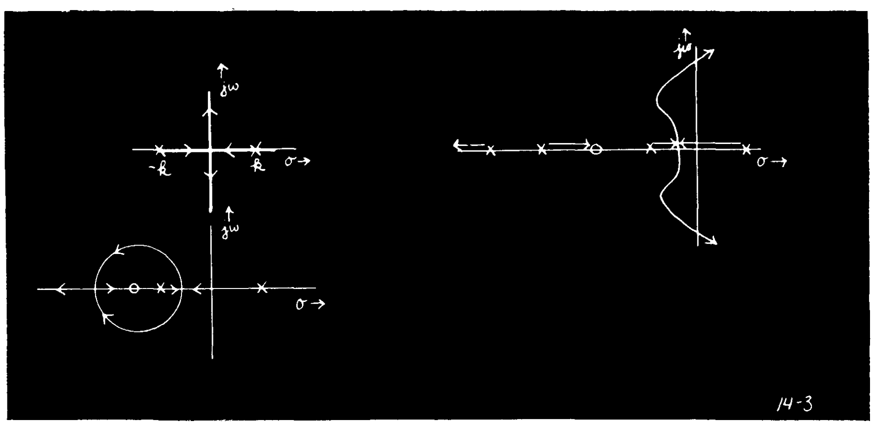
Blackboard 14.1



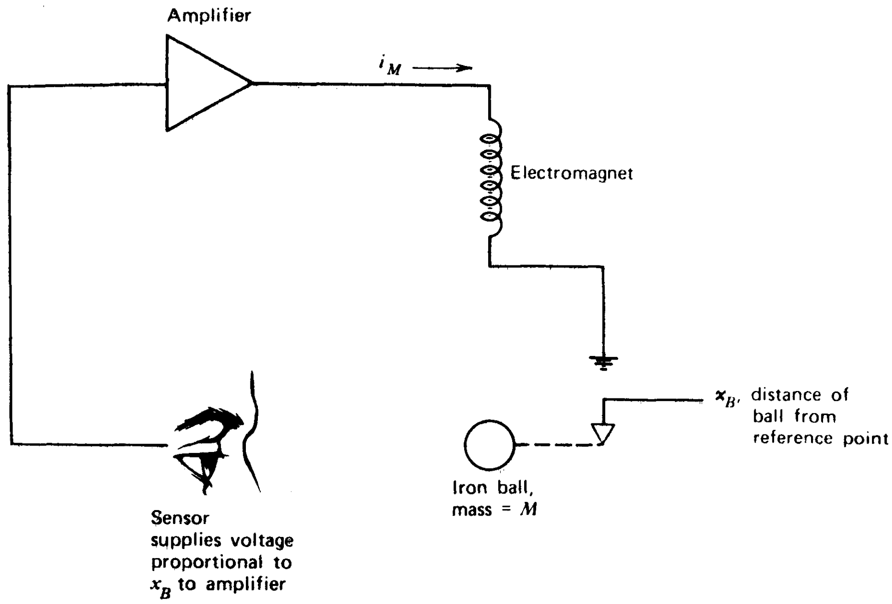
Blackboard 14.2



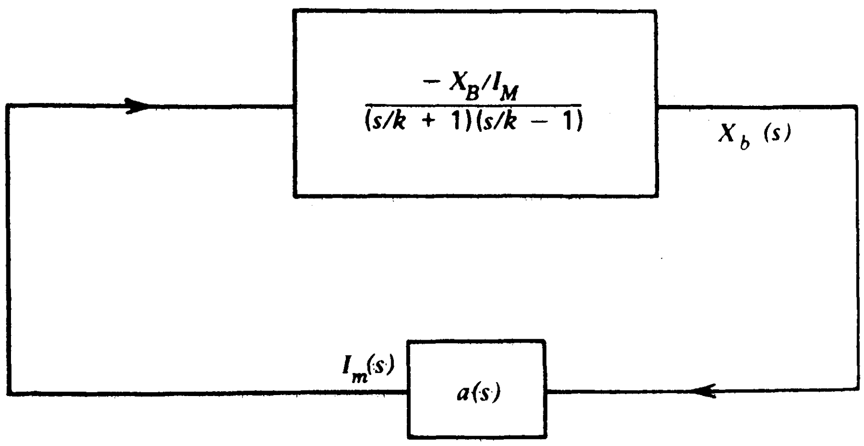
Blackboard 14.3



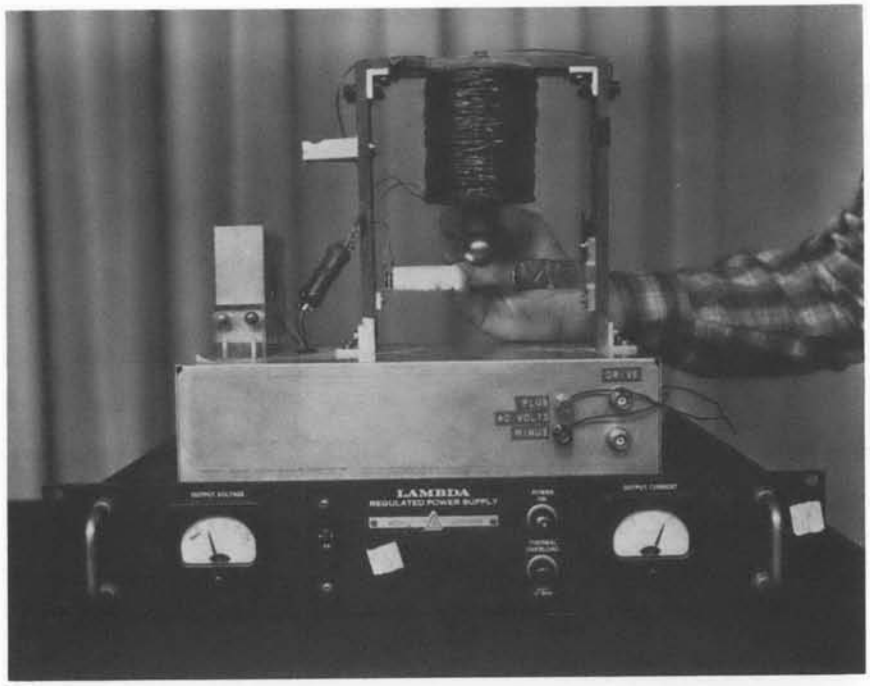
Viewgraph 14.1



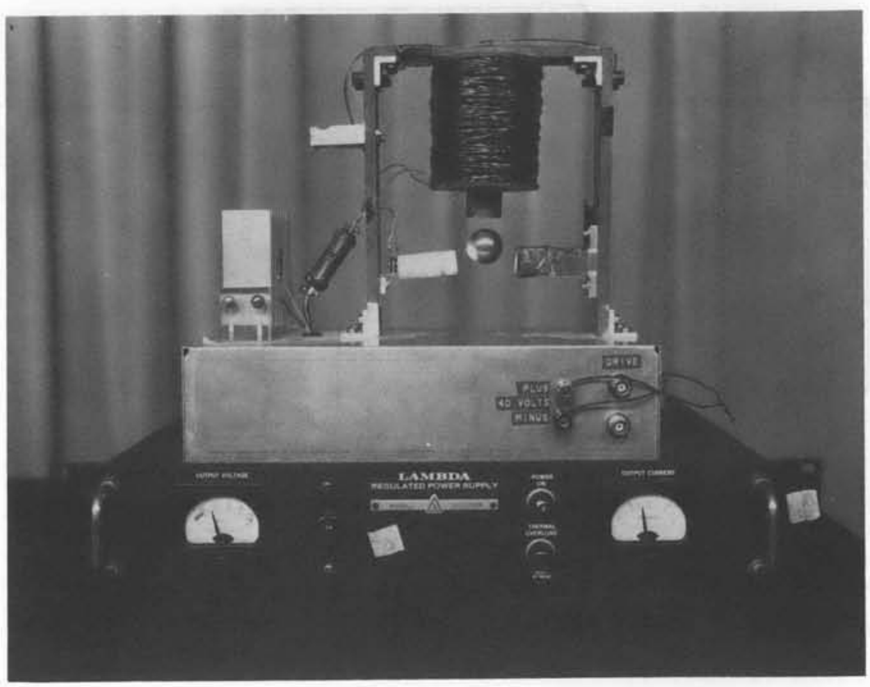
Viewgraph 14.2



Demonstration Photograph 14.1 Trying to get the magnetic-suspension system started



Demonstration Photograph 14.2 The magnetic-suspension system in operation



Comments

The analytical techniques introduced up to this point in the course make liberal use of superposition. Unfortunately, superposition does not generally apply in nonlinear systems, and consequently we need to develop new methods of analysis for systems where nonlinearity influences performance.

One possible method is to linearize the system equations about an operating point, recognizing that the linearized equations can be used to predict behavior over an appropriately restricted region around the operating point. This technique is used to determine compensation for a magnetic suspension system. The linearized equations of motion of this system have a loop-transmission pole in the right-half plane, reflecting the inherent instability that exists when the magnetic field strength is fixed.

Reading

Textbook: Chapter 6 through Section 6.2.

Problems

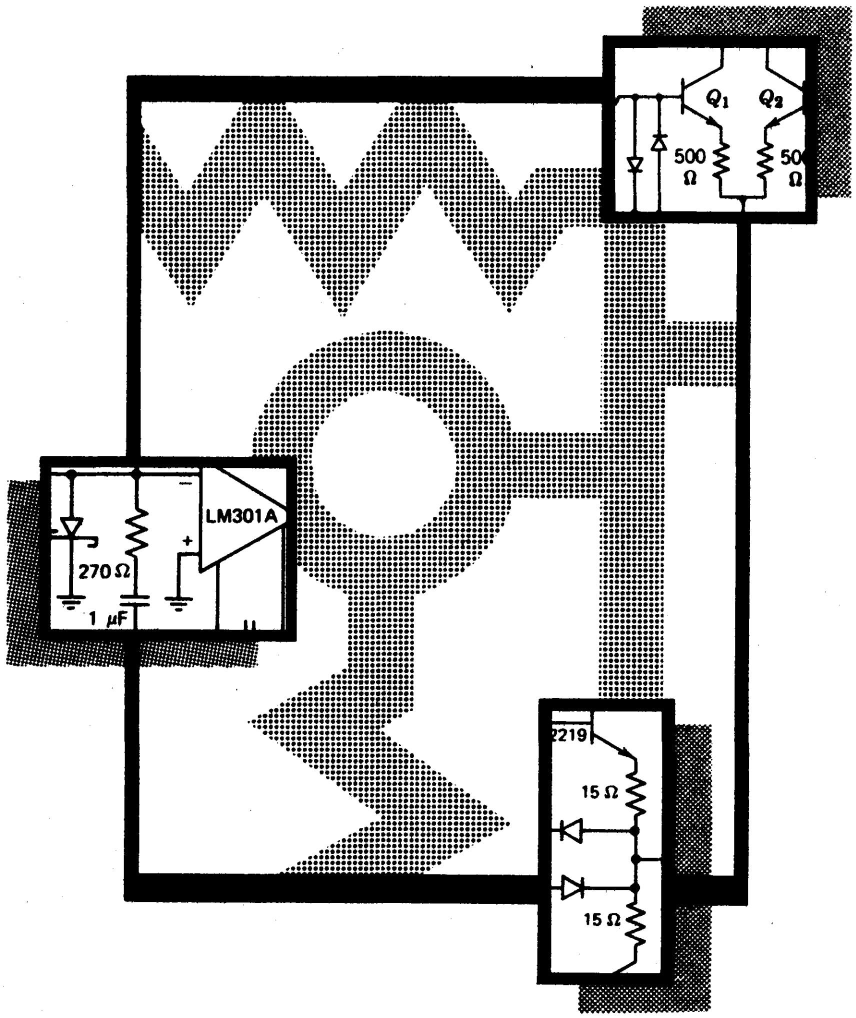
Problem 14.1 (P6.1)

Problem 14.2 (P6.2)

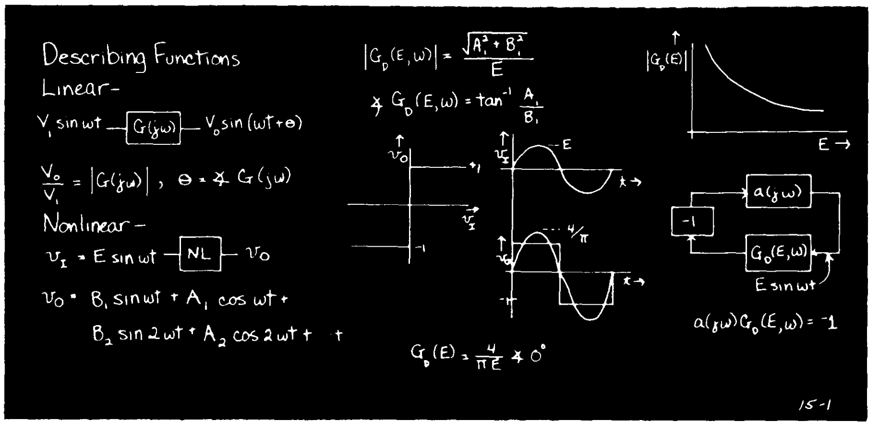
Problem 14.3 (P6.3)

VIDEO COURSE MANUAL

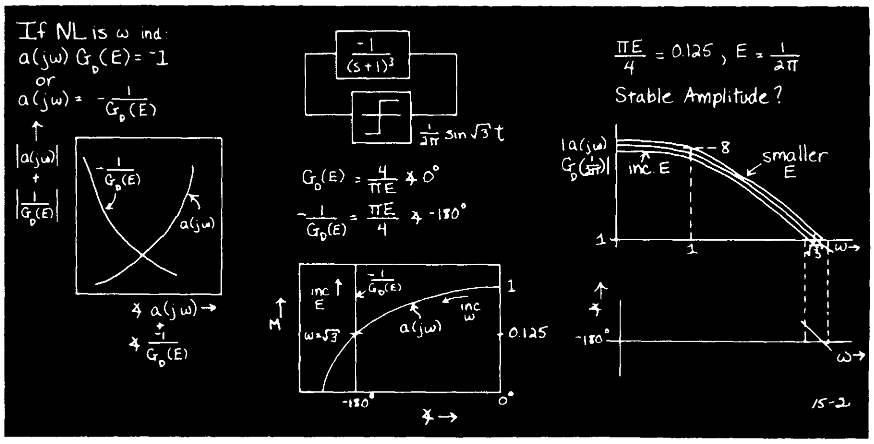
Describing Functions 15



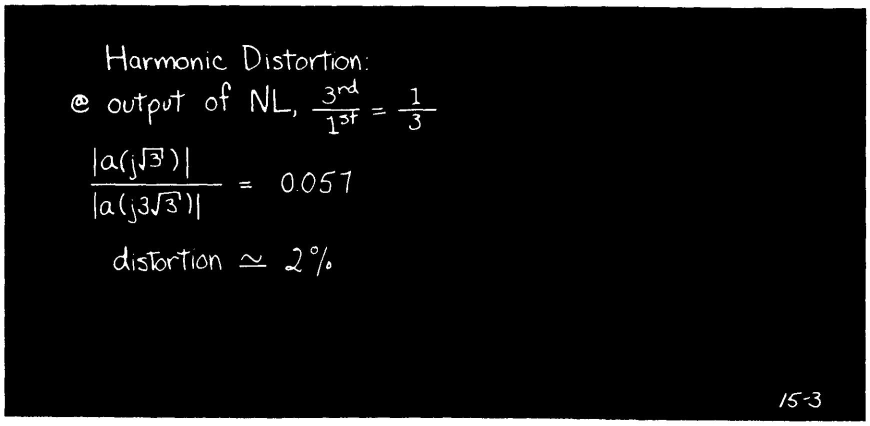
Blackboard 15.1



Blackboard 15.2



Blackboard 15.3



Comments

Describing-function analysis offers a way to apply the powerful frequency-domain methods that are so useful in linear-systems analysis to nonlinear systems. The describing function indicates the gain-and-phase shift that a nonlinear element provides to an input sinusoid, considering only the fundamental component of the output.

While describing-function analysis can be used to estimate the magnitudes of all signals in a nonlinear system that is driven with a sinusoid, the computational requirements for this type of detailed analysis are generally not justifiable. However, describing functions do provide a valuable way of estimating the amplitude, frequency, and harmonic distortion of certain kinds of oscillators.

Reading

Textbook: Sections 6.3 through 6.3.3.

Problems

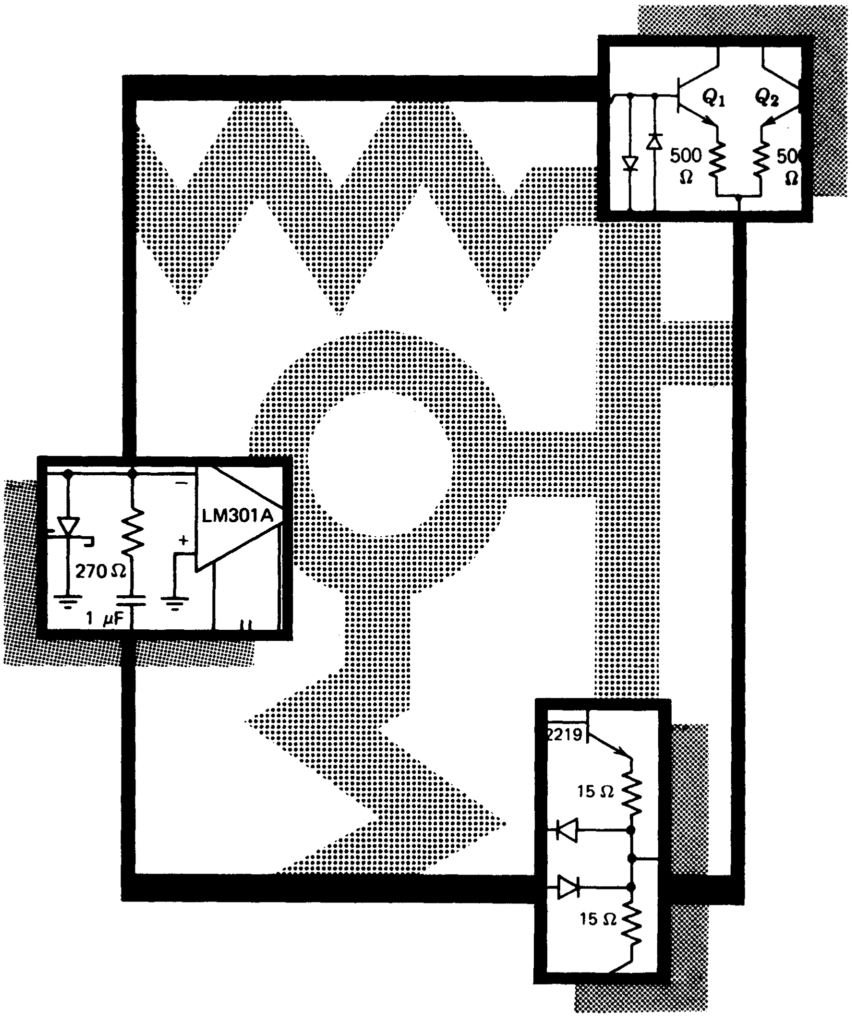
Problem 15.1 (P6.6)

Problem 15.2 (P6.7)

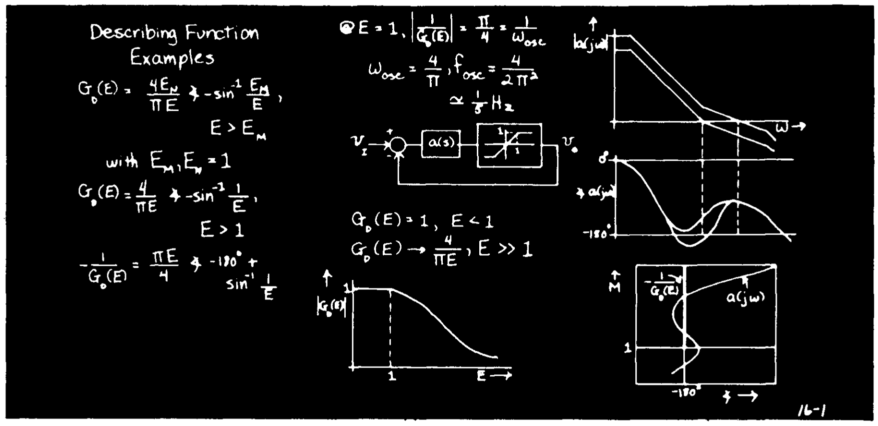
VIDEO COURSE MANUAL

Describing Functions (continued)

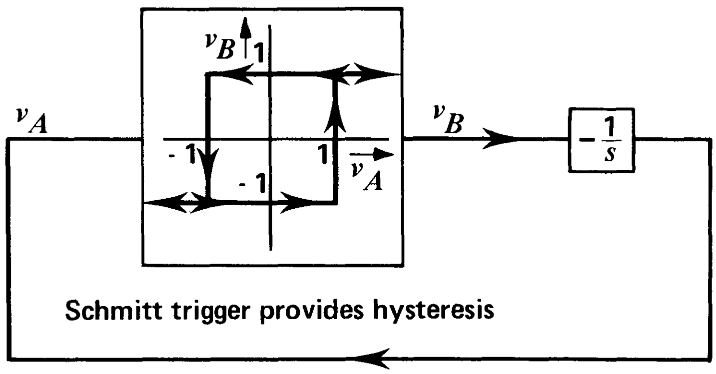
16



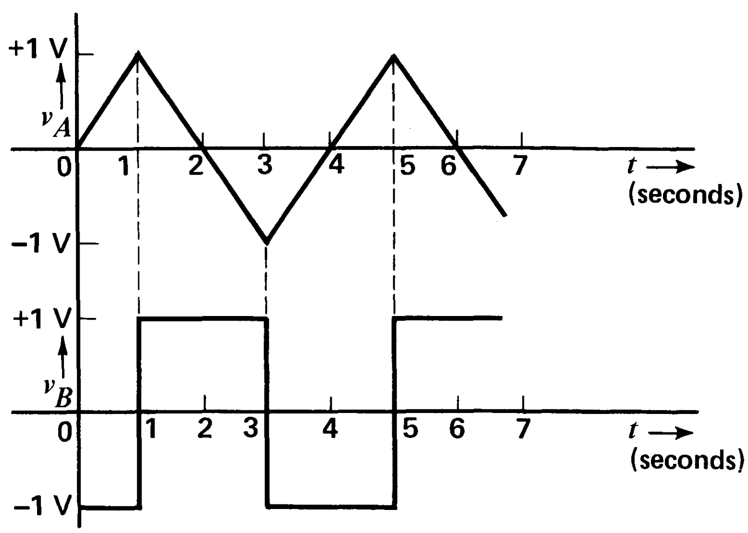
Blackboard 16.1



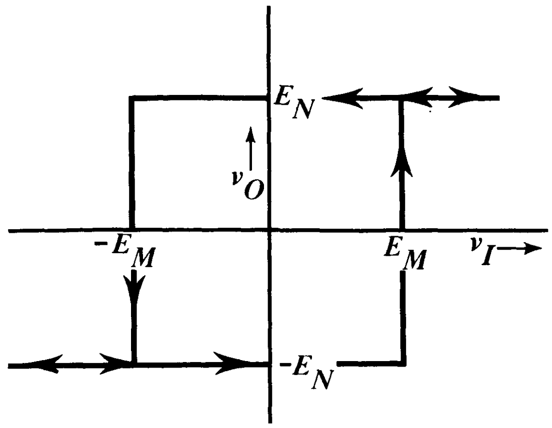
Viewgraph 16.1



Viewgraph 16.2

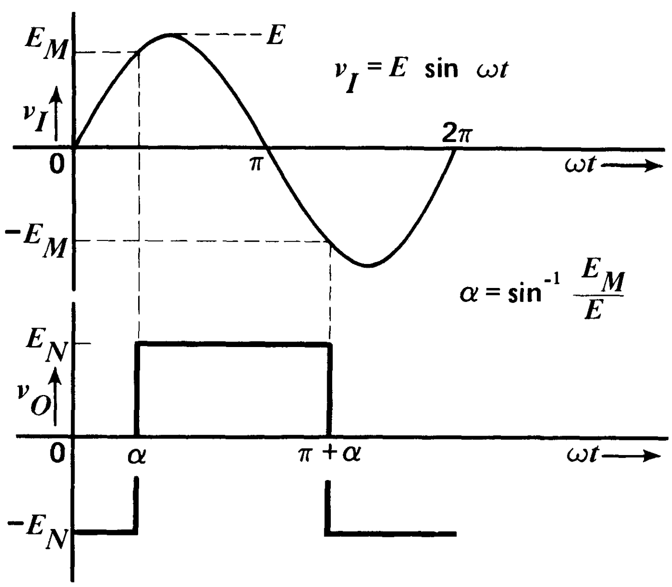


Viewgraph 16.3

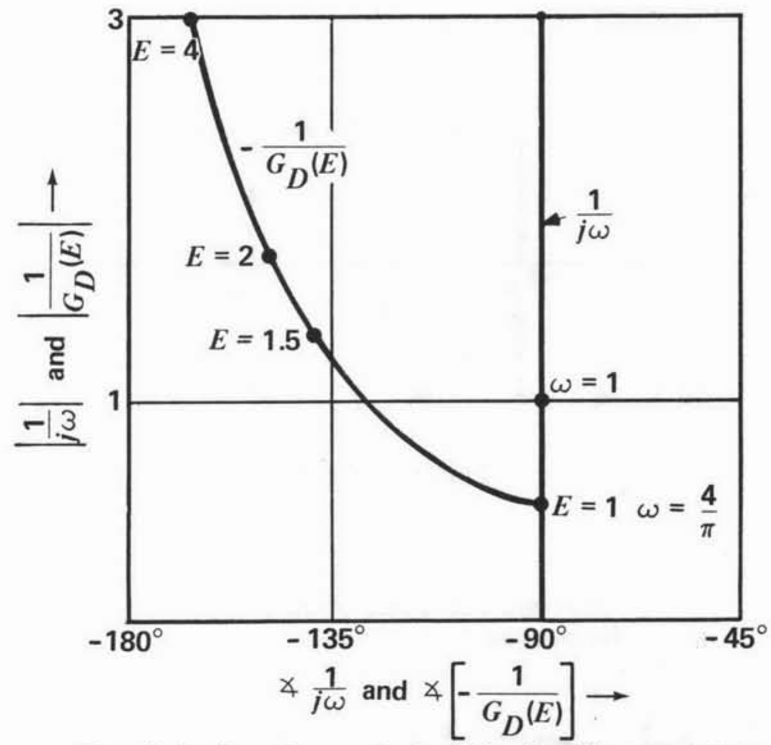


Transfer characteristics.

Input and output waveforms for sinusoidal excitation.

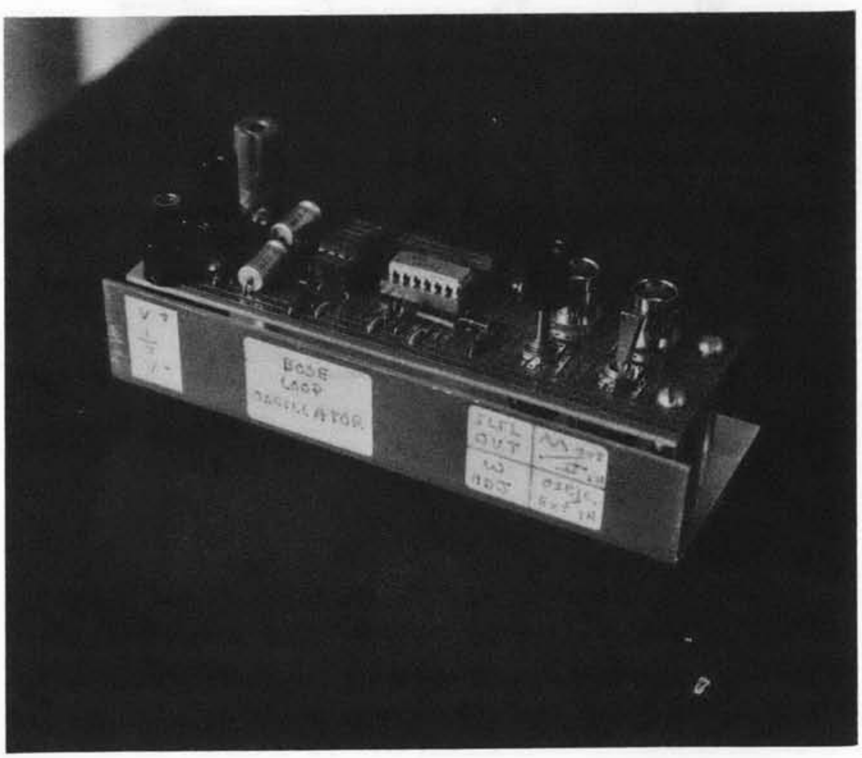


Viewgraph 16.4



Describing-function analysis of the function generator.

Demonstration Photograph 16.1 Function generator



Comments

This lecture continues examples of describing-function analysis. The method is used to predict the operation of an oscillator formed by combining an integrator with a Schmitt trigger. Reasonable agreement with the exact solution is obtained even though the conditions assumed for describing-function analysis are not particularly well satisfied by this system.

We also see an example of a system where relative stability is a function of signal levels.

Reading

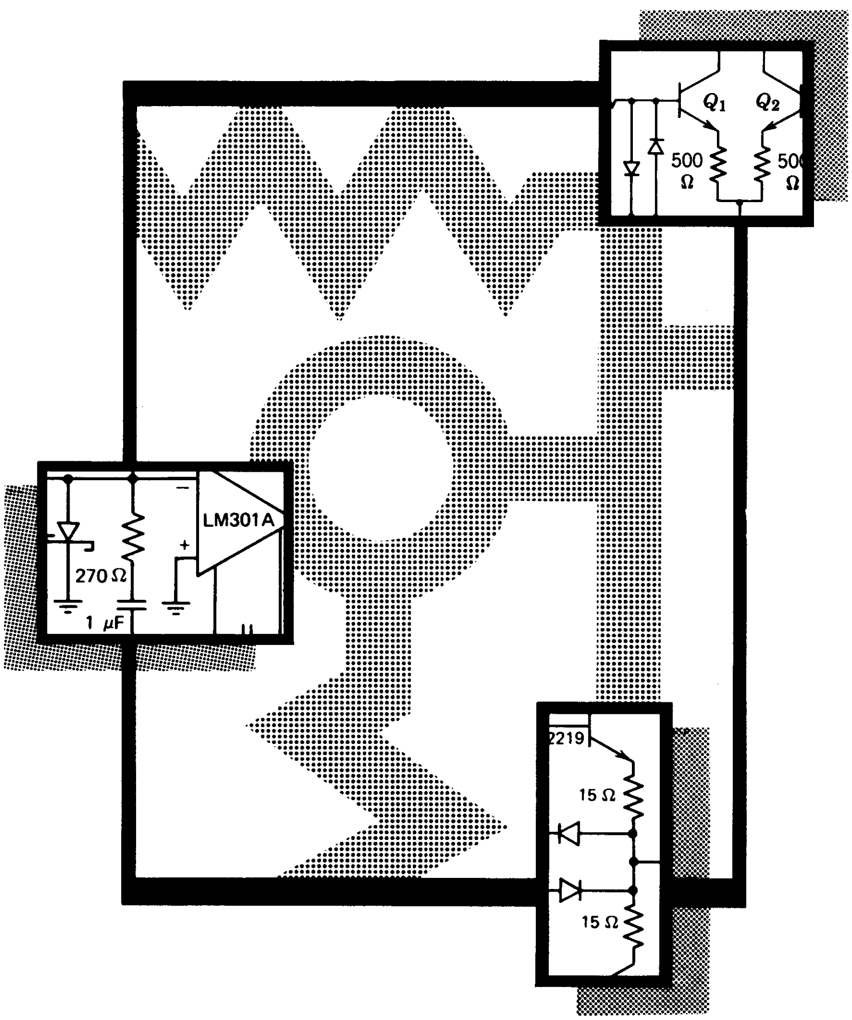
Material covered in connection with Lecture 15 provides the necessary background for this lecture.

Problem

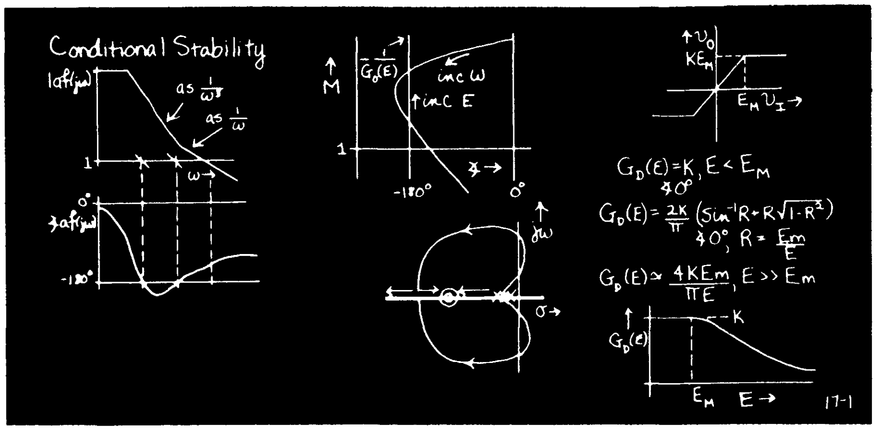
Problem 16.1 (P6.8)

VIDEO COURSE MANUAL

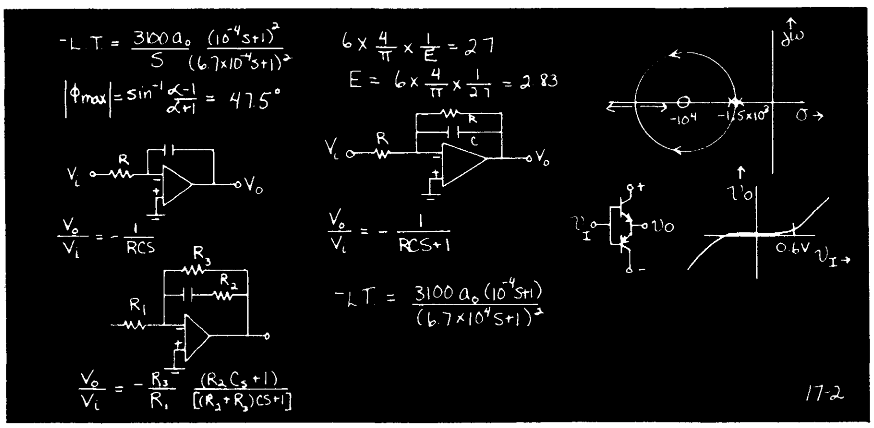
Conditional Stability 17



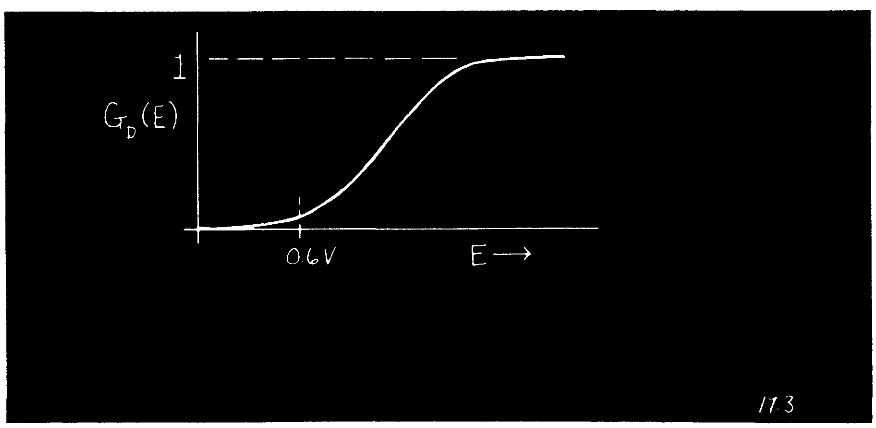
Blackboard 17.1

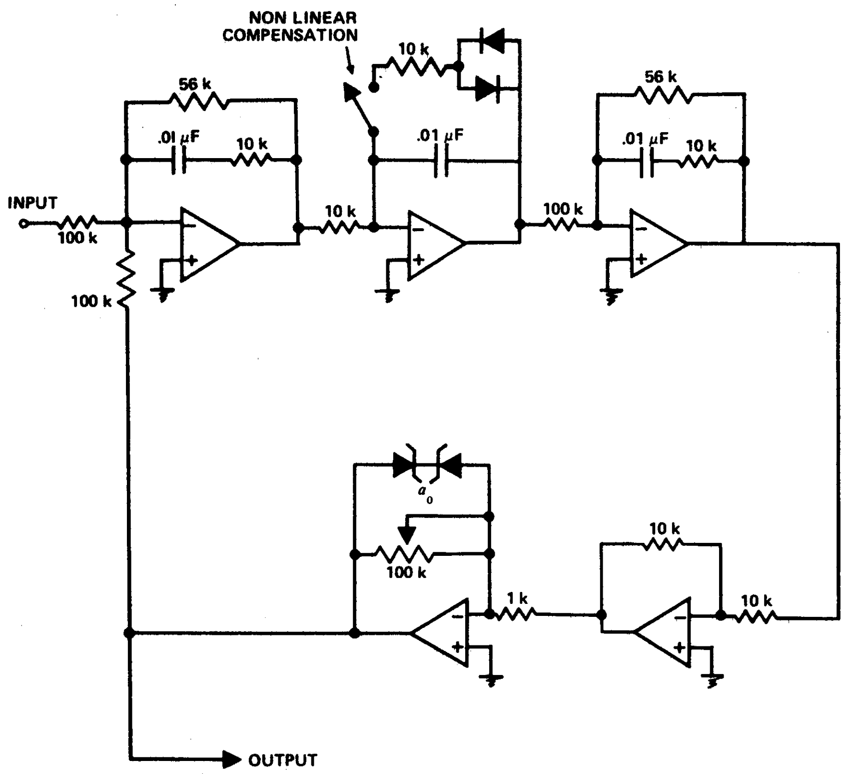


Blackboard 17.2

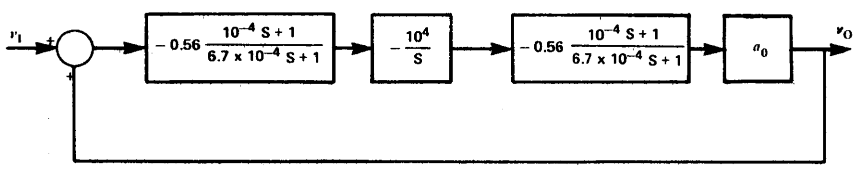


Blackboard 17.3



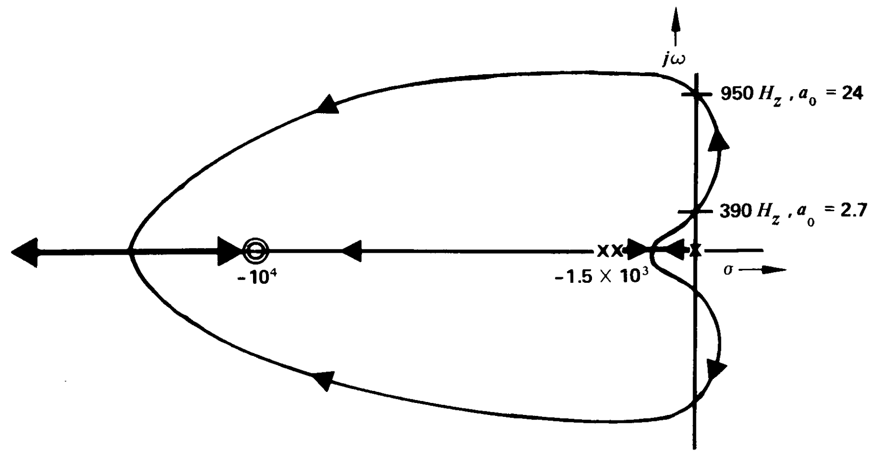


Viewgraph 17.1

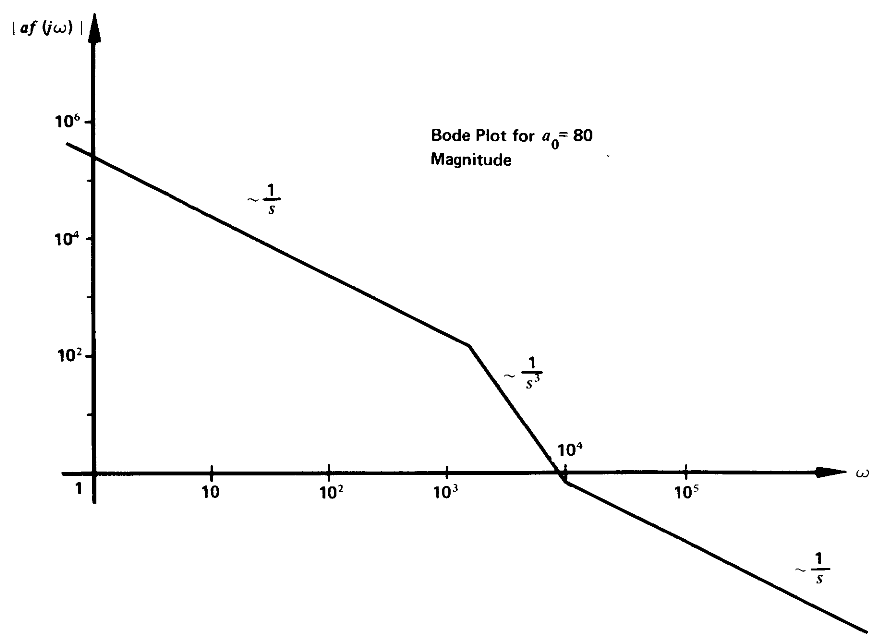


Viewgraph 17.2

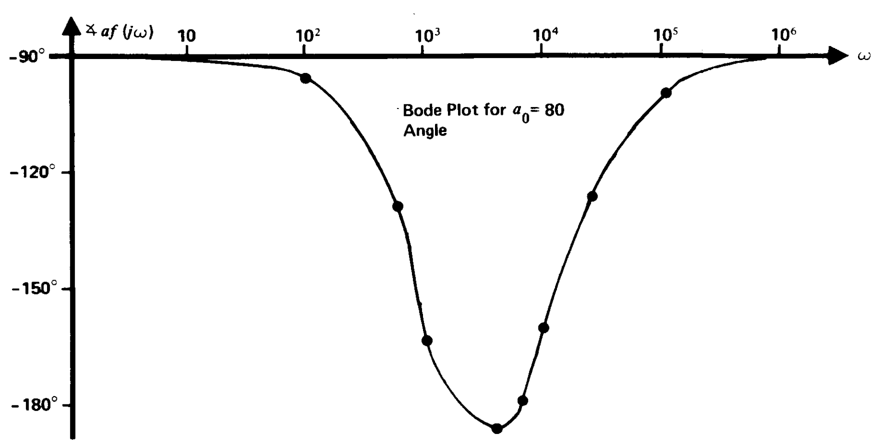
Viewgraph 17.3



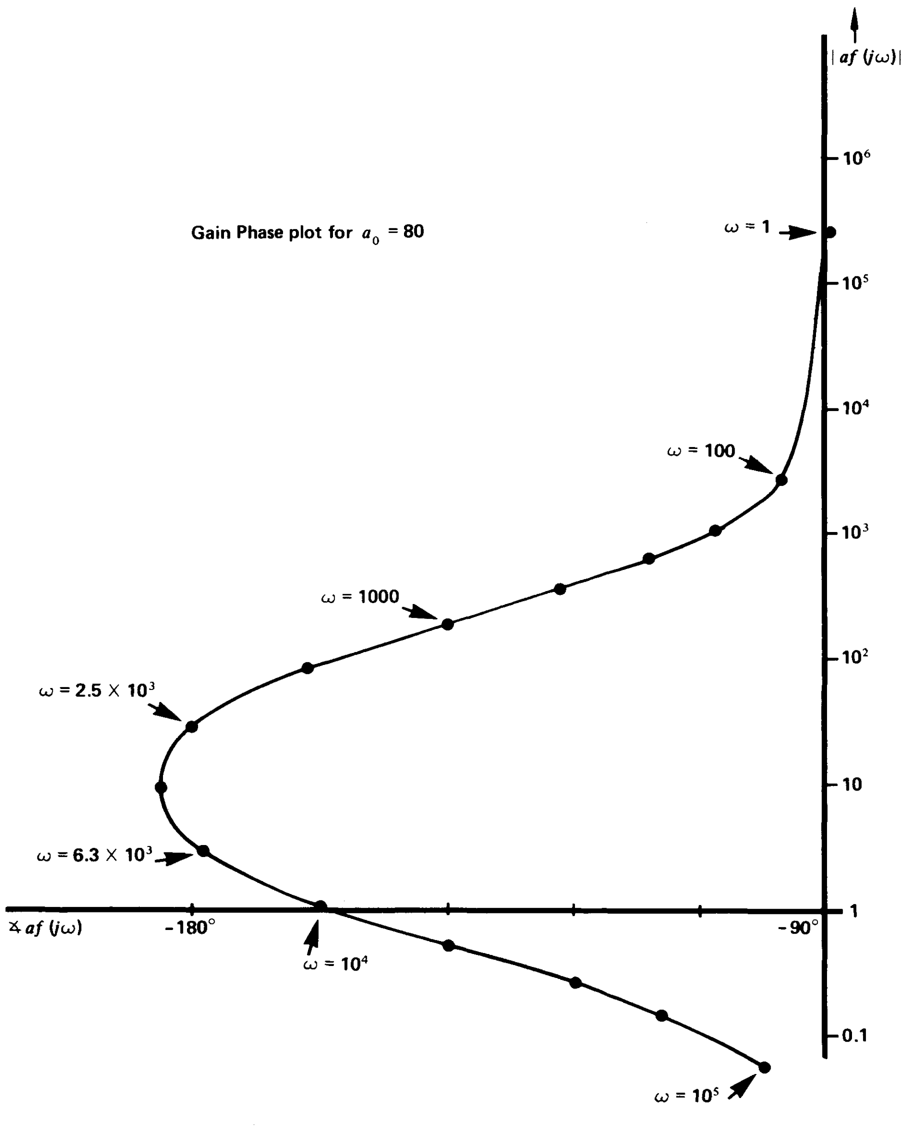
Viewgraph 17.4



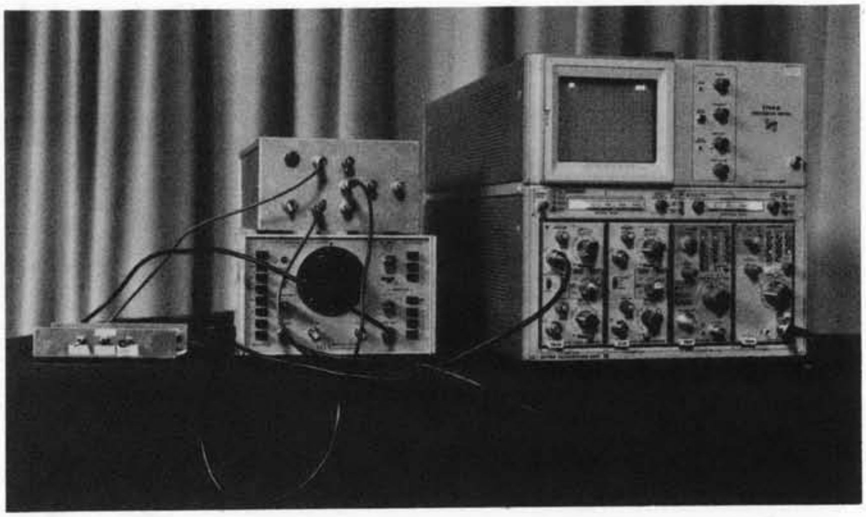
Viewgraph 17.5



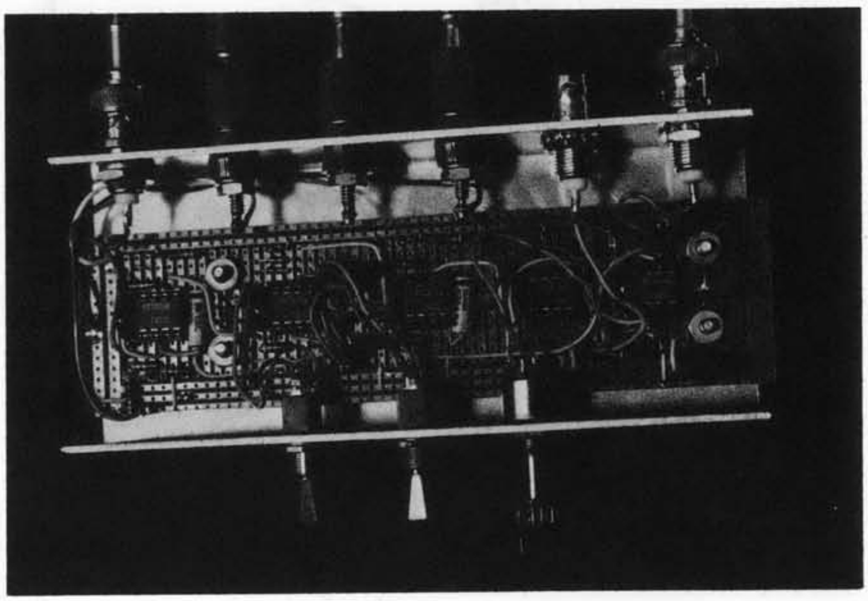
Viewgraph 17.6



Demonstration Photograph
17.1 Conditional-stability demonstration



Demonstration Photograph
17.2 Close-up of conditionally-stable system



Comments

This lecture introduces the idea of conditional stability and uses a demonstration system to illustrate important concepts. In certain systems, a loop transmission that rolls off faster than $1/s^2$ over a range of frequencies is used to achieve high desensitivity while retaining a relatively low crossover frequency. If the frequency range of fast roll-off is broad enough, the phase angle may become more negative than $-180°$ over a range of frequencies below crossover. Such systems can be well behaved when they are operating in their linear region, yet become unstable when saturation lowers crossover frequency to a region of negative phase margin.

Describing-function analysis indicates the potential for this type of behavior, predicts oscillation parameters in systems where instability is possible, and can also be used to determine appropriate nonlinear compensation methods.

Reading

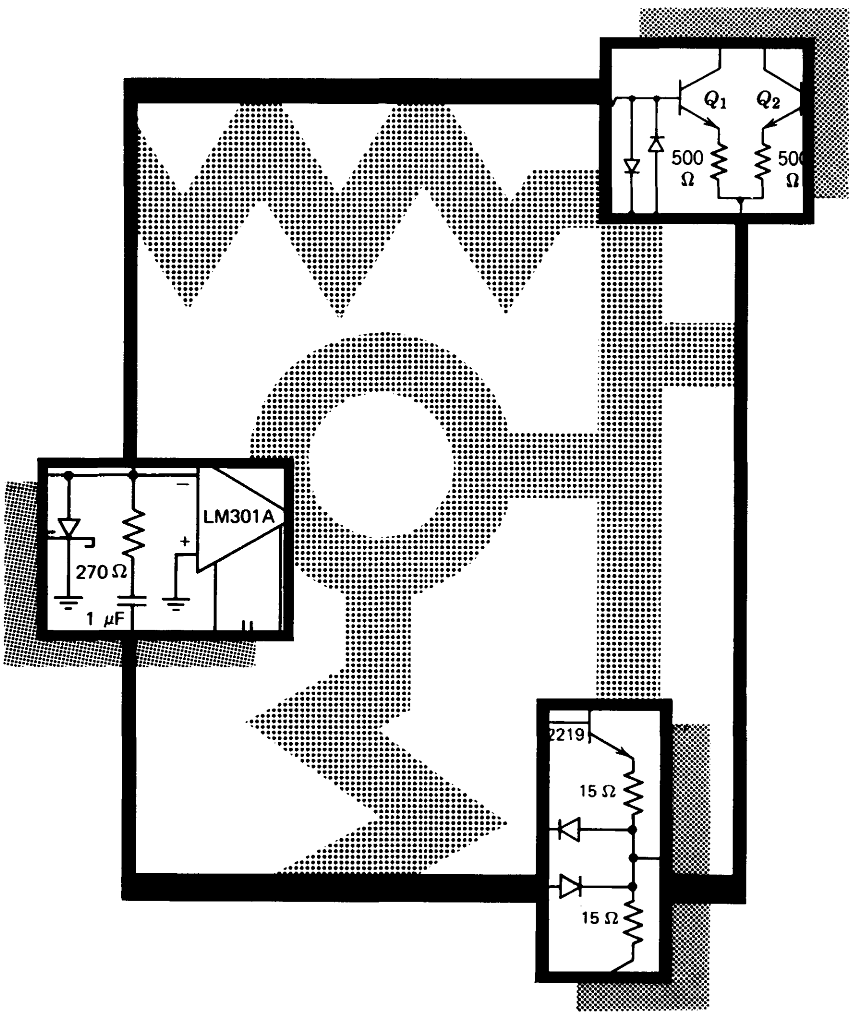
Textbook: Sections 6.3.4 and 6.3.5.

Problem

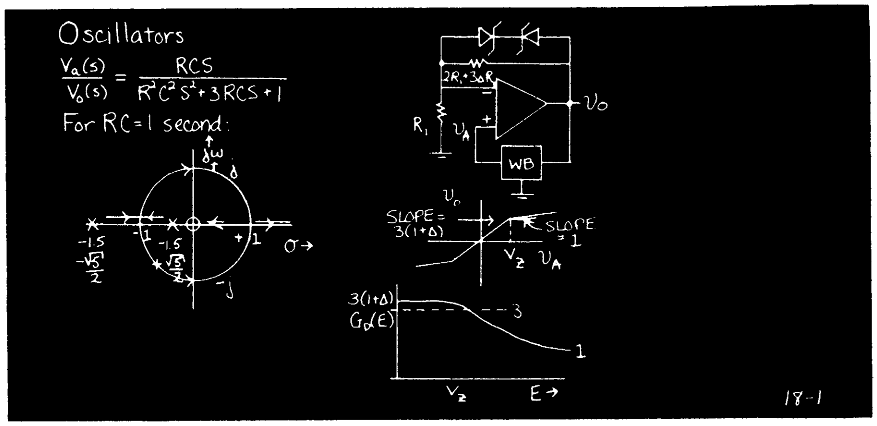
Problem 17.1 (P6.9)

VIDEO COURSE MANUAL

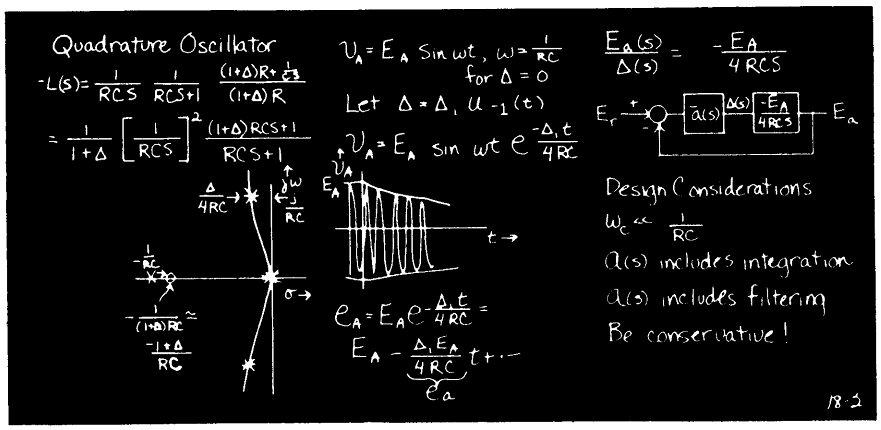
Oscillators (Intentional) 18



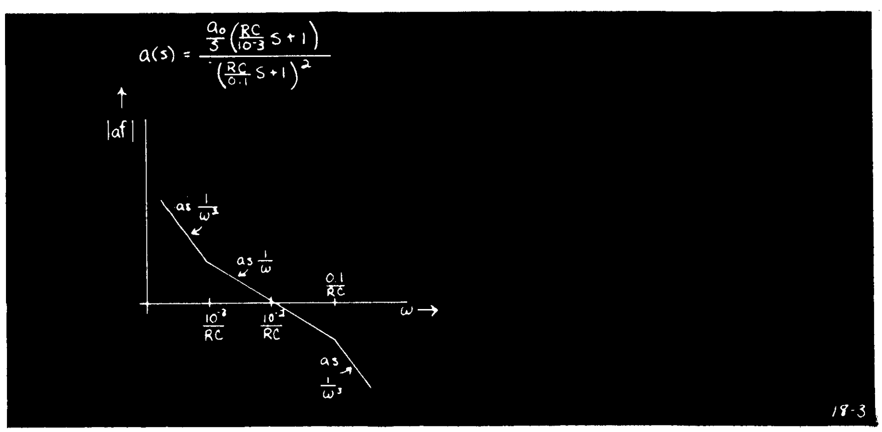
Blackboard 18.1



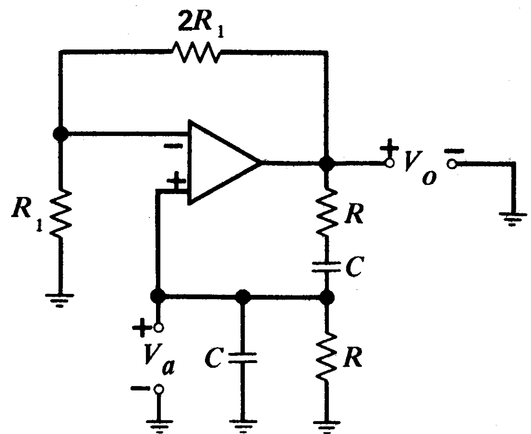
Blackboard 18.2



Blackboard 18.3

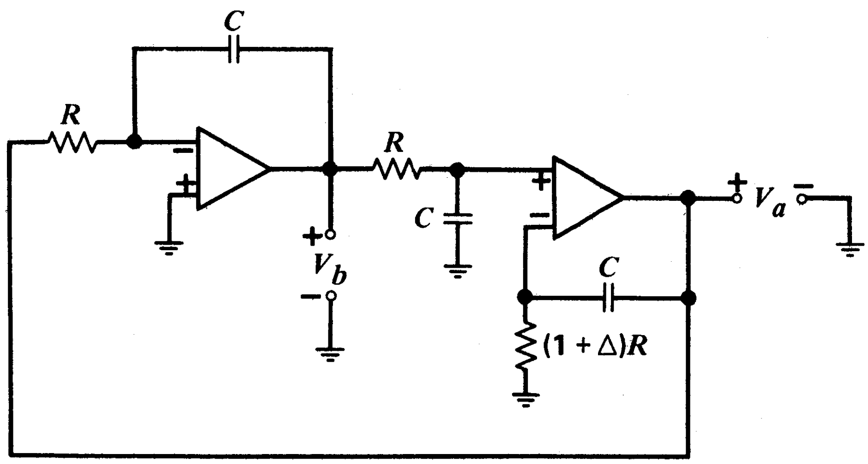


Viewgraph 18.1



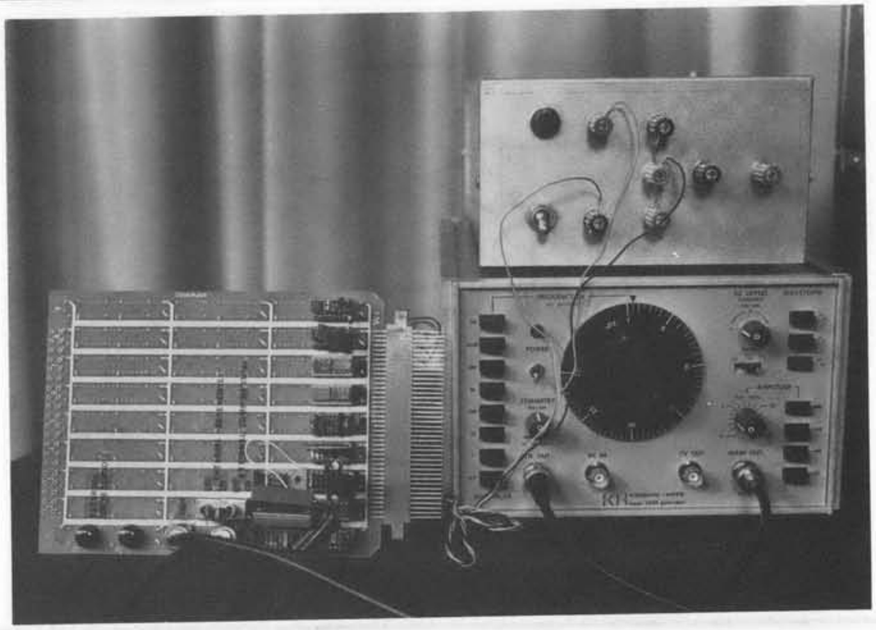
Wien-bridge oscillator

Viewgraph 18.2

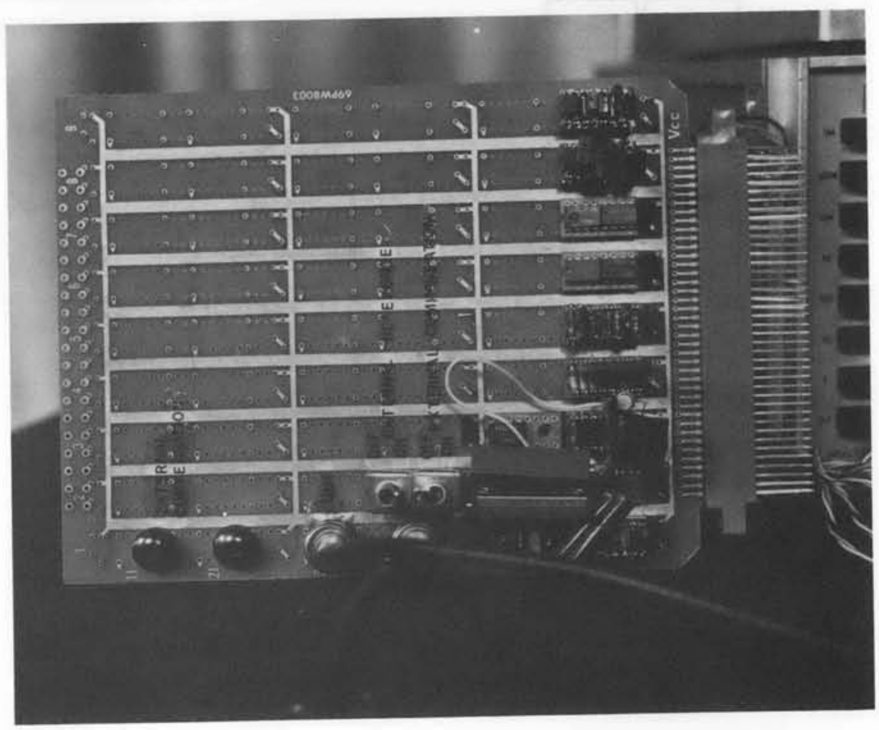


Quadrature oscillator

Demonstration Photograph
18.1 Amplitude-stabilized oscillator demonstration



Demonstration Photograph
18.2 Amplitude-stabilized oscillator close-up



Comment

To this point in the series our effort has been directed toward designing feedback systems with acceptable stability. In this lecture we look at how we might use the techniques we have learned to design oscillators that provide outputs with high spectral purity.

The basic approach is to use a very slow (compared to the frequency of oscillation) feedback loop to hold a closed-loop pair of poles of the oscillator circuit exactly on the imaginary axis. The validity of this approach is illustrated with a demonstration system.

Reading

Textbook: Section 12.1.

Problems

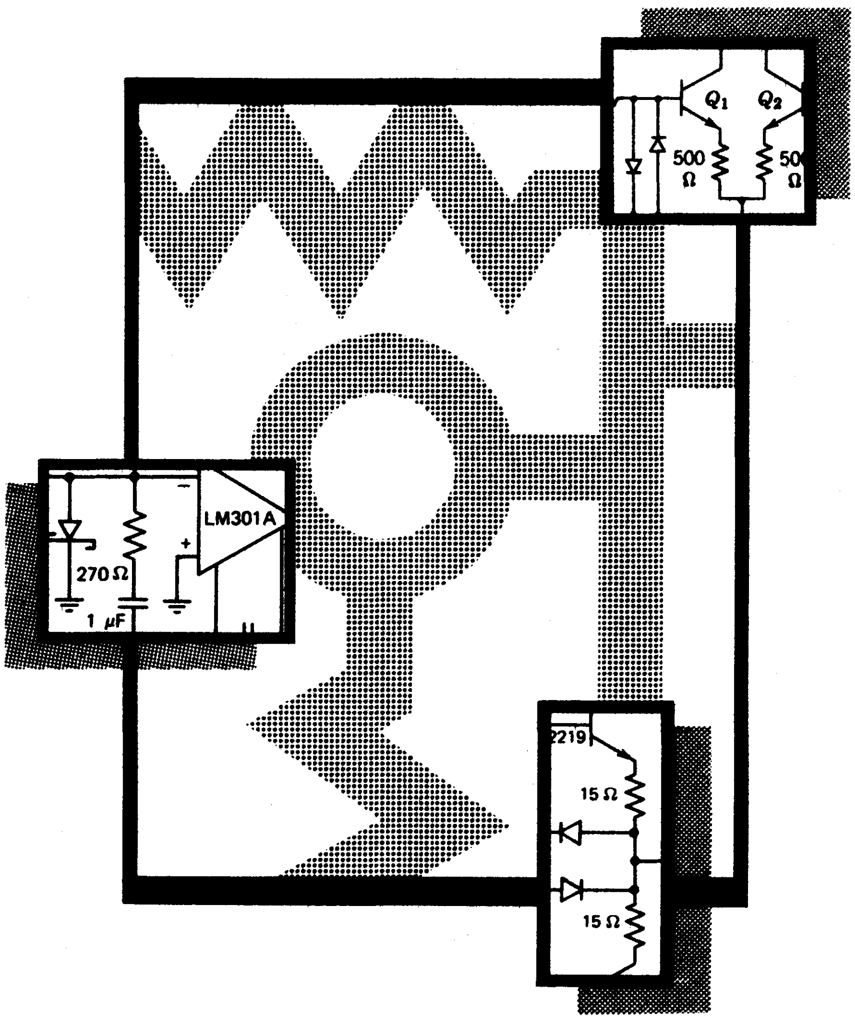
Problem 18.1 (P12.1)

Problem 18.2 (P12.2)

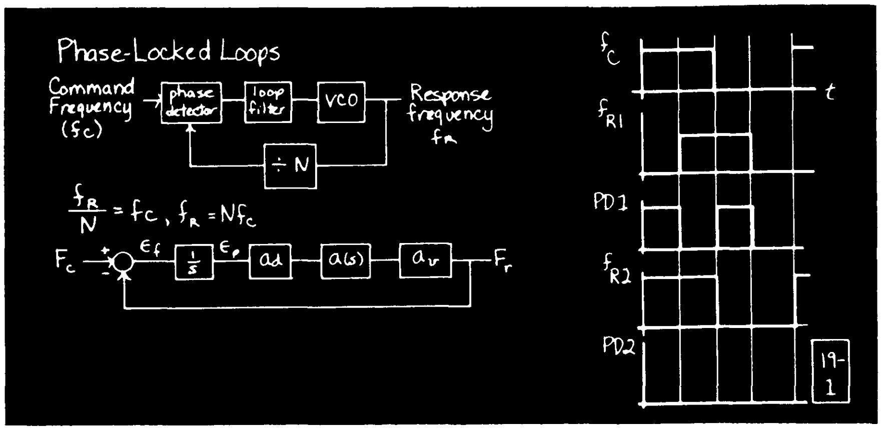
Problem 18.3 (P12.4)

VIDEO COURSE MANUAL

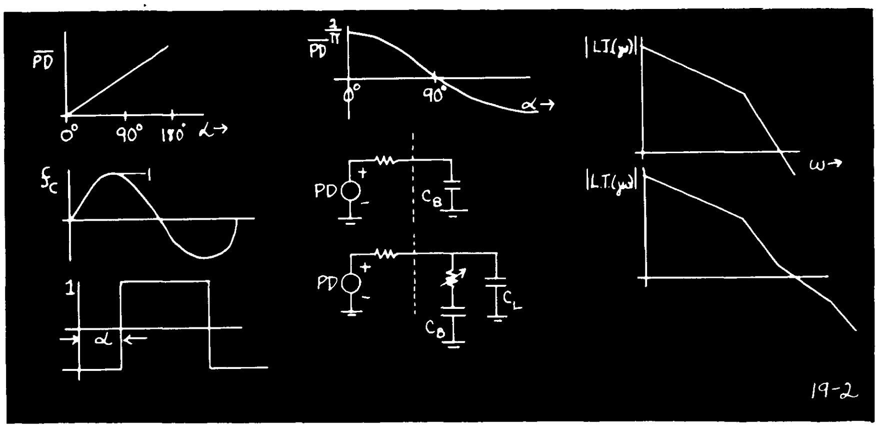
Phase-Locked Loops 19

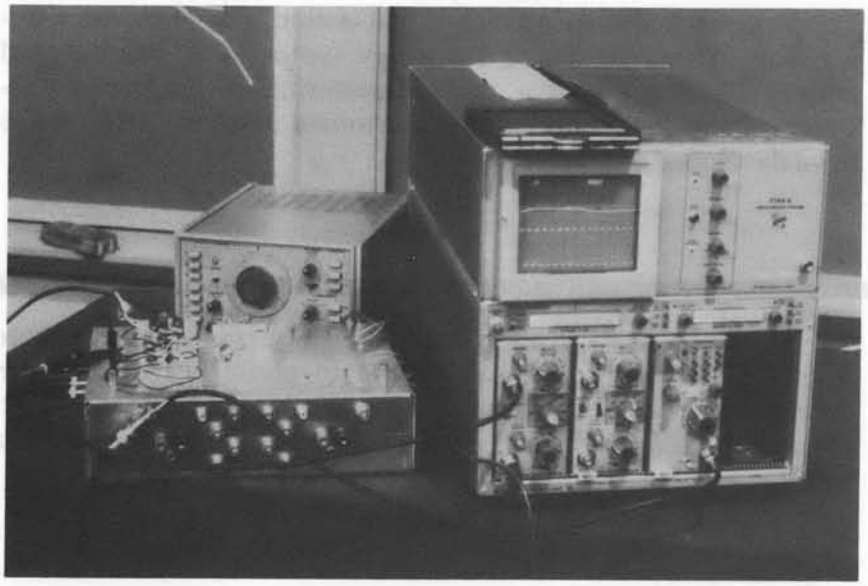


Blackboard 19.1

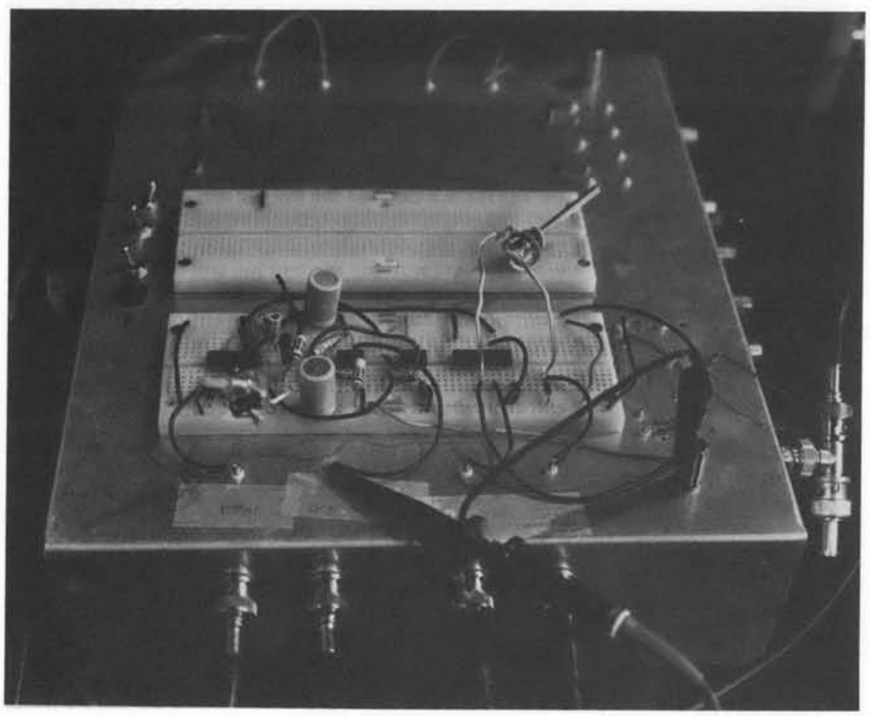


Blackboard 19.2





Demonstration Photograph 19.1 Phase-lock loop demonstration



Demonstration Photograph 19.2 Close-up of phase-lock loop

Comments

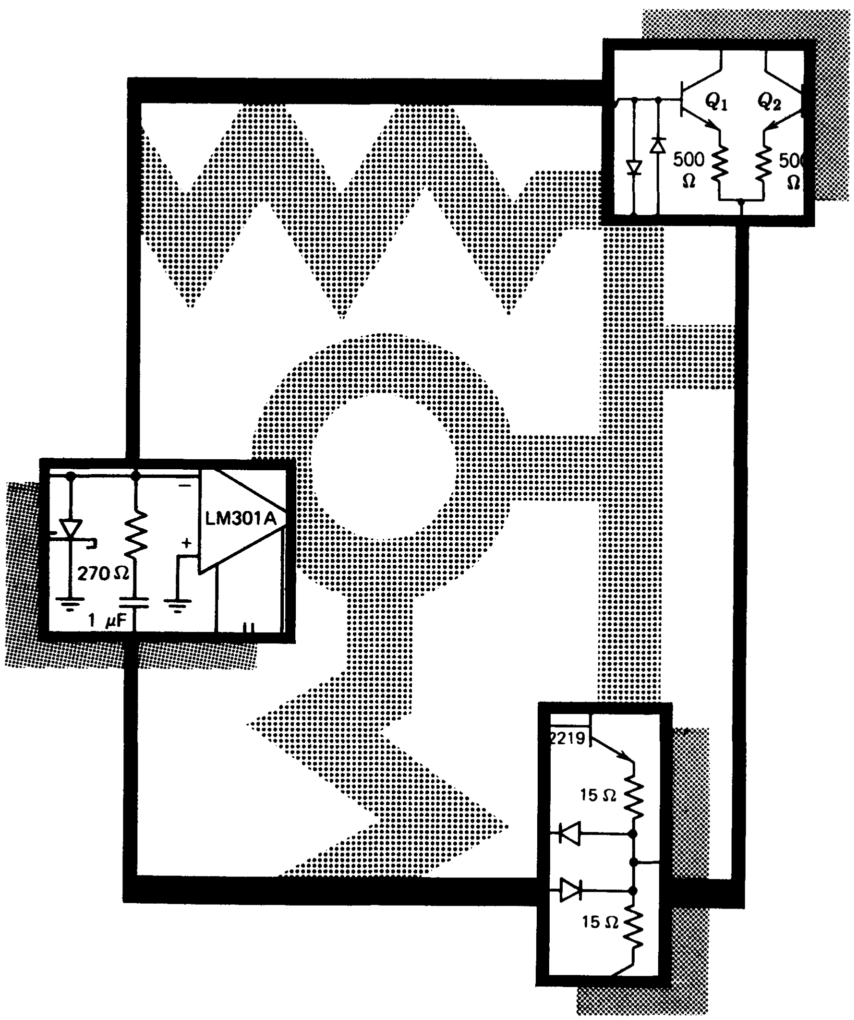
The phase-locked loop is a circuit that is used in applications ranging from demodulation to frequency synthesis. A phase-locked loop is a feedback system, and the performance and design constraints for such circuits can be determined using the methods we have developed earlier in the course.

Reading/Problems

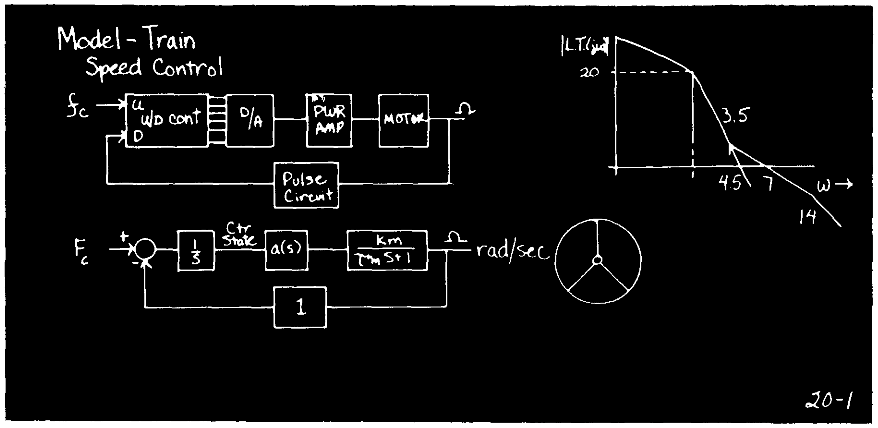
There are no specific reading assignments or problems. For those interested in a thorough background in phase-locked loops, an excellent and detailed treatment is provided in: Floyd M. Gardner, *Phaselock Techniques,* 2nd ed. (New York: John Wiley & Sons, 1979).

VIDEO COURSE MANUAL

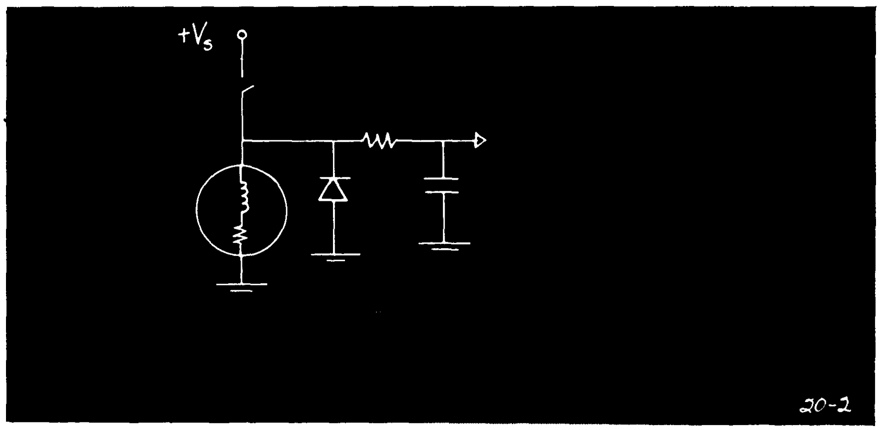
Model Train Speed Control 20

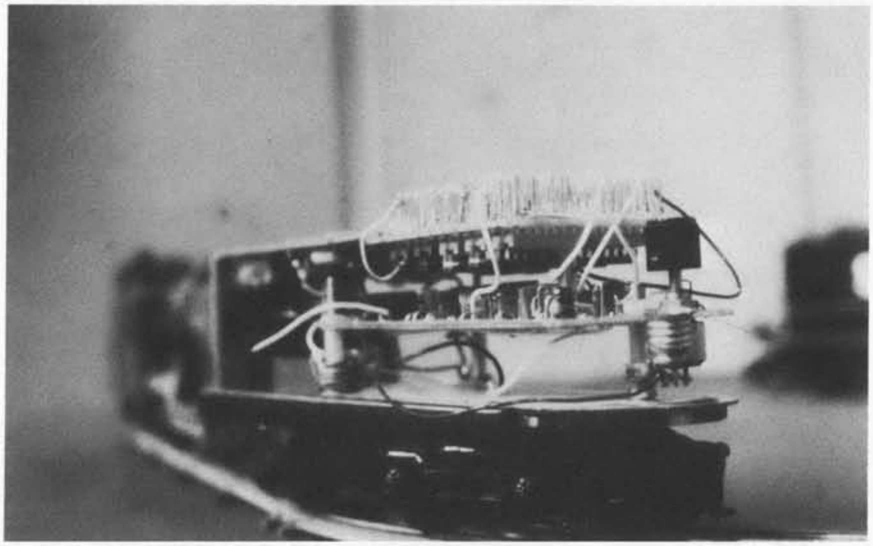


Blackboard 20.1



Blackboard 20.2





Demonstration Photograph 20.1 Here comes Super Train!

Comments

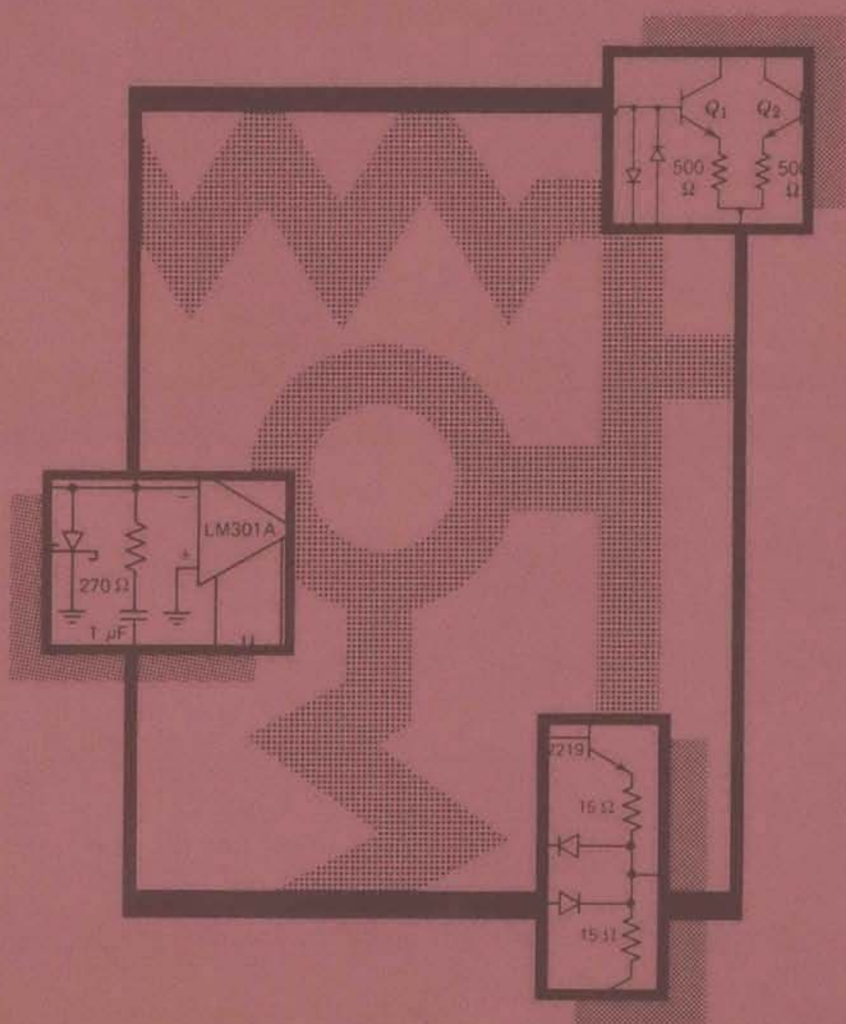
It is difficult to justify this much effort for a toy, but it is a lot of fun!

Reading/Problems

No reading or problems—school is out!

VIDEO COURSE MANUAL

Solutions to Problems



S O L U T I O N S

Introduction and Basic Concepts 1

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 1.1

Following the example of Section 1.2.3 of the textbook, with $N = 3$, we use the connection shown in Figure S1.1:

We then apply Equation 1.20 to write

$$V_o = -\frac{R_f}{R_{i1}} V_{i1} - \frac{R_f}{R_{i2}} V_{i2} - \frac{R_f}{R_{i3}} V_{i3} \qquad \textbf{(S1.1)}$$

The desired gain expression is:

$$V_o = -V_{i1} - 2V_{i2} - 3V_{i3} \qquad \textbf{(S1.2)}$$

Thus, $R_{i1} = R_f$, $R_{i2} = \frac{1}{2}R_f$, and $R_{i3} = \frac{1}{3}R_f$. By choosing $R_f = 60$ kΩ, we satisfy these conditions with $R_{i1} = 60$ kΩ, $R_{i2} = 30$ kΩ, and $R_{i3} = 20$ kΩ.

To derive the block diagram, we follow the method of Section 2.4 of the textbook to write a pair of equations in V_a, V_o, V_{i1}, V_{i2}, and V_{i3}.

We have, by superposition:

$$\begin{aligned} V_a = {} & \frac{R_f \| R_{i2} \| R_{i3}}{R_{i1} + R_f \| R_{i2} \| R_{i3}} V_{i1} + \frac{R_f \| R_{i1} \| R_{i3}}{R_{i2} + R_f \| R_{i1} \| R_{i3}} V_{i2} \\ & + \frac{R_f \| R_{i1} \| R_{i2}}{R_{i3} + R_f \| R_{i1} \| R_{i2}} V_{i3} + \frac{R_{i1} \| R_{i2} \| R_{i3}}{R_f + R_{i1} \| R_{i2} \| R_{i3}} V_o \end{aligned} \qquad \textbf{(S1.3)}$$

and

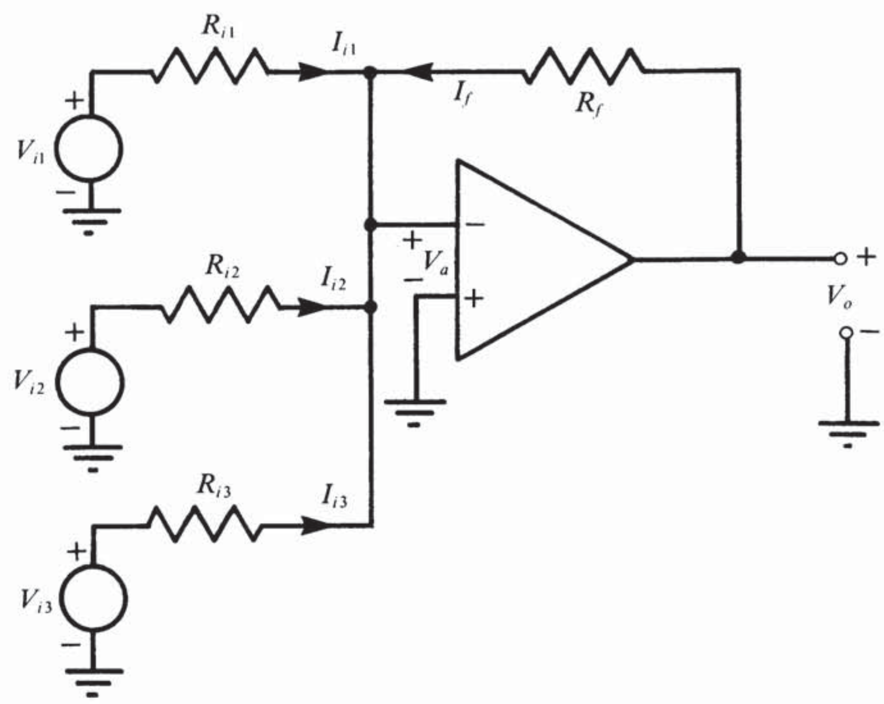
$$V_o = -aV_a \qquad \textbf{(S1.4)}$$

Substitution of numerical values into Equation S1.3 yields

$$-V_a = -\tfrac{1}{7}V_{i1} - \tfrac{2}{7}V_{i2} - \tfrac{3}{7}V_{i3} - \tfrac{1}{7}V_o \qquad \textbf{(S1.5)}$$

This, combined with Equation S1.4, yields the block diagram in Figure S1.2:

Figure S1.1 Three-input inverting amplifier.



Thus, for the specified accuracy, we require that the closed-loop gain of the output loop V_o/V' equal seven within 0.01%. That is,

$$\frac{V_o}{V'} = \frac{a}{1 + a \times 1/7} = 7 \times \frac{a}{7 + a} \geq 6.9993 \qquad \textbf{(S1.6)}$$

Thus, the minimum a is such that:

$$\frac{a}{7 + a} = (1 - 10^{-4}) \qquad \textbf{(S1.7)}$$

which is satisfied by: $a \geq 69{,}993$. That is, the loop-transmission magnitude must be greater than about 10,000.

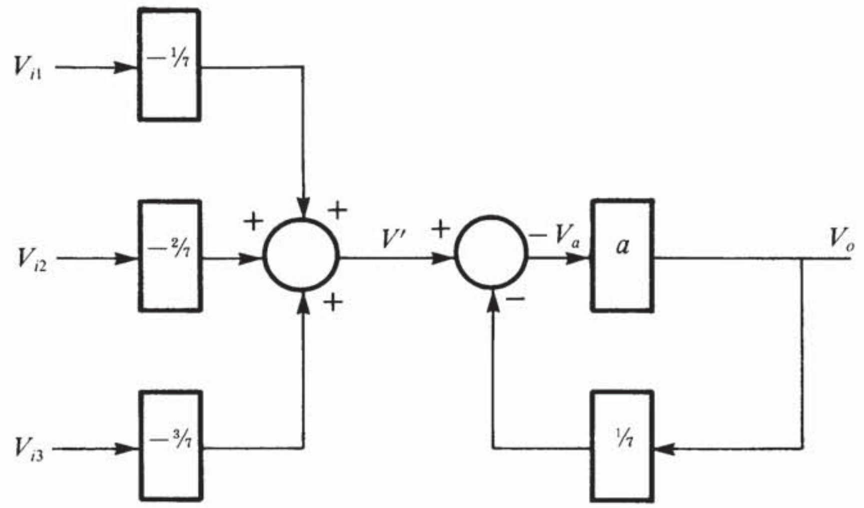


Figure S1.2 Block diagram for three-input inverting amplifier.

Solution 1.2 (P1.6)

We note that the connection of Figure 1.7*a* is simply a special case of Figure 1.7*b* with $V_{i1} = V_{i2}$. Thus, we first solve for the input–output relationship for Figure 1.7*b*. By superposition:

$$V_o = V_{i1} \times \frac{10\text{ k}\Omega}{10\text{ k}\Omega + 10\text{ k}\Omega} \times \frac{10\text{ k}\Omega + 10\text{ k}\Omega}{10\text{ k}\Omega} - V_{i2} \times \frac{10\text{ k}\Omega}{10\text{ k}\Omega} \tag{S1.8}$$

using results derived in Section 1.2.2 of the text for inverting and noninverting connections. This reduces to:

$$V_o = V_{i1} - V_{i2} \tag{S1.9}$$

Thus, this connection is a unity-gain differential amplifier. (How would you design for gains greater or less than unity?) To consider Figure 1.7*a*, we set $V_{i1} = V_{i2} = V_i$, to find:

$$V_o = 0, \text{ for all } V_i \tag{S1.10}$$

This shows the useful property that differential amplifiers reject common-mode input signals.

Solution 1.3 (P2.1)

With reference to Figure 2.20, we write

$$V_o = a_2 V_a \text{ or, equivalently, } V_a = \frac{V_o}{a_2} \tag{S1.11}$$

$$V_a = a_1 V_e \text{ or, equivalently, } V_e = \frac{V_o}{a_1 a_2} \tag{S1.12}$$

$$V_e = V_i - V_f = V_i - f_1 V_a - f_2 V_o \tag{S1.13}$$

Now, we eliminate V_a and V_e by substituting S1.11 and S1.12 into S1.13:

$$\frac{V_o}{a_1 a_2} = V_i - \frac{f_1 V_o}{a_2} - f_2 V_o$$

Collecting terms yields:

$$V_i = \left(\frac{1}{a_1 a_2} + \frac{f_1}{a_2} + f_2\right) V_o$$

or,

$$V_i = \frac{1 + a_1 f_1 + a_1 a_2 f_2}{a_1 a_2} V_o$$

Therefore,

$$\frac{V_o}{V_i} = \frac{a_1 a_2}{1 + a_1 a_2 (f_2 + f_1/a_2)} \tag{S1.14}$$

which is the desired input–output relationship.

We solve the second part by a block-diagram manipulation method. This is often easier than working through the algebra as we have done in the first half of this problem. A valid manipulation on Figure 2.20 yields the block diagram shown in Figure S1.3, which reduces further to that of Figure S1.4, which is the desired reduced form. We see that by using the relation $A = \frac{a}{1 + af}$, Equation S1.14 can be derived by inspection of this reduced block diagram.

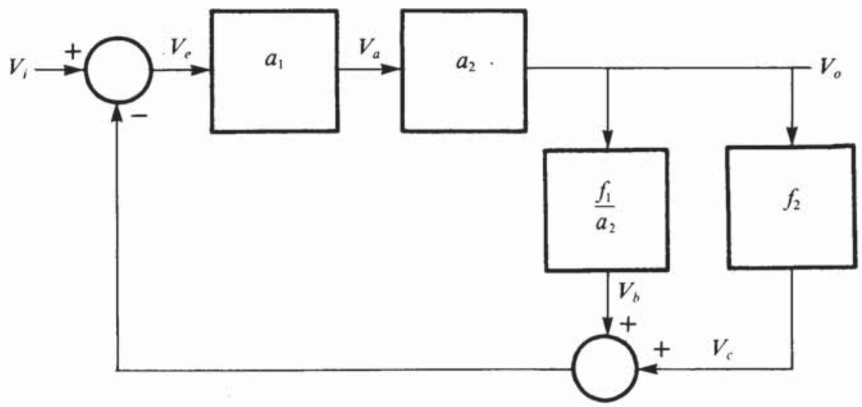


Figure S1.3 Manipulated block diagram.

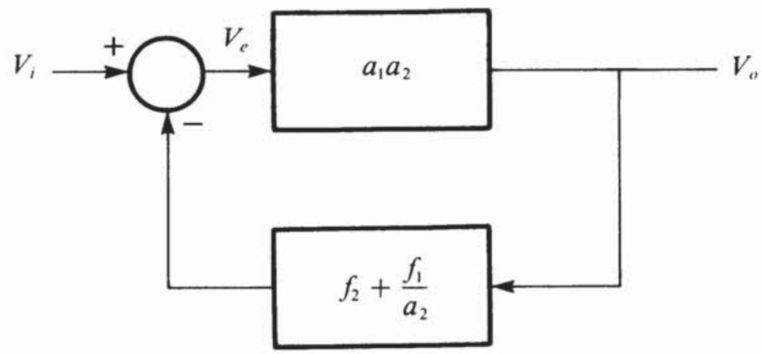


Figure S1.4 Reduced block diagram.

Solution 1.4 (P2.11)

This problem is most readily solved by removing the load and analyzing the output impedance of the op-amp connection, which we shall call R_{amp}. The output impedance with the load connected is then simply R_{amp} in parallel with R_L.

To solve for R_{amp}, we analyze the circuit of Figure S1.5:

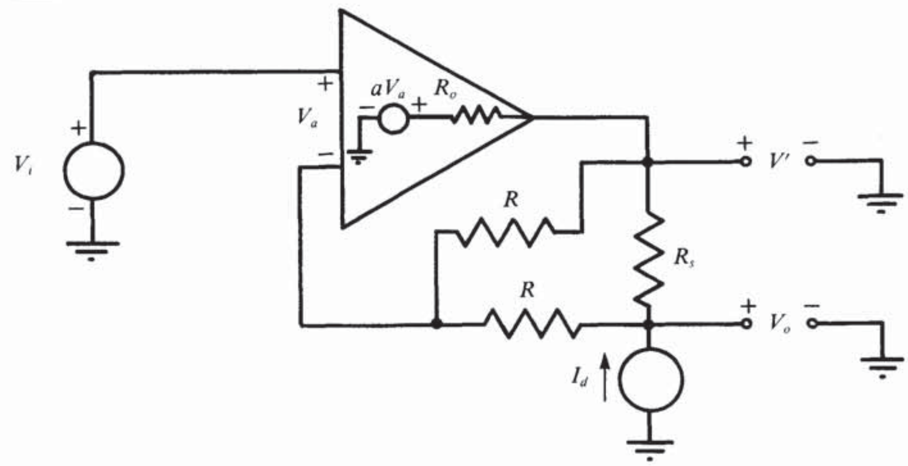


Figure S1.5 Operational amplifier connection with load removed.

Because $R \gg R_s$, we assume that all of I_d flows through the series connection of R_o and R_s. Thus

$$V_o = aV_a + I_d(R_o + R_s) \qquad \textbf{(S1.15)}$$

Now, we require an expression for V_a:

$$V_a = V_i - \frac{(V' + V_o)}{2} \qquad \textbf{(S1.16)}$$

where V' is as defined in Figure S1.5. But,

$$V' \simeq V_o - I_dR_s \qquad \textbf{(S1.17)}$$

Therefore:

$$V_a = V_i - V_o + \frac{I_dR_s}{2} \qquad \textbf{(S1.18)}$$

Equations S1.15 and S1.18 together define the block diagram shown in Figure S1.6:

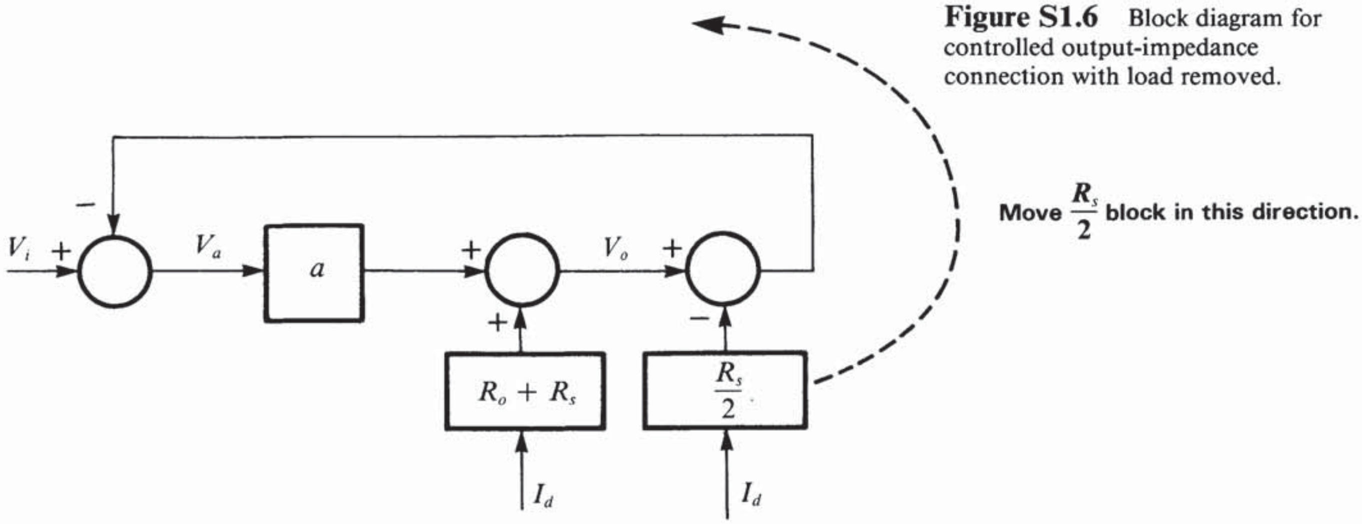


Figure S1.6 Block diagram for controlled output-impedance connection with load removed.

To calculate output-resistance R_{amp}, we need the ratio $\frac{V_o}{I_d}$. Note that V_i does not affect this ratio (as a consequence of linearity); thus, we set $V_i = 0$. We then manipulate the block diagram by propagating the $\frac{R_s}{2}$ block forward around the loop, as indicated in Figure S1.6, to arrive at the reduced block diagram of Figure S1.7:

With V'' defined as in Figure S1.7, we have $V_o = \frac{1}{1 + a} V''$, thus

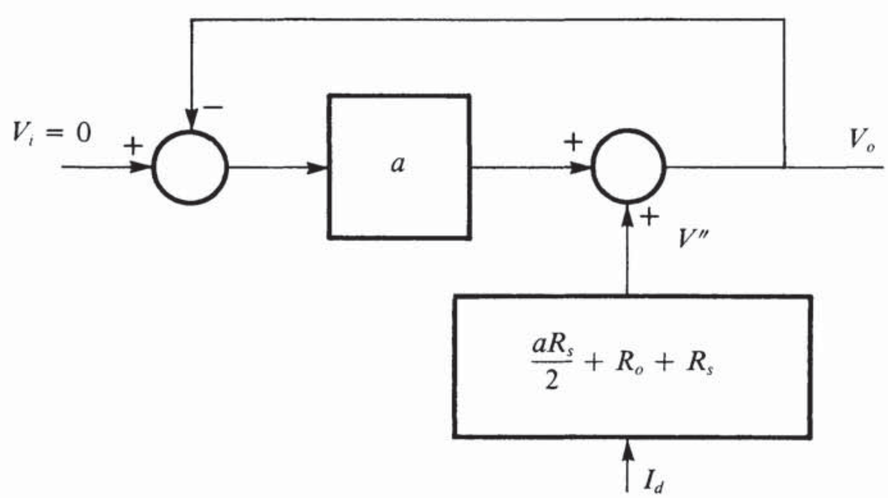


Figure S1.7 Reduced block diagram.

$$V_o = I_d\left(\frac{aR_s}{2} + R_o + R_s\right)\left(\frac{1}{1 + a}\right) \tag{S1.19}$$

and

$$R_{\text{amp}} = \frac{V_o}{I_d} = \frac{\left(\frac{a}{2} + 1\right)R_s + R_o}{1 + a} \tag{S1.20}$$

If we define R_{out} as the output impedance with the load connected, we have:

$$R_{\text{out}} = R_{\text{amp}} \| R_L = \frac{\left(\frac{a}{2} + 1\right)R_s + R_o}{1 + a} \| R_L \tag{S1.21}$$

In the limit of large a, we have

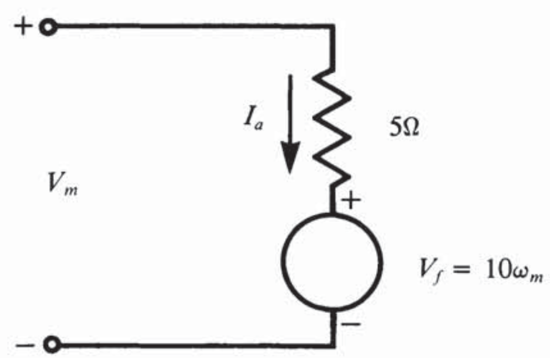
$$R_{\text{out}} = \lim_{a \to \infty}\left[\frac{\left(\frac{a}{2} + 1\right)R_s + R_o}{1 + a} \| R_L\right] = \frac{R_s}{2} \| R_L \tag{S1.22}$$

Solution 1.5 (P2.9)

Let's start by modeling the physical system of motor and antenna (the "plant" in control terminology), which has V_m as its input and θ_o as its output.

The motor is modeled as shown in Figure S1.8:

Figure S1.8 Motor model.



where ω_m is the motor rotational speed in radians per second and V_f represents the motor's back e.m.f. voltage.

Now, motor torque T_m is related to I_a by:

$$T_m = 10 \times I_a \tag{S1.23}$$

From Figure S1.8, we write:

$$I_a = \frac{1}{5\Omega}(V_m - 10\omega_m) \qquad \textbf{(S1.24)}$$

Thus, the first part of the block diagram appears as shown in Figure S1.9.

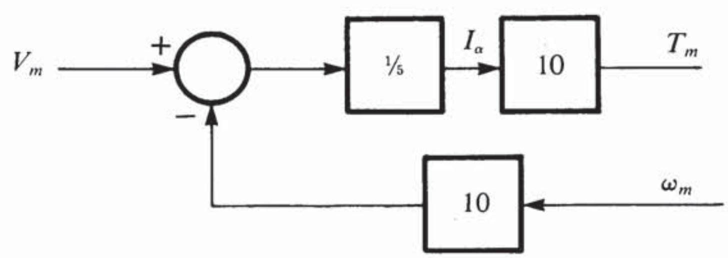


Figure S1.9 Partial block diagram for Problem 1.5 (P2.9).

Now, we require a relation between T_m and ω_m. The rotational equivalent of Newton's law $F = ma$ is $T = I\alpha$ where I is the rotational inertia and α is the angular acceleration. Applying this relationship with $\alpha_m = \frac{d\omega_m}{dt}$ and $I_m = 2$ kg-m we have

$$\frac{d\omega_m}{dt} = \frac{T_m}{2} \qquad \textbf{(S1.25)}$$

Thus, the transfer function between T_m and ω_m is

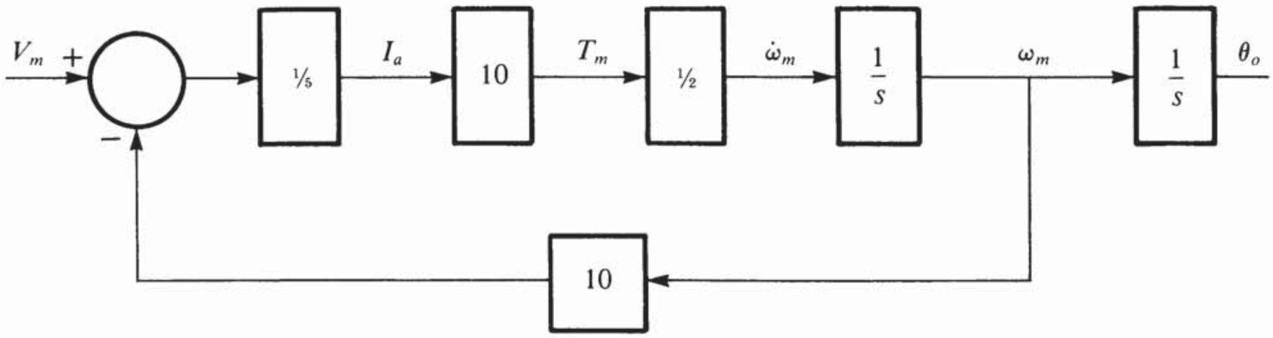
$$\omega_m = \frac{T_m}{2} \times \frac{1}{s} \qquad \textbf{(S1.26)}$$

Angular position is the integral of angular velocity. Therefore:

$$\theta_o = \frac{1}{s} \times \omega_m \qquad \textbf{(S1.27)}$$

Now, we can apply relations S1.26 and S1.27 to draw the complete block diagram for the motor and antenna:

Figure S1.10 Motor and antenna block diagram.

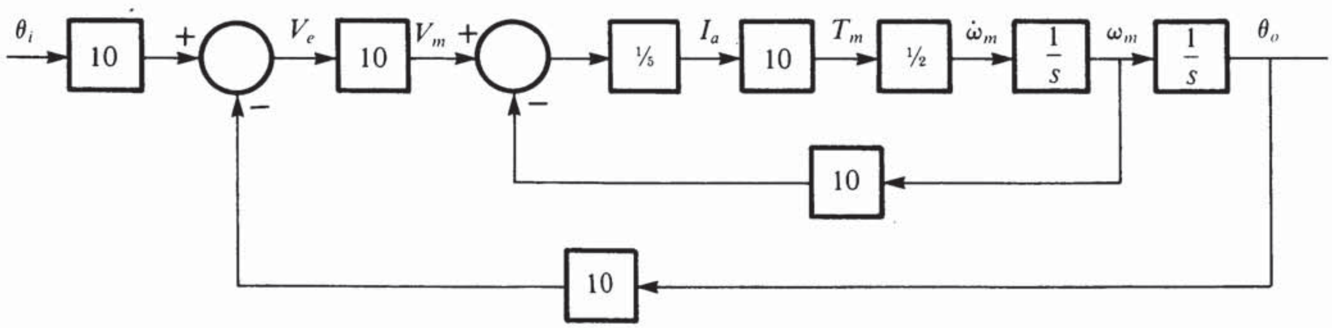


Lastly, we add the differential amplifier. With the error signal defined as in Figure 2.27, and an amplifier gain of 10, we have

$$V_m = 10[10(\theta_i - \theta_o)] \quad \textbf{(S1.28)}$$

This relationship allows us to draw the complete block diagram as shown in Figure S1.11.

Figure S1.11 Complete block diagram for antenna-rotator system.



This may be greatly simplified. The transfer function of the inner loop is:

$$\frac{a}{1+af} = \frac{\frac{1}{s}}{1+\frac{10}{s}} = \frac{1}{s+10} \qquad \textbf{(S1.29)}$$

Then, the reduced block diagram is as shown in Figure S1.12:

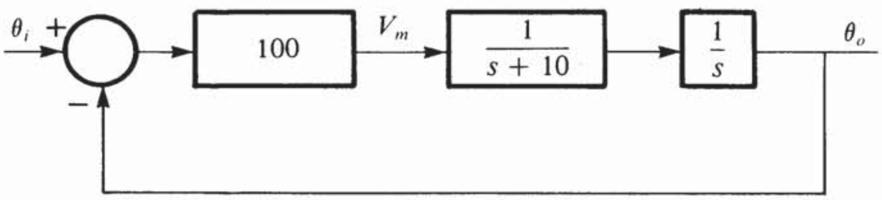


Figure S1.12 Reduced antenna-rotator block diagram.

By inspection of Figure S1.12, the transfer function between θ_o and θ_i is given by:

$$\frac{\theta_o}{\theta_i} = \frac{\frac{100}{s(s+10)}}{1+\frac{100}{s(s+10)}} = \frac{100}{s^2+10s+100} = \frac{1}{\frac{s^2}{100}+\frac{s}{10}+1} \qquad \textbf{(S1.30)}$$

To consider a wind disturbance, we sum a disturbance torque T_d at the point labeled T_m in Figure S1.11 to get Figure S1.13:

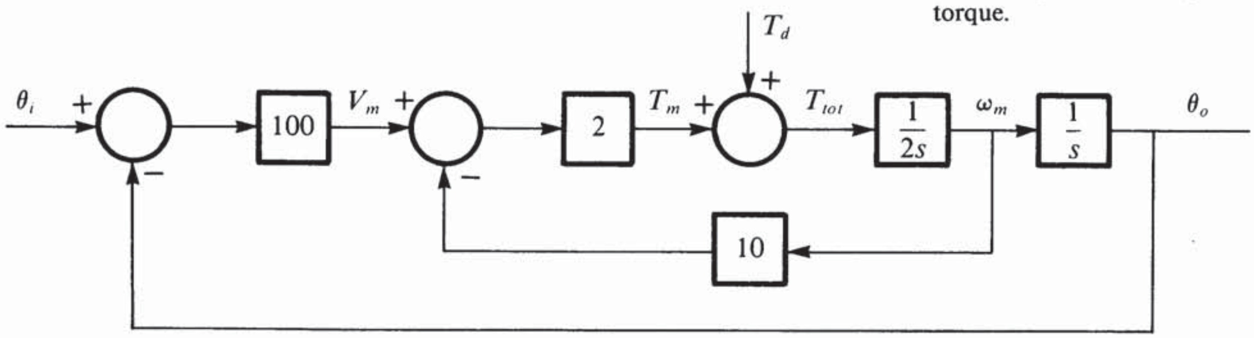


Figure S1.13 Antenna-rotator block diagram including disturbance torque.

At this point, we save further calculations by using a diagram manipulation to move the T_d input to the same summing junction as θ_i, yielding Figure S1.14:

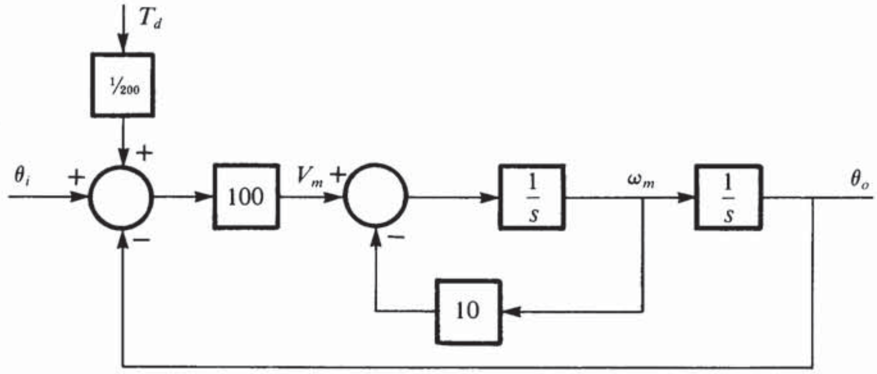


Figure S1.14 Reduced block diagram.

Thus, we see that

$$\frac{\theta_o}{T_d} = \frac{1}{200}\frac{\theta_o}{\theta_i} = \frac{\frac{1}{200}}{\frac{s^2}{100} + \frac{s}{10} + 1} \qquad \textbf{(S1.31)}$$

For a constant input of $T_d = 1$ N-m, we evaluate the transfer function at $s = 0$ to find $\theta_o = \frac{1}{200}$ radians.

In closing, there are two useful points to notice. First, both $\frac{\theta_o}{\theta_i}$ and $\frac{\theta_o}{T_d}$ have the same transfer function denominator. This is the term $1 + af$, which is the system characteristic equation and does not depend on where an input drives the system. Secondly, the input T_d is attenuated relative to θ_i by a factor of 200, which is simply the forward path gain that precedes the torque disturbance.

S O L U T I O N S

Effects of Feedback on Noise and Nonlinearities 2

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 2.1 (P2.3)

There are two regions of operation for this circuit. When $|v_O|$ is less than 1, the feedback element has an incremental gain of 1. Thus:

$$\frac{v_o}{v_i} = \frac{1000}{1 + 1000} \simeq 1, \ |v_O| \leq 1 \tag{S2.1}$$

When $|v_O|$ is greater than 1, the feedback element has an incremental gain of zero. Thus:

$$\frac{v_o}{v_i} = \frac{1000}{1 + 0} = 1000, \ |v_O| > 1 \tag{S2.2}$$

These two equations describe the complete range of operation. The closed-loop transfer characteristics are plotted in Figure S2.1:

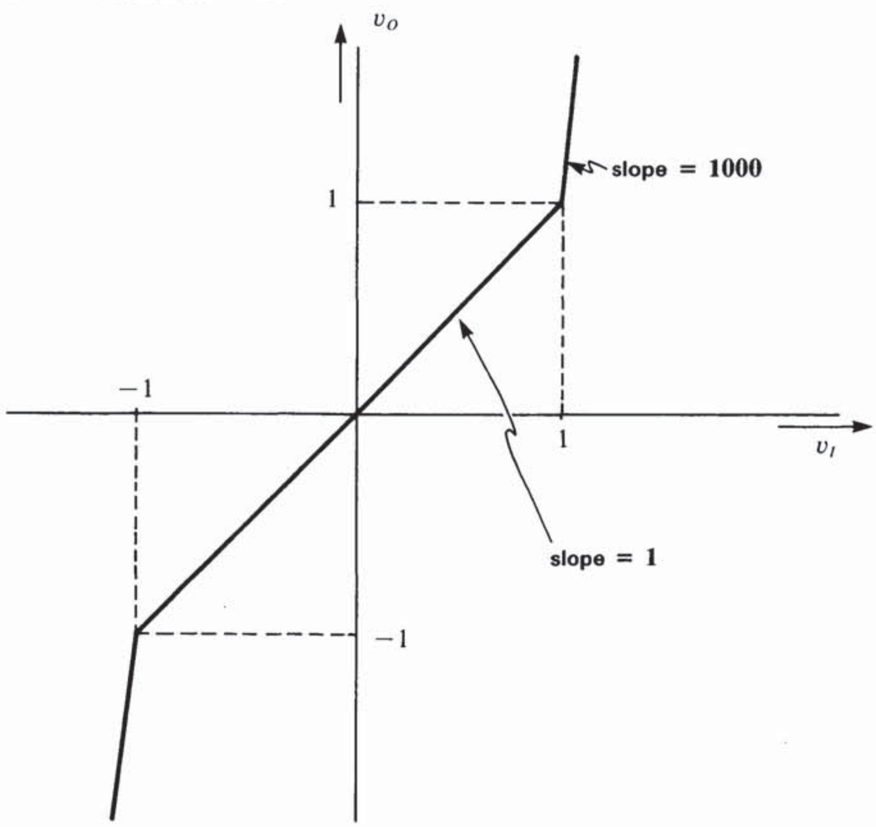


Figure S2.1 Closed-loop transfer characteristics for Problem 2.1 (P2.3).

Solution 2.2 (P2.4)

(a) The complementary emitter–follower has the transfer characteristic shown in Figure S2.2. Note the ± 0.6 volt deadzone.

(b) For a closed-loop gain of $+5$, we use the noninverting connection with a value for the feedback signal of ⅕ of the output voltage, as shown in Figure S2.3:

(c) Here we must consider two nonlinear effects:

- the ± 0.6 volt deadzone
- saturation due to the finite value of V_C

Because the open-loop gain of the operational amplifier is high (10^5), we expect the effect of the deadzone to be greatly reduced, as was shown in Section 2.3.2 of the textbook.

The transistor stage has two regions of operation that we analyze separately. For $|v_I| \leq 0.6$ volts, the output is zero. For $|v_I| \leq 0.6$, we must have $v_A - v_F \leq \dfrac{0.6}{10^5}$. However, in this range, $v_F = 0$. Therefore $v_O = 0$ when v_A is less than 6×10^{-6} volts. For $v_A > 6 \times 10^{-6}$ volts, the complementary emitter–follower will be driven in its linear range, with $v_I > 0.6$ volts. Here, it has an incremental gain of 1. Thus,

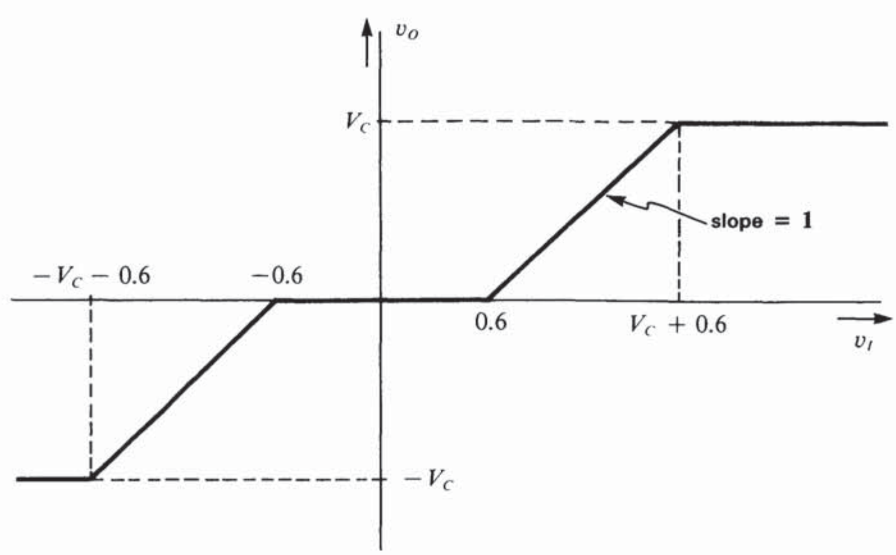


Figure S2.2 Complementary emitter–follower transfer characteristics.

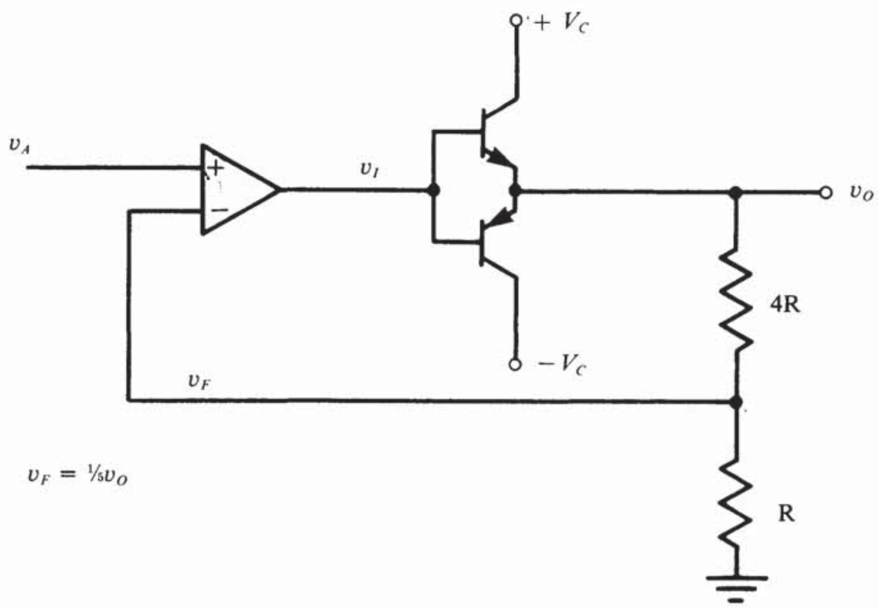


Figure S2.3 Circuit with closed-loop gain of 5.

$$\frac{v_o}{v_a} = \frac{10^5}{1 + \frac{1}{5}10^5} \simeq 5 \qquad |v_A| > 6 \times 10^{-6} \qquad \textbf{(S2.3)}$$

Finally, we account for the finite value of V_C. When v_O reaches $\pm V_C$ the output will saturate. This will occur for $v_A \simeq \frac{V_c}{5}$. The complete closed-loop characteristic is plotted in Figure S2.4. As predicted, the deadzone has been greatly reduced (by a factor of 10^5).

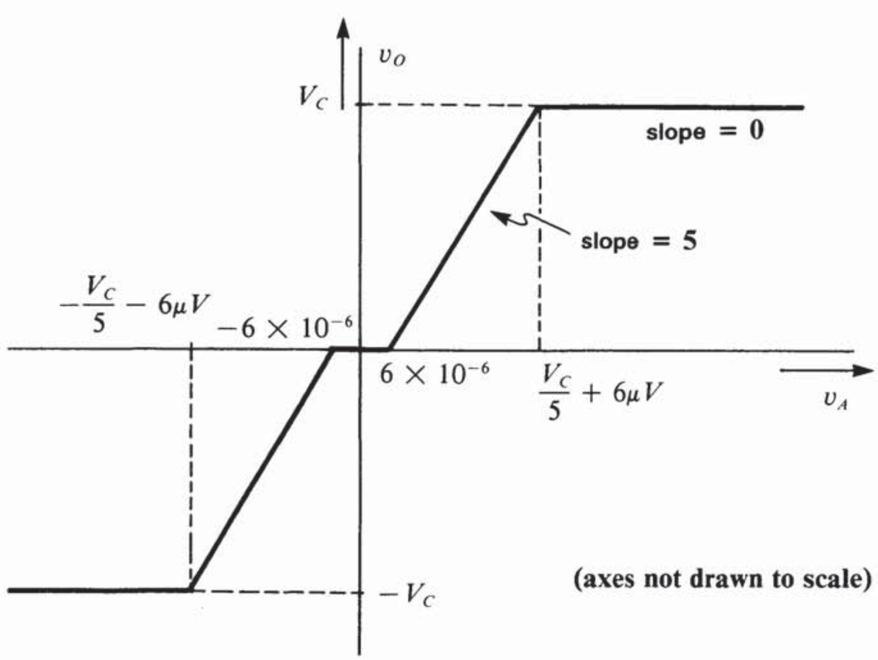
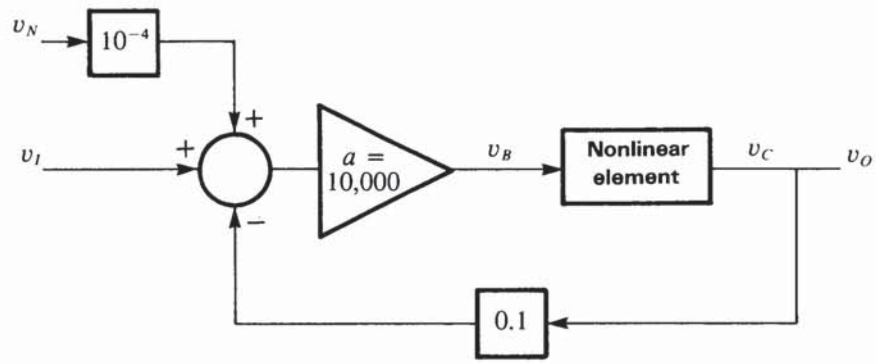


Figure S2.4 Transfer characteristics for circuit of Figure S2.3.

Solution 2.3 (P2.5)

We start this problem by redrawing the block diagram of Figure 2.24. We move the summation point for v_N to the input, as shown in Figure S2.5:

Figure S2.5 Manipulated block diagram for Problem 2.3 (P2.5)



Note that due to the large gain of $a = 10{,}000$, the sinusoidal v_N is attenuated by a factor of 10,000 relative to v_I. As stated in the problem assignment, in parts (a) and (b) we assume that $v_N = 0$.

(a) Now, for $V_I = 0.5$, V_O will be approximately 5, and the nonlinear element has an incremental gain of 1. Thus, the incremental gain is:

$$\frac{v_o}{v_i} = \frac{10^4}{1 + 10^4 \times 0.1} = \frac{10{,}000}{1001} = 9.99 \qquad \textbf{(S2.4)}$$

For $V_I = 1.25$, V_O will be approximately 12.5, and the nonlinear element has an incremental gain of ½. Thus, the overall incremental gain is:

$$\frac{v_o}{v_i} = \frac{5 \times 10^3}{1 + 5 \times 10^3 \times 0.1} = \frac{5000}{501} = 9.98 \qquad \textbf{(S2.5)}$$

(b) The signal v_A is interesting to examine, because we will see that it is acting in such a way as to reduce the effect the nonlinearity has on the output signal v_O.

In the absence of the sinusoid, v_N, for $|v_I| = 1.001$, $|v_o| = 10$. For $|v_o| < 10$, the nonlinear element has an incremental gain of 1. Thus,

$$\frac{v_a}{v_i} = \frac{10^4}{1 + 10^4 \times 0.1} = 9.99 \qquad 0 \leq |v_I| \leq 1.001 \quad \textbf{(S2.6)}$$

For $|v_I| = 1.502$, $|v_O| = 15$. For $10 < |v_O| < 15$, the nonlinear element has an incremental gain of ½. Thus,

$$\frac{v_a}{v_i} = \frac{10^4}{1 + 10^4 \times \frac{1}{2} \times 0.1} = 19.96 \qquad 1.001 < |v_I| \leq 1.502$$

(S2.7)

For $|v_I| > 1.502$, the nonlinear element saturates, and $|v_O| = 15$. Here, the nonlinear element has an incremental gain of zero. Thus,

$$\frac{v_a}{v_i} = 10^4 \qquad |v_I| > 1.502$$

$$|v_A| < 30 \quad \textbf{(S2.8)}$$

Finally, when $|v_A| = 30$ volts, the amplifier saturates.

Then, with $v_I(t) = t$, $t \geq 0$, we can use the above expressions to solve for $v_A(t)$.

$$v_A(t) = 9.99t \qquad 0 \leq t \leq 1.001 \quad \textbf{(S2.9)}$$

$$v_A(t) = 10 + 19.96(t - 1.001) \qquad 1.001 < t \leq 1.502 \quad \textbf{(S2.10)}$$

$$v_A(t) = 20 + 10^4(t - 1.502) \qquad 1.502 < t < 1.503 \quad \textbf{(S2.11)}$$

$$v_A(t) = 30 \qquad t \geq 1.503 \quad \textbf{(S2.12)}$$

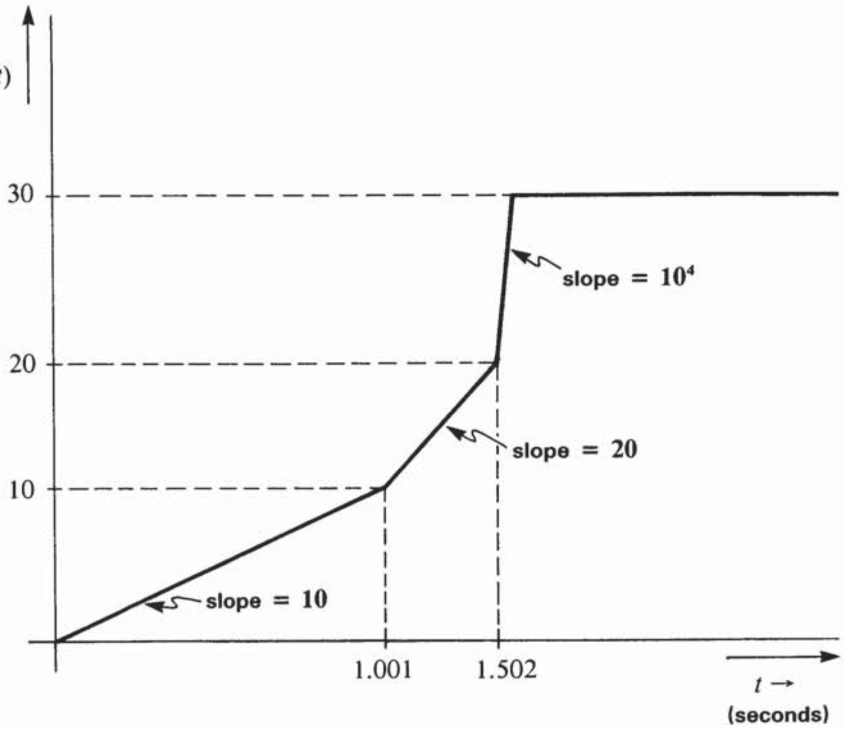
The resulting $v_A(t)$ is plotted in Figure S2.6.

(c) For $v_I = 0$, the incremental gain $\frac{v_o}{v_i} = 9.99$, as calculated earlier. As mentioned earlier, the gain for v_N is a factor of 10^4 less, or

$$\frac{v_o}{v_N} = 9.999 \times 10^{-4} \quad \textbf{(S2.13)}$$

So the sinusoidal component of v_O has an amplitude of about 10^{-3} volts.

Figure S2.6 Plot of $v_A(t)$ for $v_I(t) = t, t \geq 0$.



Introduction to Systems with Dynamics 3

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 3.1 (P3.1)

From Figure 3.6 on page 79 of the textbook, a first-order system has a step response as shown in Figure S3.1:

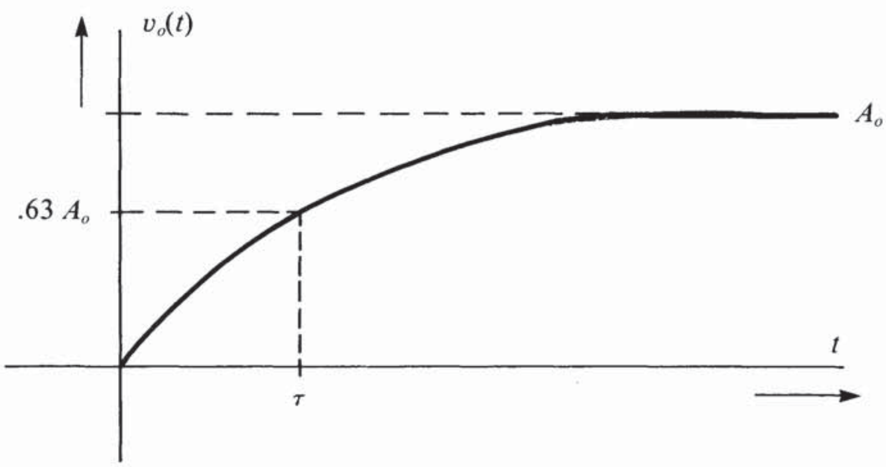


Figure S3.1 First-order step response.

where A_o is the d-c gain of the system. This time response is described by

$$v_O(t) = A_o(1 - e^{-t/\tau}) \qquad \textbf{(S3.1)}$$

and the transfer function for this system is

$$\frac{V_o}{V_i}(s) = \frac{A_o}{\tau s + 1} \qquad \textbf{(S3.2)}$$

Here, we are given an operational amplifier connected for a non-inverting gain of 10, as shown in Figure S3.2:

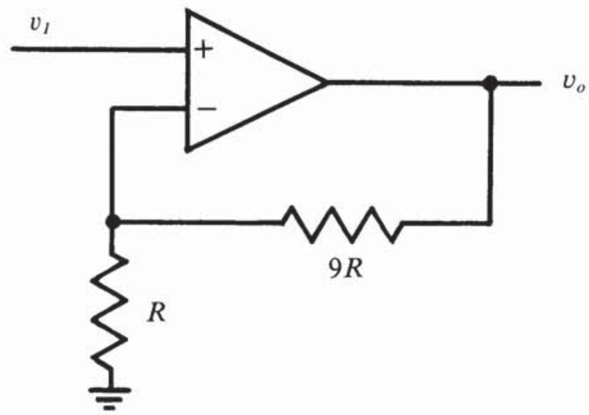


Figure S3.2 Gain-of-ten connection.

This corresponds to the block diagram of Figure S3.3:

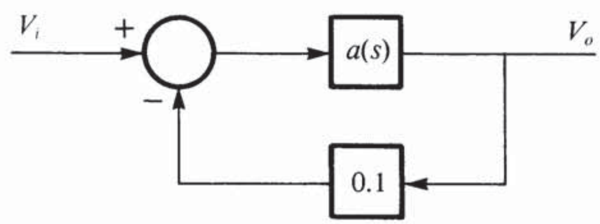


Figure S3.3 Block diagram for the gain-of-ten connection.

This connection has a transfer function of

$$\frac{V_o}{V_i}(s) = \frac{a(s)}{1 + 0.1 \times a(s)} = \frac{10\ a(s)}{10 + a(s)} \tag{S3.3}$$

We are given that this system is first order with $A_o = 10$, and $\tau = 10^{-6}$ sec. Thus, using Equation S3.2, we have

$$\frac{10\ a(s)}{10 + a(s)} = \frac{10}{10^{-6}s + 1} \tag{S3.4}$$

Solve this for $a(s)$:

$$a(s) = \frac{10 + a(s)}{10^{-6}s + 1} \tag{S3.5}$$

Collect terms:

$$a(s)\left(1 - \frac{1}{10^{-6}s + 1}\right) = \frac{10}{10^{-6}s + 1} \qquad \textbf{(S3.6)}$$

or

$$a(s)\left(\frac{10^{-6}s}{10^{-6}s + 1}\right) = \frac{10}{10^{-6}s + 1} \qquad \textbf{(S3.7)}$$

which yields:

$$a(s) = \frac{10^{7}}{s} \qquad \textbf{(S3.8)}$$

That is, the op amp is modeled as a pole at the origin, which represents an integrator.

Solution 3.2 (P3.2)

First, let's examine the pole locations for this system. There is a complex pair at $s_1 = -0.25 + j0.97$ and $s_2 = -0.25 - j0.97$. There is a real axis pole at $s_3 = -10$. These poles are shown on the s plane in Figure S3.4.

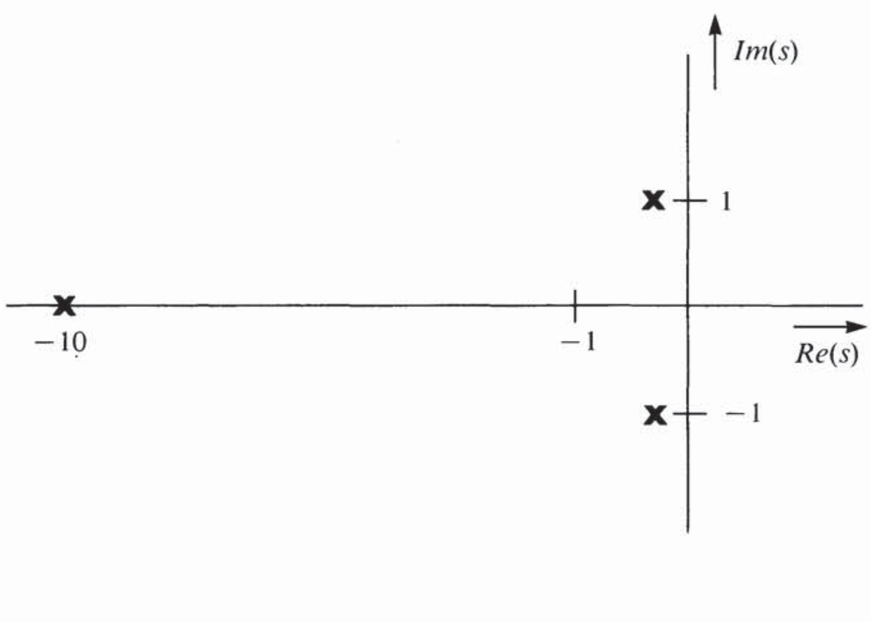


Figure S3.4 Pole locations for Problem 3.2 (P3.2).

Following the discussion of Section 3.3.2 of the textbook, because the real axis pole is a factor of 10 farther from the origin than the complex pair, the system is well approximated by the complex pair alone.

The complex pair has $\omega_n = 1$, and $\zeta = 0.25$. We consider the system to be approximated by the transfer function:

$$A(s) \simeq \frac{1}{s^2 + 0.5s + 1} \tag{S3.9}$$

This has a step response given by Equation 3.41 of the textbook. That is:

$$v_O(t) = a_o\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \Phi\right)\right]$$

where

$$\Phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{S3.10}$$

Here $a_o = 1$, and $\Phi = 1.32$ radians. Thus,

$$v_O(t) = [1 - 1.03e^{-t/4} \sin(0.97t + 1.32)] \tag{S3.11}$$

For this second-order system, we can estimate the peak overshoot by using Equation 3.58 of the textbook. That is,

$$P_o = 1 + e^{-\pi/\tan\Phi} = 1.45 \tag{S3.12}$$

Thus, there is a 45% overshoot.

Solution 3.3 (P3.5)

A system that is second order, with a d-c gain of 1, has a transfer function of the form

$$\frac{V_o}{V_i}(s) = \frac{1}{\dfrac{s^2}{\omega_n^2} + \dfrac{2\zeta s}{\omega_n} + 1} \tag{S3.13}$$

Given that $P_o = 1.38$, we can use Equation 3.58 of the textbook to solve for ζ. We have:

$$P_o = 1 + e^{-\pi/\tan\theta} = 1.38 \tag{S3.14}$$

Stability 4

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 4.1 (P4.1)

Because the coefficients of the polynomial

$$s^5 + s^4 + 3s^3 + 4s^2 + s + 2 \tag{S4.1}$$

are all present and of the same sign, the necessary condition for all roots to have negative real parts is satisfied. The Routh array is:

$$\begin{array}{ccc}
1 & 3 & 1 \\
1 & 4 & 2 \\
\dfrac{(1\times 3)-(1\times 4)}{1} = -1 & \dfrac{(1\times 1)-(1\times 2)}{1} = -1 & 0 \\
\dfrac{(-1\times 4)-(1\times -1)}{-1} = 3 & \dfrac{(-1\times 2)-(1\times 0)}{-1} = 2 & 0 \\
\dfrac{(3\times -1)-(-1\times 2)}{3} = -\dfrac{1}{3} & 0 & 0 \\
\dfrac{(-\frac{1}{3}\times 2)-(3\times 0)}{-\frac{1}{3}} = 2 & 0 & 0
\end{array} \tag{S4.2}$$

Redrawing the array for clarity, we have

$$\begin{array}{ccc}
1 & 3 & 1 \\
1 & 4 & 2 \\
-1 & -1 & 0 \\
3 & 2 & 0 \\
-\frac{1}{3} & 0 & 0 \\
2 & 0 & 0
\end{array} \tag{S4.3}$$

There are four sign changes in the first column, and thus four right-half-plane zeros of the polynomial.

Solution 4.2 (P4.2)

Following the development on pp. 116 and 117 of the textbook, we write the characteristic equation as 1 minus the loop transmission. That is,

$$\begin{matrix}\text{Characteristic} \\ \text{equation}\end{matrix} = 1 - L(s) = 1 + \frac{a_o}{(\tau s + 1)^4} \qquad \textbf{(S4.4)}$$

After clearing fractions, the characteristic polynomial is

$$\begin{aligned} P(s) &= (\tau s + 1)^4 + a_o \\ &= \tau^4 s^4 + 4\tau^3 s^3 + 6\tau^2 s^2 + 4\tau s + 1 + a_o \end{aligned} \qquad \textbf{(S4.5)}$$

The Routh array associated with this polynomial is

$$\begin{matrix} \tau^4 & 6\tau^2 & 1 + a_o \\ 4\tau^3 & 4\tau & 0 \\ 5\tau^2 & 1 + a_o & 0 \\ 4\tau\left(\dfrac{4 - a_o}{5}\right) & 0 & 0 \\ 1 + a_o & 0 & 0 \end{matrix} \qquad \textbf{(S4.6)}$$

The reader should check the algebra used to derive this array.

Assuming τ is positive, roots with positive real parts occur for $a_o < -1$ (one right-half-plane zero), and for $a_o > 4$ (two right-half-plane zeros). Recall that the problem asks for the a_o that results in a pair of complex roots on the imaginary axis. Only the value $a_o = 4$ satisfies this condition. With $a_o = 4$, the entire fourth row is zero, and we can solve for the pole locations by using the auxiliary equation. Using the coefficients of the third row, the auxiliary equation is

$$5\tau^2 s^2 + 5 = 0 \qquad \textbf{(S4.7)}$$

The equation has solutions at

$$s = \pm \frac{j}{\tau} \qquad \textbf{(S4.8)}$$

indicating that with $a_o = 4$, the system will oscillate at $\dfrac{1}{\tau}$ rad/sec.

Now that we have found two of the poles for $a_o = 4$, they can be factored out to find the two other roots of the characteristic equation. That is, we can factor $P(s)$ as the term $\tau^2 s^2 + 1$ multiplied by a quadratic. This quadratic can be found by applying synthetic division to $P(s)$ as shown below.

$$
\begin{array}{r r l}
 & \tau^2 s^2 + 4\tau s + 5 & \\
\tau^2 s^2 + 1 \,\big) & \overline{\tau^4 s^4 + 4\tau^3 s^3 + 6\tau^2 s^2 + 4\tau s + 5} & \\
 & \underline{\tau^4 s^4 + \tau^2 s^2} & \\
 & 4\tau^3 s^3 + 5\tau^2 s^2 + 4\tau s + 5 & \quad \textbf{(S4.9)} \\
 & \underline{4\tau^3 s^3 + 4\tau s} & \\
 & 5\tau^2 s^2 + 5 & \\
 & \underline{5\tau^2 s^2 + 5} & \\
 & 0 &
\end{array}
$$

Then the two remaining poles are solutions of

$$\tau^2 s^2 + 4\tau s + 5 = 0 \tag{S4.10}$$

which is solved by the quadratic formula to give

$$s = \frac{-2 + j}{\tau} \tag{S4.11a}$$

and

$$s = \frac{-2 - j}{\tau} \tag{S4.11b}$$

as the two other closed-loop pole locations when $a_o = 4$.

S O L U T I O N S

Root locus 5

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 5.1 (P4.5)

(a) Rule 2 is all that is required to find the branches, and Rule 1 tells us that the branches terminate on the two zeros. Thus the root locus is as sketched in Figure S5.1*a*.

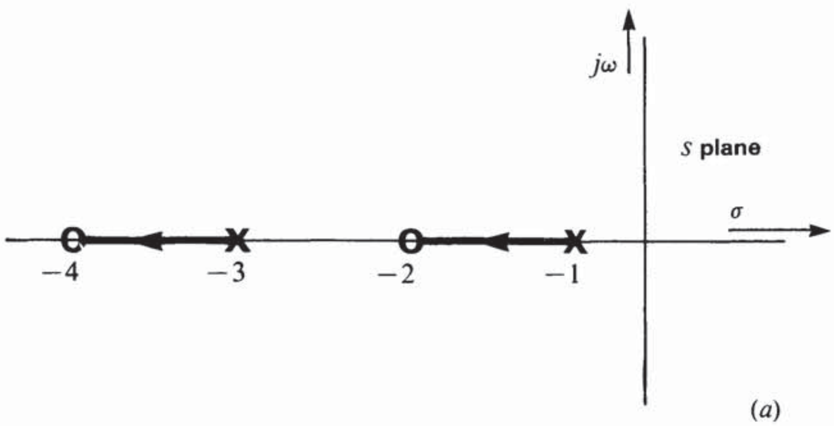


Figure S5.1 Root loci for Problem 5.1 (P4.5). (*a*) Root locus for Figure 4.27*a*.

(b) Rule 4 tells us that the average distance of the poles from the origin remains constant at $\bar{s} = -\dfrac{2+1+1}{3} = -4/3$. By Rule 5, the two complex poles approach asymptotes of $\pm 60°$. By Rule 6, the angle of the branch in the vicinity of the upper complex pole is

$$\begin{aligned}\theta_p &= 180° + \Sigma \sphericalangle z - \Sigma \sphericalangle p \\ &= 180° + 0 - 90° - 45° \qquad \textbf{(S5.1)} \\ &= 45°\end{aligned}$$

At the lower complex pole the angle will be $-45°$.

The point at which the complex pair crosses the imaginary axis can be estimated by using the asymptotes that intersect the real axis at $s = -4/3$. They intersect the imaginary axis at $s = \pm 4/3 \tan 60° = \pm j2.31$. The complex pole pair will cross the imaginary axis near this point.

The exact intersection of the root loci with the imaginary axis can be solved for using the Routh criterion. By inspection of the pole-zero diagram the characteristic equation (after clearing fractions) is

$$\begin{aligned} P(s) &= (s + 2)(s + 1 + j)(s + 1 - j) + a_o f_o \\ &= (s + 2)(s^2 + 2s + 2) + a_o f_o \\ &= s^3 + 4s^2 + 6s + 4 + a_o f_o \end{aligned} \tag{S5.2}$$

The Routh array is

$$\begin{array}{cc} 1 & 6 \\ 4 & 4 + a_o f_o \\ 5 - \dfrac{a_o f_o}{4} & 0 \\ 4 + a_o f_o & 0 \end{array} \tag{S5.3}$$

The value $a_o f_o = 20$ will make the third row all zero. With this value of $a_o f_o$, we use the second row to write the auxiliary equation

$$4s^2 + 24 = 0 \tag{S5.4}$$

This has solutions at

$$s = \pm j\sqrt{6} = \pm j2.45 \tag{S5.5}$$

This exact value is close to our earlier estimate of $s = \pm j2.31$, as expected. In many cases, such an estimate will be sufficiently accurate, and the Routh computation may be avoided.

Using all of the above information we sketch the root locus in Figure S5.1*b*.

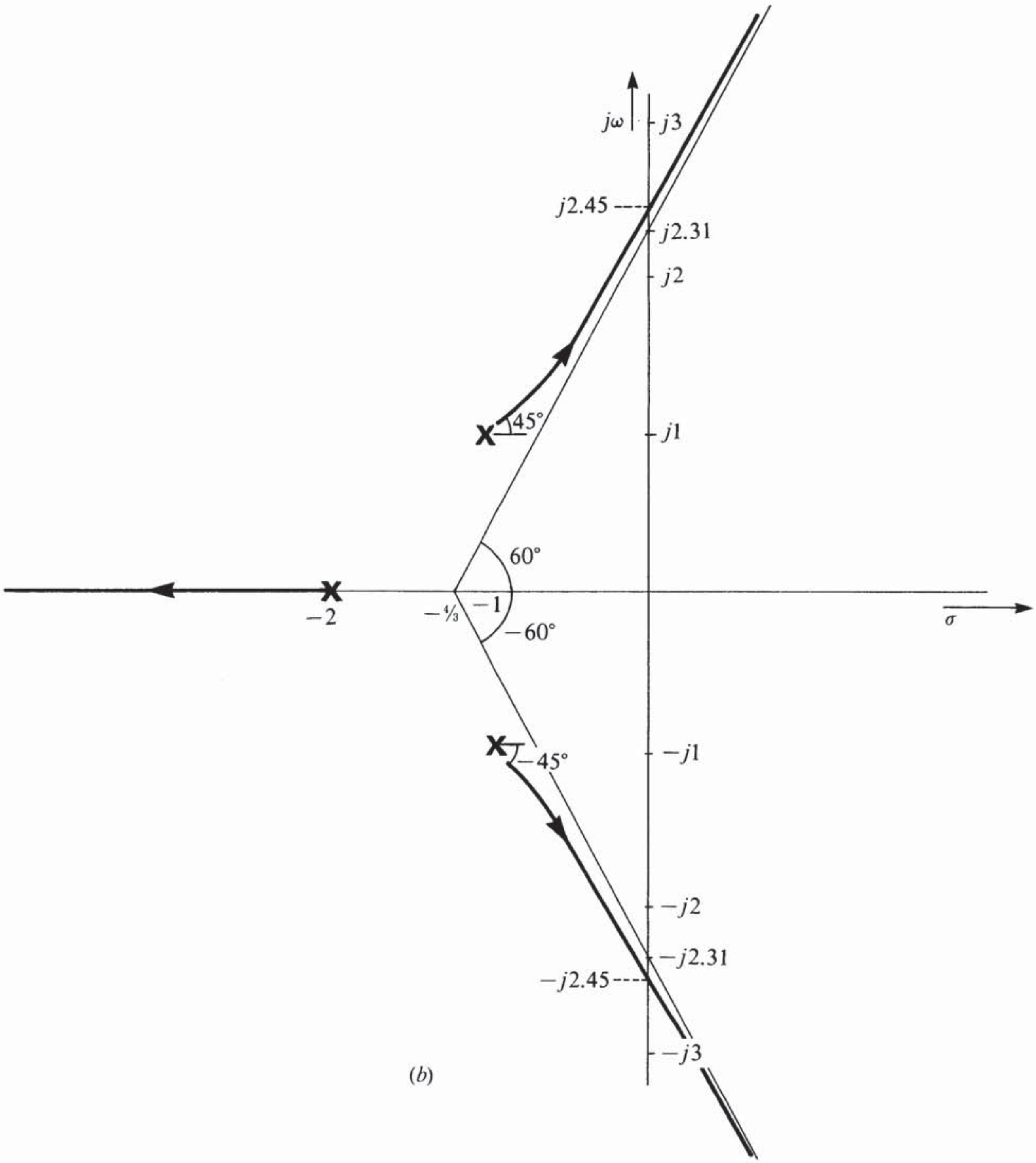
(c) Following the development in the textbook of Rule 7, for moderate values of $a_o f_o$, we may ignore the pole at $s = -1000$, and sketch the locus of the three other poles. Using Rule 3, the breakaway point between the poles at $s = -1$ and $s = -2$ will occur at the solution of

$$\frac{d((s + 1)(s + 2)(s + 3))}{ds} = 0 \tag{S5.6}$$

Multiplying gives

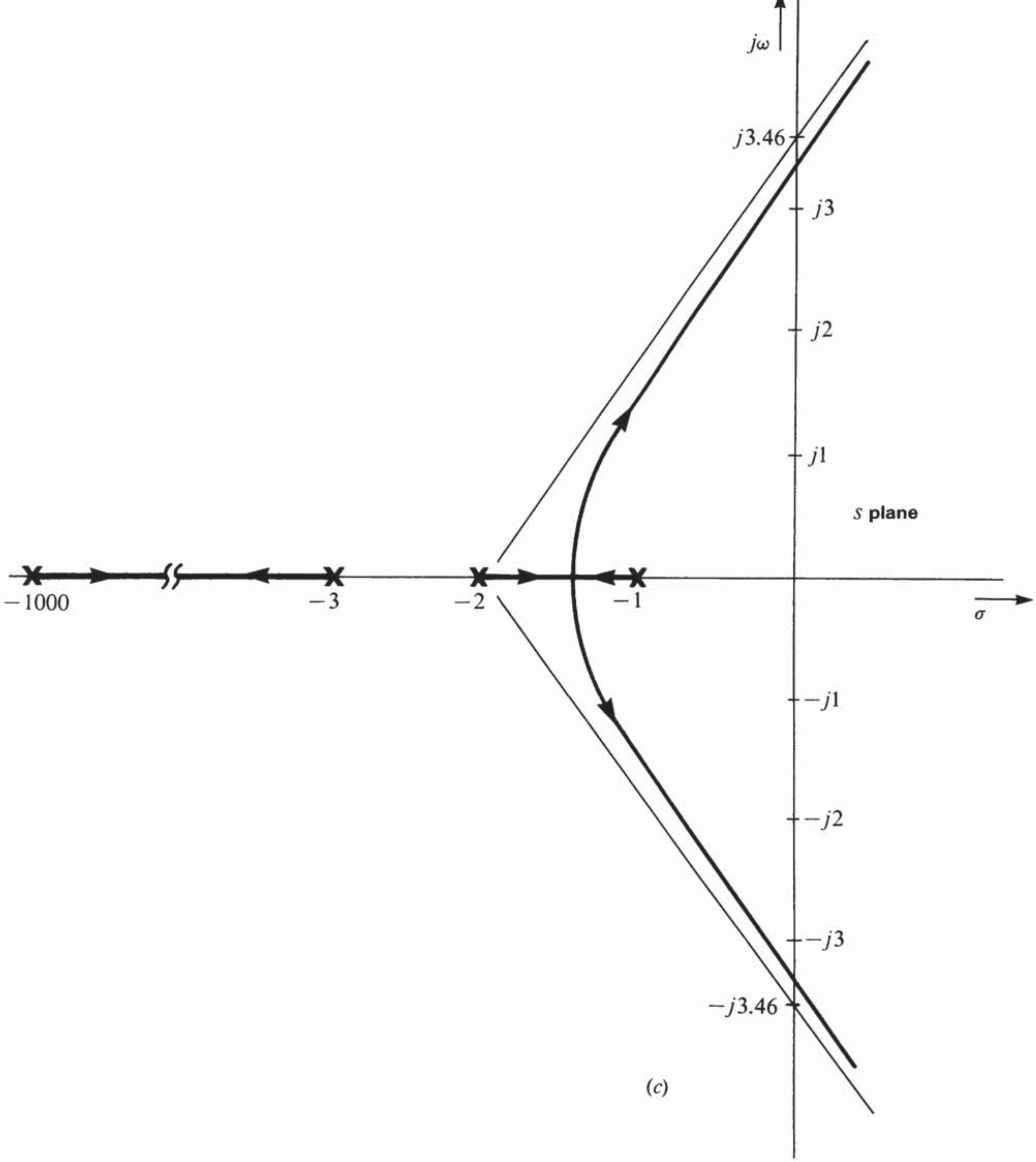
$$\frac{d(s^3 + 6s^2 + 11s + 6)}{ds} = 0 \tag{S5.7}$$

Figure S5.1 (*b*) Root locus for Figure 4.27*b*.



(*b*)

Figure S5.1 (*c*) Root locus for Figure 4.27*c*.



which gives

$$3s^2 + 12s + 11 = 0 \tag{S5.8}$$

This is solved by

$$s = -1.42$$

and

$$s = -2.58 \tag{S5.9}$$

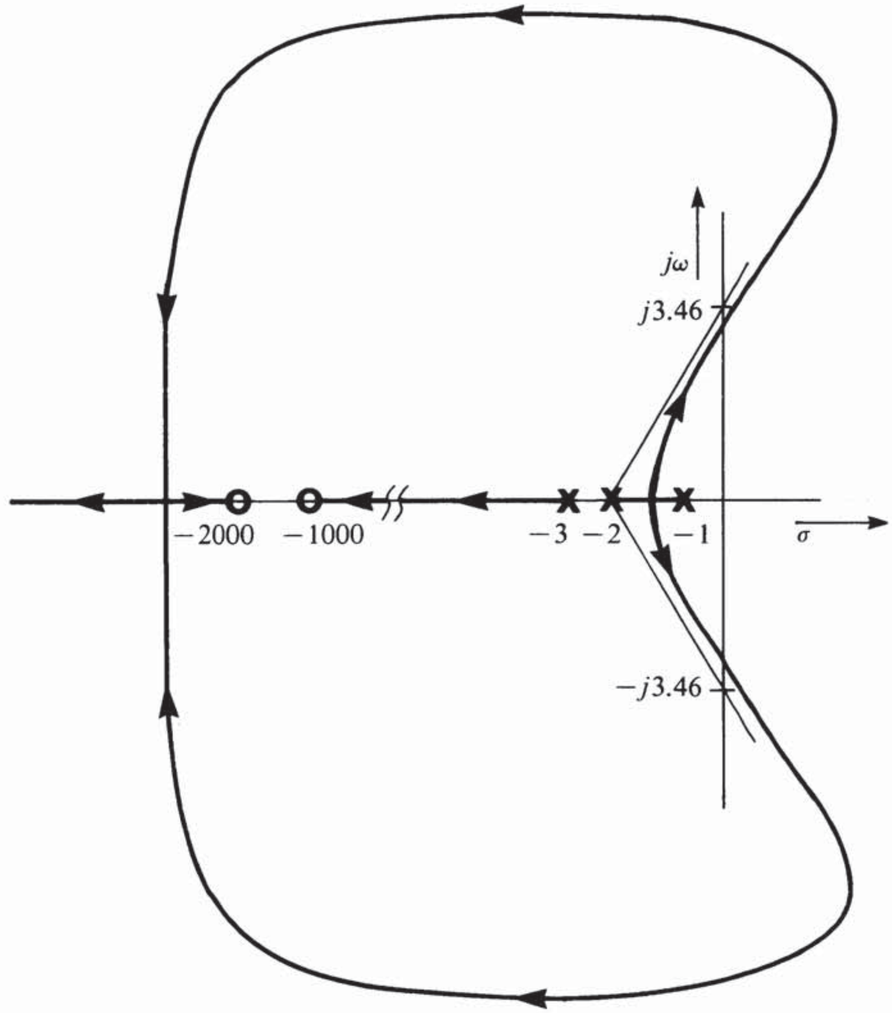
Only the solution at $s = -1.42$ is meaningful here (see p. 127 in the textbook), so the breakaway point is at $s = -1.42$.

By Rule 5, the loci approach asymptotes of $\pm 60°$, and these asymptotes intersect the real axis at $s = -2$. The asymptotes cross the imaginary axis at $s = \pm j2 \tan 60° = \pm j3.46$. Using this information, we sketch the root locus as shown in Figure S5.1*c*.

(d) By Rule 7, for low values of $a_o f_o$, the root locus will be the same as in part c. However, by Rule 2, we know that there is a branch of the locus to the left of the zero at $s = -2000$. Thus, while the two complex poles initially enter the right half of the s plane, they must at some point turn around, reenter the left half of the s plane, and rejoin the real axis at some point to the left of the zero at $s = -2000$. After rejoining the real axis, one pole will move to the right to the zero at $s = -2000$, and the other pole will move off to the left to infinity. Thus, the root locus appears as sketched in Figure S5.1*d*.

This system has the interesting property of being stable for low and high values of $a_o f_o$, but unstable for intermediate values of $a_o f_o$.

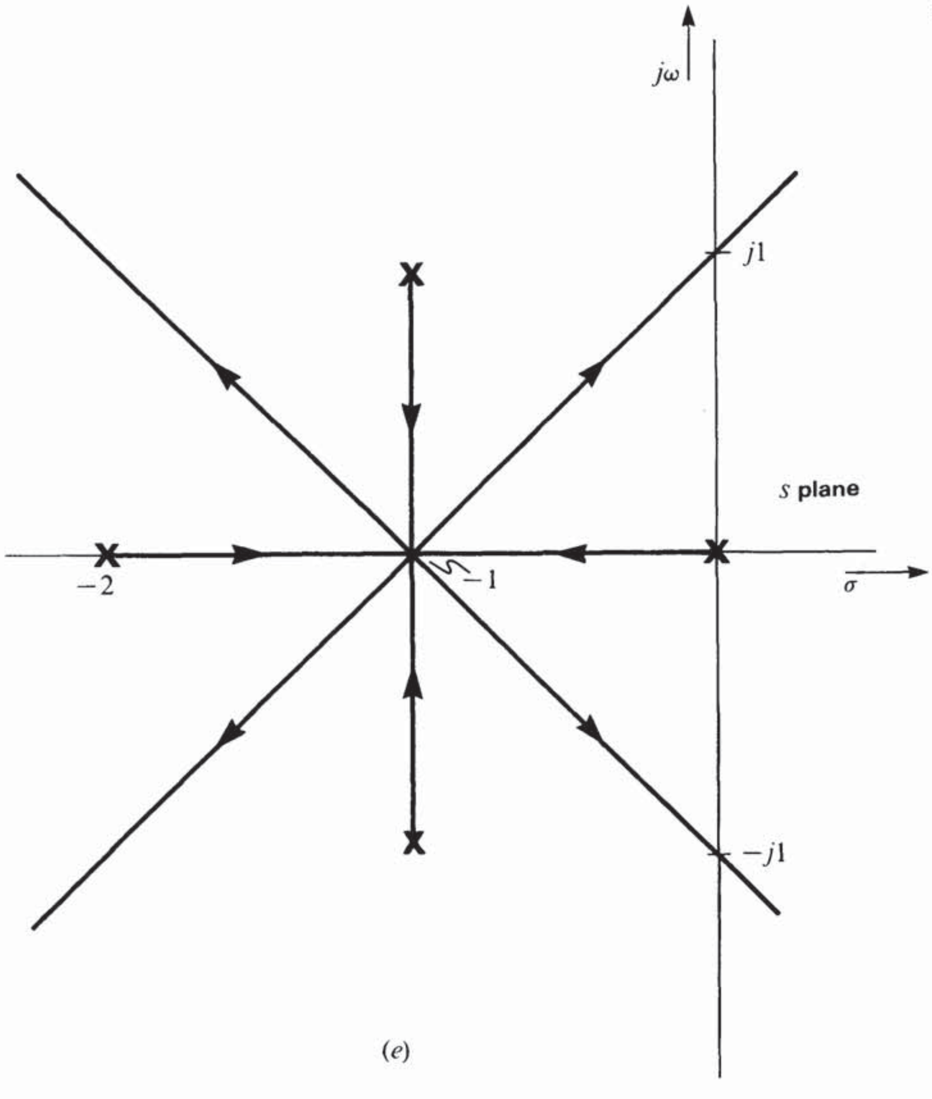
Figure S5.1 (*d*) Root locus for Figure 4.27*d*.



(*d*)

(e) By Rule 2, branches exist on the real axis between $s = 0$ and $s = -2$. By Rule 5, because there are four poles and no zeros, all four poles eventually (for large $a_o f_o$) approach infinity along asymptotes of $\pm 45°$ and $\pm 135°$. The asymptotes intersect at $s = -2$. By Rule 6, the loci leave the pole at $s = -1 + j$ at an angle of $-90°$, and leave the pole at $s = -1 - j$ at an angle of $+90°$. Combining this information we sketch the root locus as in Figure S5.1*e*.

Figure S5.1 (*e*) Root locus for Figure 4.27*e*.



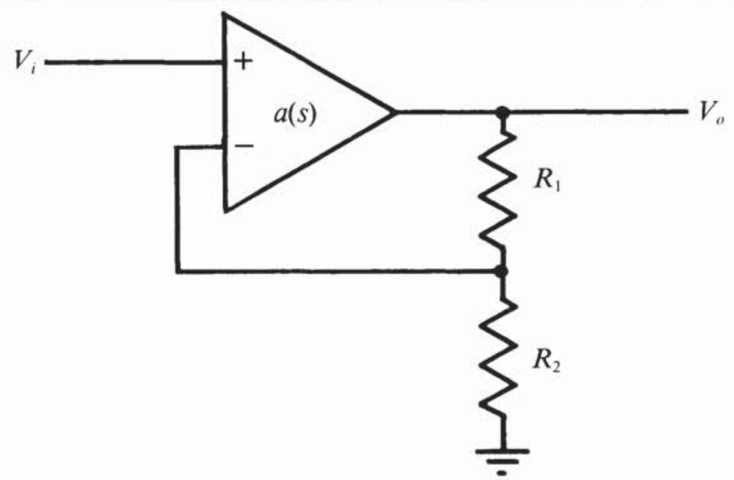
(*e*)

Solution 5.2 (P4.7)

The noninverting configuration is as shown in Figure S5.2.

The loop transmission for this topology is $-a(s)f_o$, where $f_o = \dfrac{R_2}{R_1 + R_2}$. As long as the $a_o f_o$ product is large, that is, as long as $f_o \gg 10^{-6}$, the closed-loop gain is well approximated by $\dfrac{1}{f_o} = \dfrac{R_1 + R_2}{R_2}$.

Figure S5.2 Noninverting amplifier.



At this point, a sketch of the root locus is helpful for understanding the system behavior as f_o is varied. The closed-loop poles start at the poles of $a(s)$ for small values of f_o, and move as shown in Figure S5.3 as f_o is increased. (Note that the axes are not drawn to scale.)

The root locus has been drawn by recognizing that two of the poles will approach asymptotes of $\pm 60°$, and eventually enter the right half of the s plane, as f_o grows. The third pole follows the real axis off to the left. Other details, such as breakaway points, are not important for this analysis, and we do not solve for them.

The dashed lines indicate points where the damping ratio for a complex pair is equal to 0.5. From the root-locus sketch it is apparent that there is some value of f_o for which the poles will lie on this line. Following the development on p. 128 of the textbook, when the system damping ratio is 0.5, its characteristic equation is given by Equation 4.62 as

$$P'(s) = s^3 + (\gamma + 2\beta)s^2 + 2\beta(\gamma + 2\beta)s + 4\gamma\beta^2 \quad \textbf{(S5.10)}$$

The closed-loop characteristic equation for the amplifier is 1 minus the loop transmission, which is (when cleared of fractions)

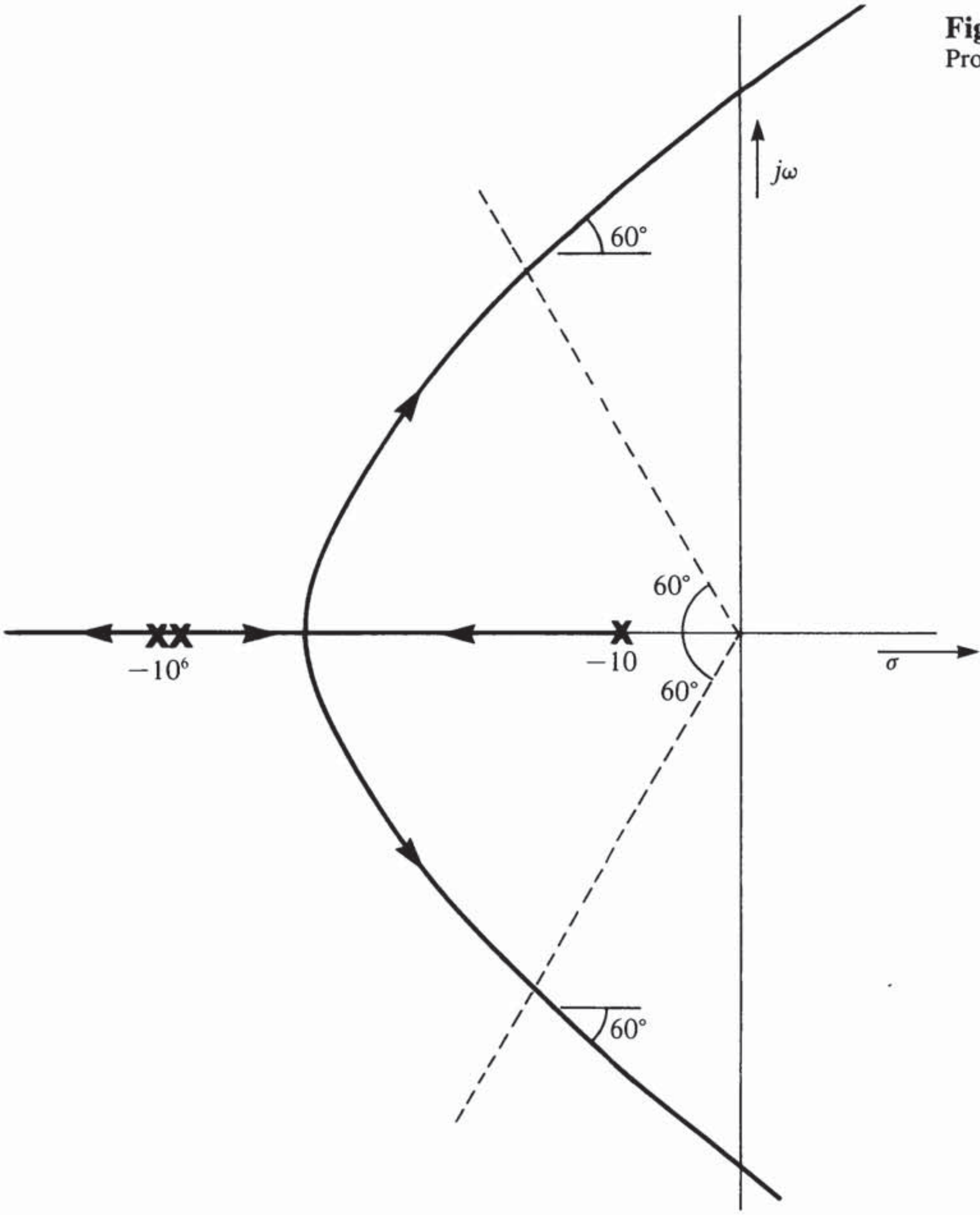


Figure S5.3 Root locus for Problem 5.2 (P4.7).

$$P(s) = (0.1s + 1)(10^{-6}s + 1)^2 + 10^6 f_o \simeq 10^{-13}s^3 + 2 \times 10^{-7}s^2 + 0.1s + 10^6 f_o \tag{S5.11}$$

where three insignificant terms have been dropped. Now, we multiply $P(s)$ through by 10^{13} and equate with $P'(s)$

$$s^3 + 2 \times 10^6 s^2 + 10^{12}s + 10^{19}f_o = s^3 + (\gamma + 2\beta)s^2 + 2\beta(\gamma + 2\beta)s + 4\gamma\beta^2 \tag{S5.12}$$

Equating the coefficients of s^2 yields

$$\gamma + 2\beta = 2 \times 10^6 \tag{S5.13}$$

Equating the coefficients of s, and substituting in from Equation S5.13 gives

$$2\beta(2 \times 10^6) = 10^{12} \tag{S5.14}$$

Thus,

$$\beta = 2.5 \times 10^5$$

and

$$\gamma = 1.5 \times 10^6 \tag{S5.15}$$

Then the remaining term is used to find that

$$f_o = 3.75 \times 10^{-2} \tag{S5.16}$$

With this value of f_o, the low-frequency loop transmission is large, and the low-frequency closed-loop gain is given by $\frac{1}{f_o}$ which is equal to 26.7. Resistor values to realize this gain can be chosen by setting

$$\frac{R_1 + R_2}{R_2} = 26.7 \tag{S5.17}$$

or

$$R_1 = 25.7R_2$$

More Root Locus 6

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 6.1 (P4.6)

For the first transfer function $a(s)$, the root locus is shown in Figure S6.1*a*.

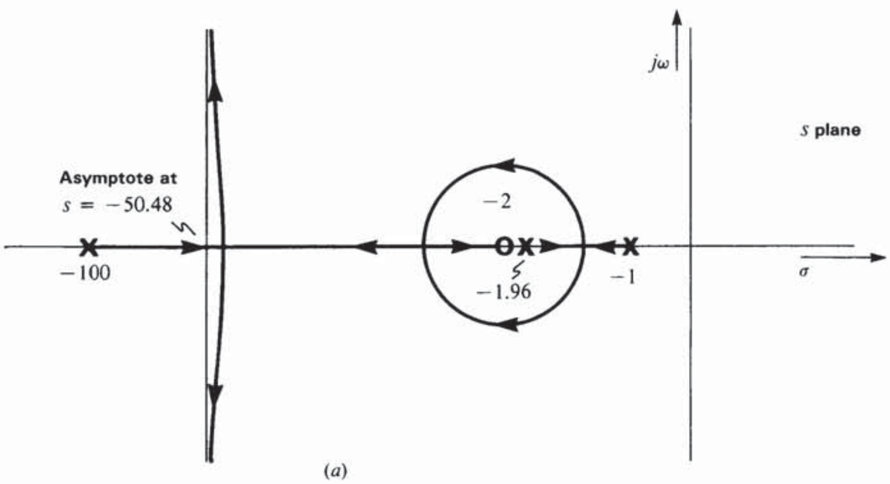


Figure S6.1 Root loci for Problem 6.1 (P4.6). (*a*) Root locus for $a(s) = \dfrac{a_o(0.5s + 1)}{(s + 1)(0.01s + 1)(0.51s + 1)}$.

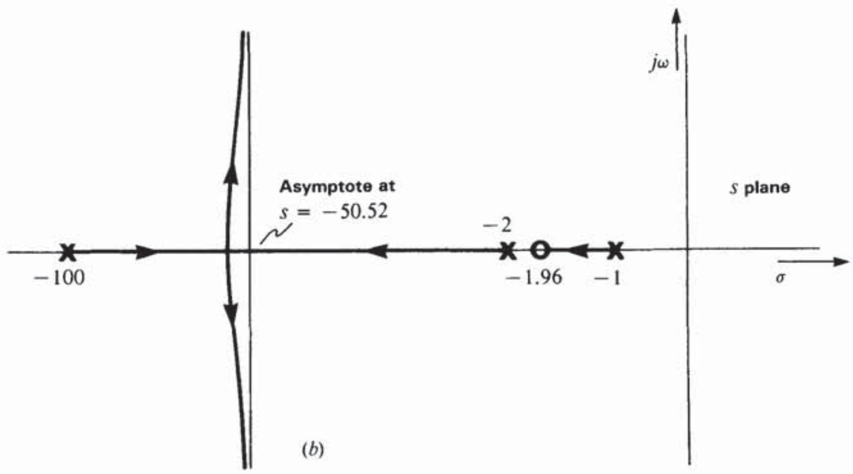
For moderate values of a_o, we evaluate the root locus by ignoring the pole at $s = -100$. Then, the locus is similar to the locus of Figure 4.8 in the textbook. The exact locations of the breakaway and reentry points are not necessary for this problem, and we do not solve for them. By Rule 5, two of the poles approach asymptotes of $\pm 90°$ from the real axis. These asymptotes intersect the real axis at

$$s = \frac{-100 - 1.96 - 1 + 2}{2} = -50.48 \tag{S6.1}$$

Considering that the third pole is moving to the right towards the zero at $s = -2$, the two poles must break away slightly to the right of the asymptotes, in order to satisfy Rule 4.

For the second transfer function $a'(s)$, the root locus is shown in Figure S6.1*b*.

Figure S6.1 (*b*) Root locus for $a'(s) = \dfrac{a_o(0.51s + 1)}{(s + 1)(0.01s + 1)(0.5s + 1)}$.



By Rule 5, two of the poles will approach asymptotes of $\pm 90°$ from the real axis. These asymptotes intersect the real axis at

$$s = \frac{-100 - 2 - 1 + 1.96}{2} = -50.52 \qquad \textbf{(S6.2)}$$

Because the third pole is moving to the left towards the zero at $s = -1.96$, the two poles must break away from the real axis slightly to the left of the asymptotes, in order to satisfy Rule 4.

The root locus for the third transfer function $a''(s)$ is shown in Figure S6.1*c*.

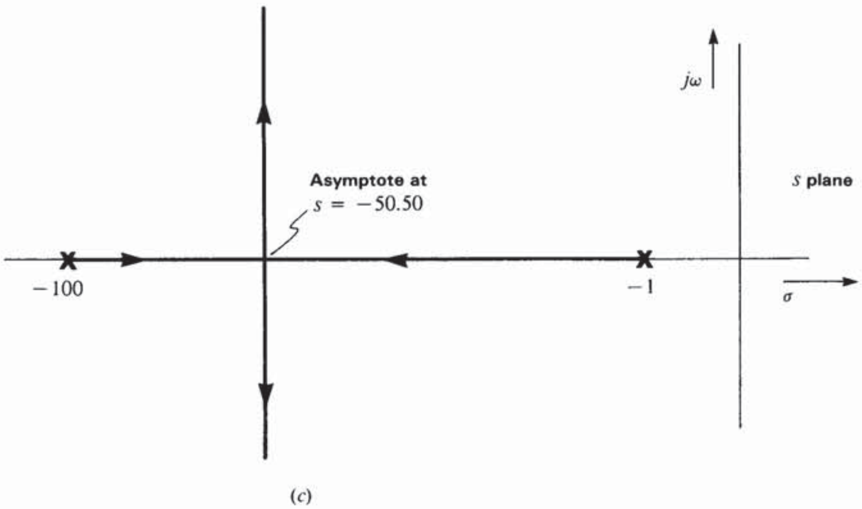


Figure S6.1 (c) Root locus for
$$a''(s) = \frac{a_o}{(s+1)(0.01s+1)}.$$

Again, the asymptotes are at $\pm 90°$, and intersect the real axis at

$$s = \frac{-100 - 1}{2} = -50.50 \qquad \textbf{(S6.3)}$$

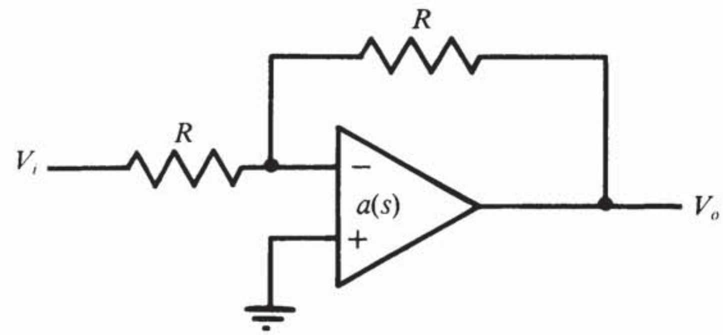
To satisfy Rule 4, the poles break away from the axis exactly on the asymptotes.

Inspection of the three root-locus diagrams indicates that the three systems will have very similar behavior for moderate to large values of a_o. This is true because the complex pairs approach nearly identical asymptotes in all three cases. Further, the low-frequency pole-zero doublets effectively cancel out. Thus, intuition is verified by the root-locus behavior.

Solution 6.2 (P4.8)

The unity-gain inverter connection is shown in Figure S6.2.

Figure S6.2 Unity-gain inverter.



The loop transmission for this connection is $-\frac{1}{2}a(s)$. Thus the characteristic equation of 1 minus the loop transmission is

$$1 + \tfrac{1}{2}a(s) = 0 \qquad \textbf{(S6.4)}$$

or, substituting in for $a(s)$:

$$1 + \frac{5 \times 10^4}{(\tau s + 1)(10^{-6}s + 1)} = 0 \qquad \textbf{(S6.5)}$$

Clearing fractions and multiplying terms gives

$$10^{-6}\tau s^2 + (\tau + 10^{-6})s + 1 + 5 \times 10^4 = 0 \qquad \textbf{(S6.6)}$$

Following Equation 4.75 in the textbook, we make the associations

$$p'(s) = s(10^{-6}s + 1) \qquad \textbf{(S6.7a)}$$

and

$$q'(s) \simeq 10^{-6}s + 5 \times 10^4 \qquad \textbf{(S6.7b)}$$

Then, the root contour method indicates that we should form a root locus with poles at the zeros of $q'(s)$ and zeros at the zeros of $p'(s)$. Thus, we have a pole at $s = -5 \times 10^{10}$, and zeros at $s = 0$, and $s = -10^6$. This configuration of singularities may seem strange, because it represents a physically impossible system having more zeros than poles in the finite s plane. Remember, however, that the zeros associated with the root contour technique are *not* the closed-loop zeros for the system under study. (The inverting connection in question here certainly doesn't have a zero at the origin.) The root contour does, however, accurately represent the location of the closed-loop poles as τ is varied.

We construct the root contour by recognizing that there is a pole at infinity that will move in from the left. Thus, the contour is as shown in Figure S6.3.

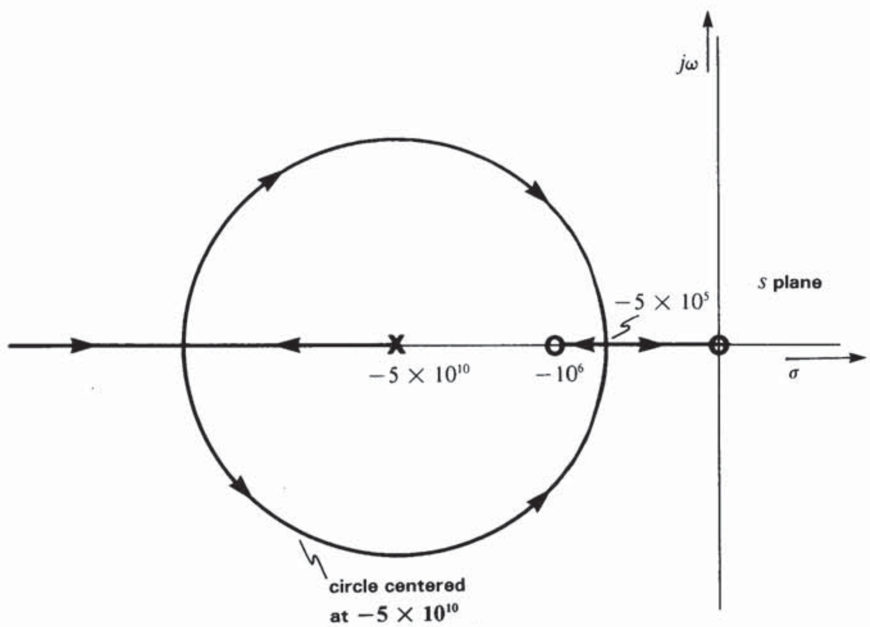


Figure S6.3 Root contour for Problem 6.2 (P4.8).

The angle condition imposes the geometric constraint that the branches circle the pole at -5×10^{10}. The entrance point on the real axis at $s = -5 \times 10^5$ is solved for by applying Rule 7 to the group of singularities near the origin (i.e., the zeros at $s = 0$ and $s = -10^6$). That is, the breakaway point found by considering only the zeros is at the point

$$s = \frac{-10^6 + 0}{2} = -5 \times 10^5 \qquad \textbf{(S6.8)}$$

For some people (including the author of this solution), this root contour will still seem contrary to common sense. What is perhaps a more intuitive solution may be found by making the substitution $\alpha = 1/\tau$. Then, the characteristic equation (after clearing fractions) is

$$s(10^{-6}s + 1) + \alpha(10^{-6}s + 5 \times 10^4) = 0 \qquad \textbf{(S6.9)}$$

This root contour has poles where the contour of Figure S6.3 has zeros and vice versa, which will look more natural to many readers. The location of the contour is identical in both cases.

Returning to the contour of Figure S6.3, we solve for the value of τ required to set $\zeta = 0.707$. In the vicinity of the origin, where the closed-loop poles have a damping ratio of 0.707, the root contour is well approximated by a vertical line through the point $s = -5 \times 10^5$, as shown in Figure S6.4.

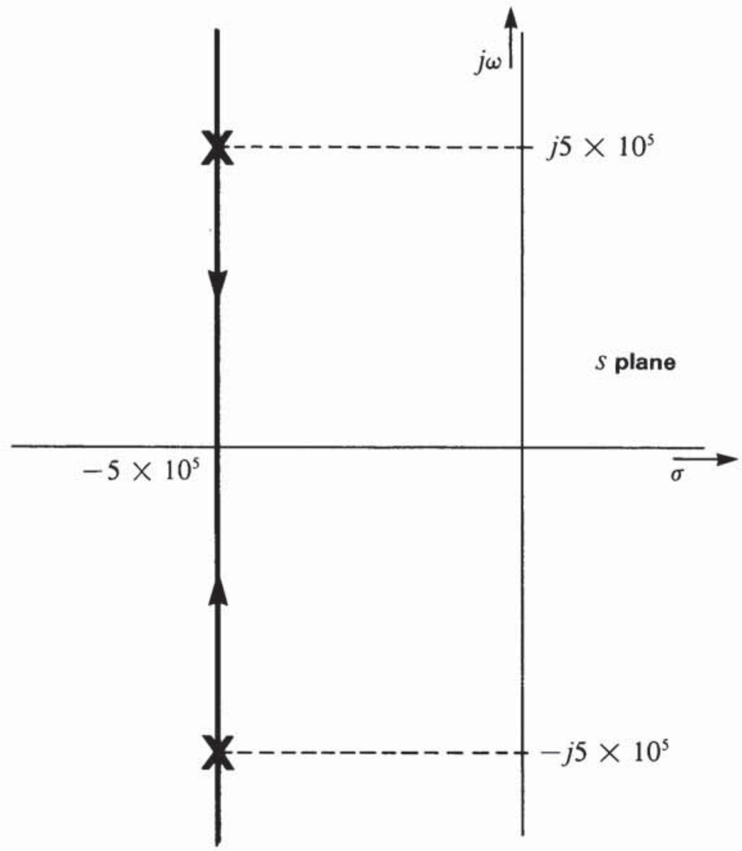


Figure S6.4 Root contour for Problem 6.2 (P4.8) in the vicinity of the origin.

Poles on this contour with a damping ratio of 0.707 will be at $s = -5 \times 10^5 \pm j5 \times 10^5$ as shown. Then, Rule 8 is used to solve for the value of τ which will result in this damping ratio. From Equation 4.56, the required value is

$$
\begin{aligned}
\tau &= \left|\frac{q'(s)}{p'(s)}\right|_{s=-5\times10^5(1+j)} \\
&= \left|\frac{10^{-6}s + 5\times10^4}{s(10^{-6}s+1)}\right|_{s=-5\times10^5(1+j)} \qquad \textbf{(S6.10)} \\
&= 0.1
\end{aligned}
$$

In closing, we note that this problem could be solved quite directly by putting Equation S6.6 into the standard form $\frac{s^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}s + 1 = 0$. Then, simply set $\zeta = 0.707$, and solve for τ. Such an approach verifies the results we have obtained via the root contour.

S O L U T I O N S

Stability via Frequency Response 7

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 7.1 (P4.9)

The time-delay term has a constant magnitude of 1, and a phase of -0.01ω radians. (It is a common mistake to use units of degrees here.) Thus a pure delay is equivalent to a negative phase shift that varies linearly with ω. Applying Equations 3.46 and 3.47 from the textbook gives

$$|L(j\omega)| = \frac{a_o}{\sqrt{\omega^2 + 1}} \tag{S7.1a}$$

and

$$\measuredangle L(j\omega) = -\tan^{-1}\omega - 0.01\omega \text{ radians} \tag{S7.1b}$$

These two expressions are used to sketch a Nyquist diagram as shown in Figure S7.1.

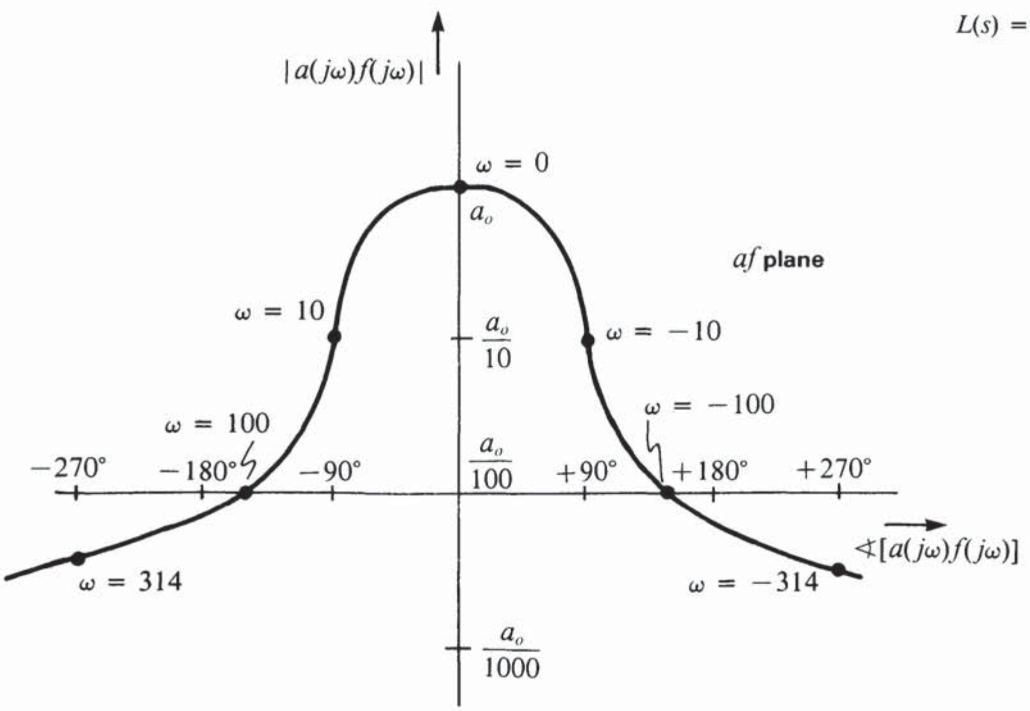


Figure S7.1 Nyquist diagram for $L(s) = -\dfrac{a_o e^{-0.01s}}{(s+1)}$.

Because we use degrees as the units for the phase axis, it is helpful to remember that there are 57.3 degrees per radian. As $\omega \rightarrow \infty$, the phase is unbounded. Thus, for a sufficiently large value of a_o, the $\pm 180°$ points will be enclosed in the contour, and the system will be unstable. The maximum value of a_o for stability is such that the $\pm 180°$ points are intersected by the af contour. Inspection of the Nyquist diagram indicates that this point will occur for $\omega > 100$. In this region, the magnitude and phase are well approximated by

$$|L(j\omega)| \simeq \frac{a_o}{\omega} \qquad \omega \gg 1 \tag{S7.2a}$$

and

$$\measuredangle L(j\omega) \simeq -\frac{\pi}{2} - 0.01\omega \qquad \omega \gg 1 \tag{S7.2b}$$

Applying Equation S7.2*b*, the frequency at which the phase is $-180°$ is:

$$-\pi = -\frac{\pi}{2} - 0.01\omega_1$$

or

$$\omega_1 = 157 \text{ rad/sec} \tag{S7.3}$$

Then, to intersect the $-180°$ point, we must have $|L(j\omega)|_{\omega=157} = 1$. Then, by Equation S7.2*a*, $a_o = 157$ is the maximum value that results in a stable system.

Because the feedback path is frequency independent, we may apply Equation 4.88 from the textbook to solve for the value of a_o, which results in $M_p = 1.4$.

$$1.4 \simeq \frac{1}{\sin \phi_m} \tag{S7.4}$$

Thus

$$\phi_m \simeq \sin^{-1}\left(\frac{1}{1.4}\right) \simeq 45° \tag{S7.5}$$

For a 45° phase margin, Equation S7.2*b* requires crossover at a frequency such that

$$-\frac{3\pi}{4} = -\frac{\pi}{2} - 0.01\omega$$

or

$$\omega \simeq 79 \text{ rad/sec} \tag{S7.6}$$

To have crossover at $\omega = 79$ rad/sec, Equation S7.2*a* requires that $a_o = 79$.

Solution 7.2 (P4.10)

Before drawing the Nyquist plot, it is helpful to draw a Bode plot for this system. Then, the Nyquist plot may be sketched directly from the Bode plot. Figure S7.2 is a Bode plot for the transfer function of interest

$$L(s) = -\frac{a_o s^3}{(s + 1)(0.1s + 1)^2} \qquad \textbf{(S7.7)}$$

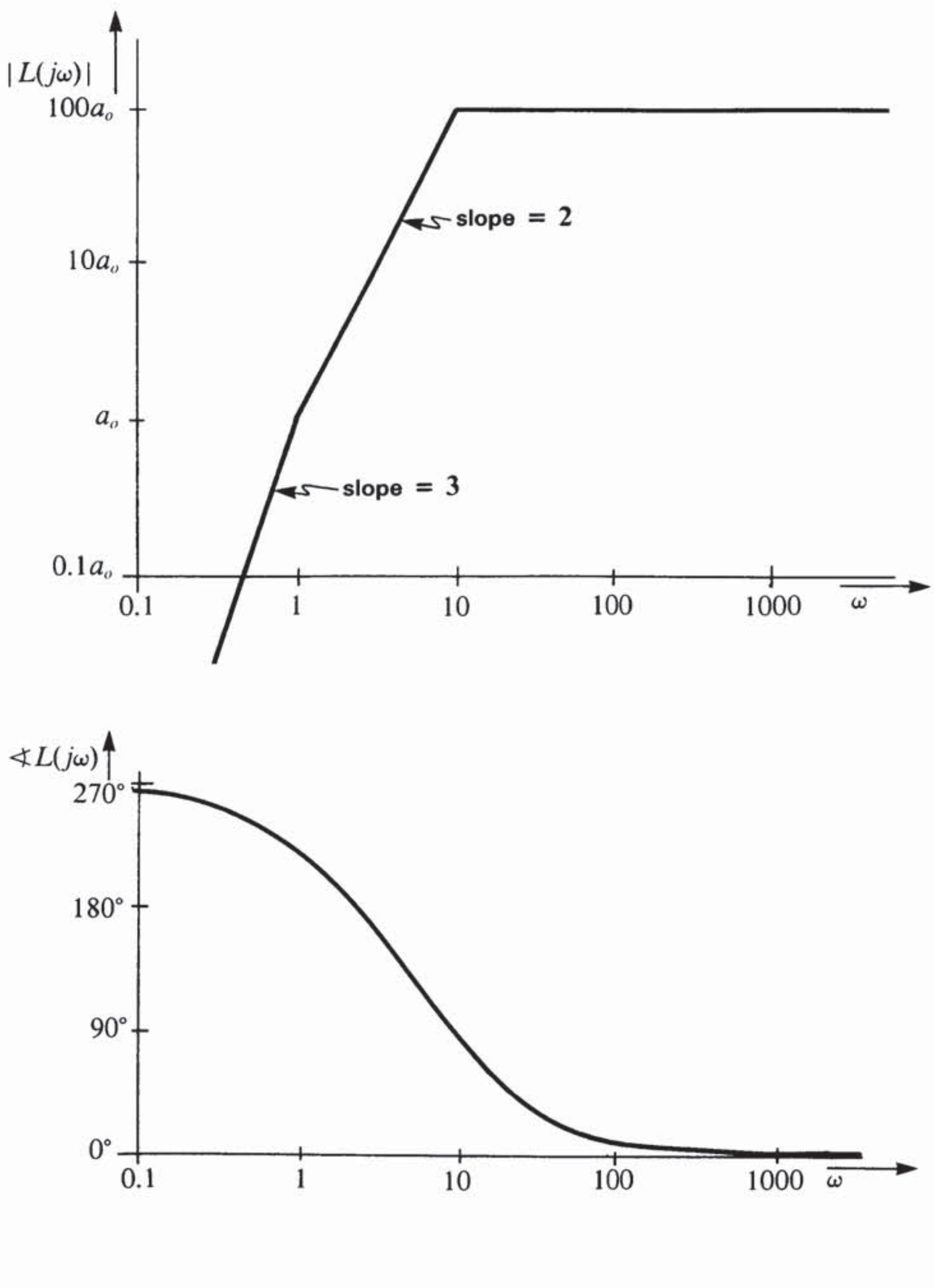
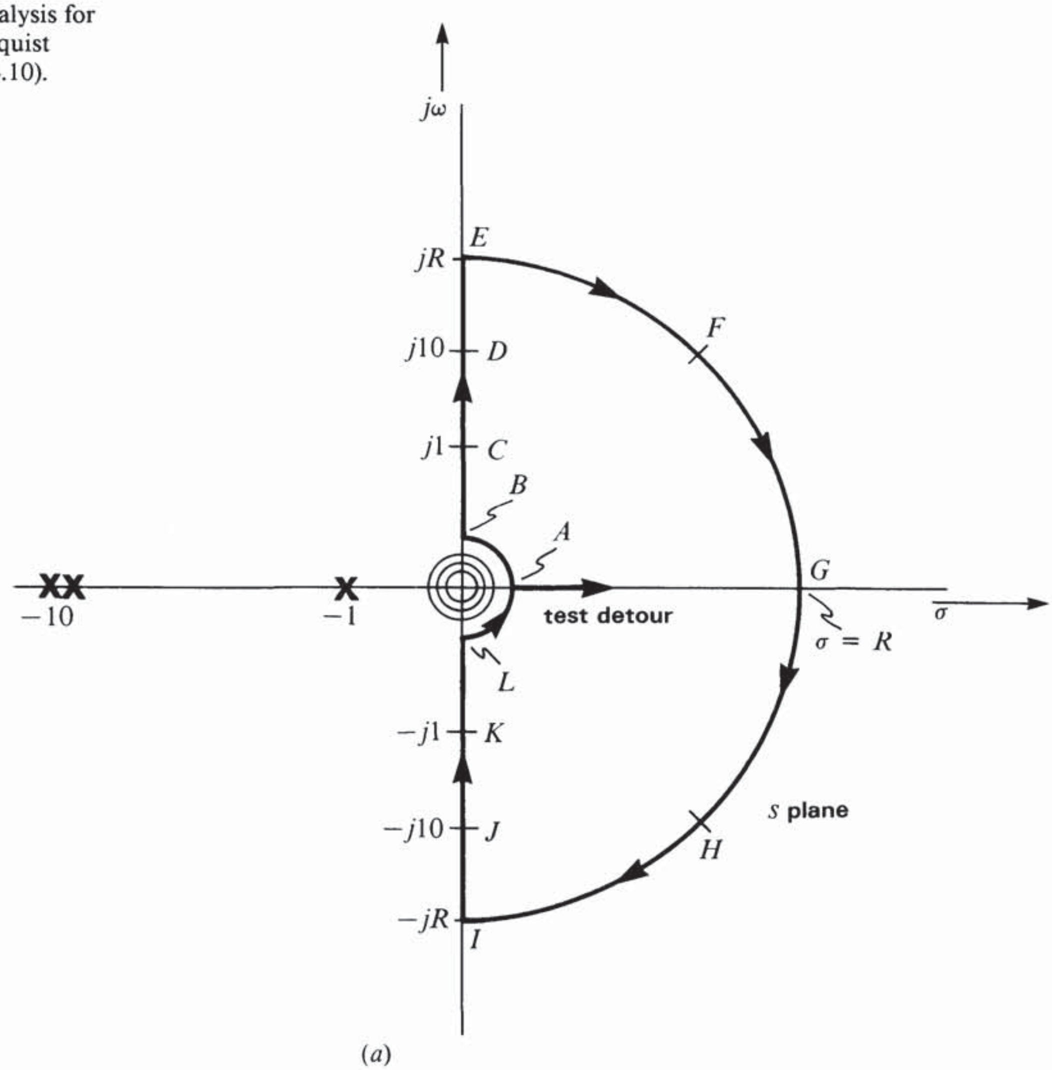


Figure S7.2 Bode plot for Problem 7.2 (P4.10).

Because there are singularities at the origin, we choose the contour shown in Figure S7.3*a*. The resulting Nyquist plot is as shown in Figure S7.3*b*. The points labeled *A* through *L* in the *s* plane map to the points equivalently labeled in the *af* plane. There are several important features to notice. For points near the origin in the *s* plane, the magnitude of $L(s)$ is very small. Thus, the point *A* in the *s* plane maps to the negative imaginary axis in the *af* plane as shown. For $|s| \gg 10$ (i.e., for points in the *s* plane far from the origin)

$$L(s) \simeq 100a_o \qquad \textbf{(S7.8)}$$

Figure S7.3 Nyquist analysis for Problem 7.2 (P4.10). (*a*) Nyquist contour for Problem 7.2 (P4.10).



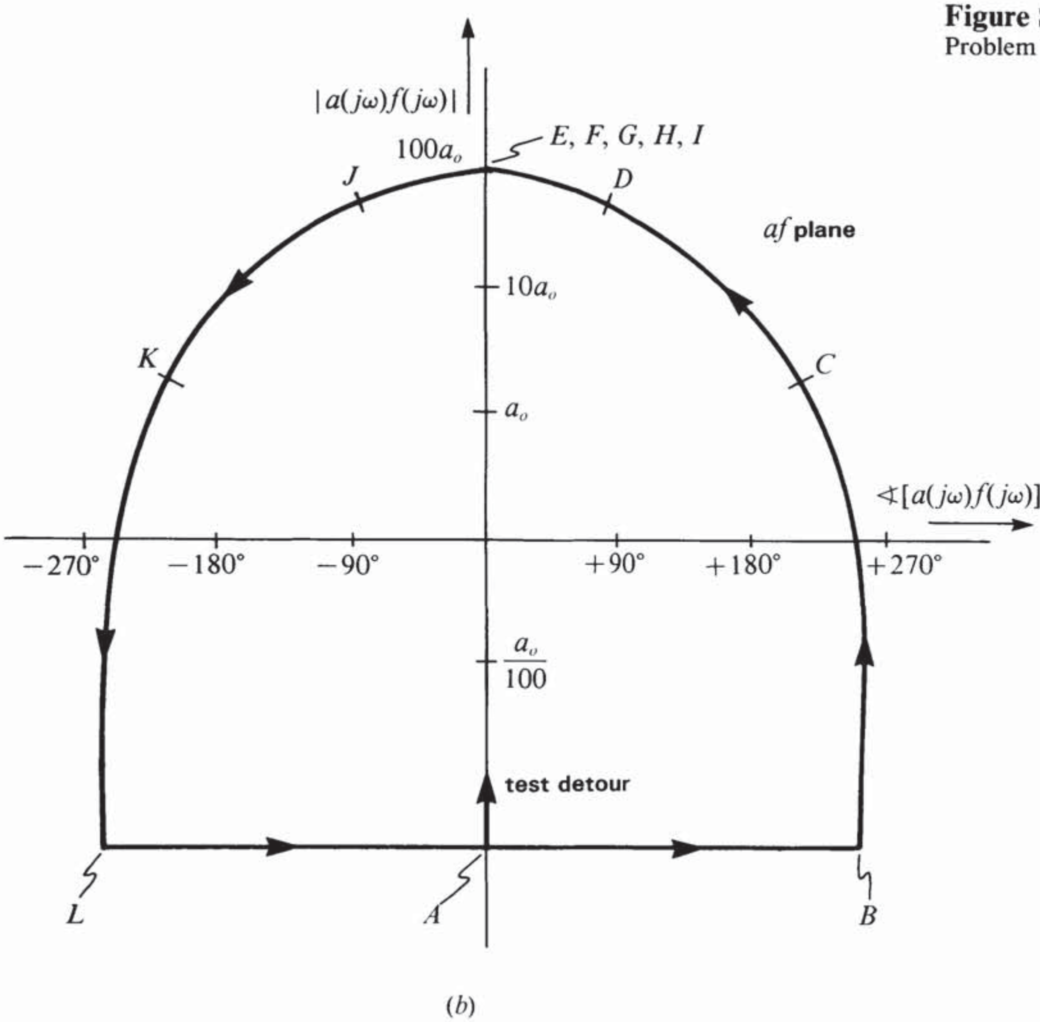


Figure S7.3 (*b*) Nyquist plot for Problem 7.2 (P4.10).

This is true all the way around the semicircle in the right half of the s plane. Thus, the points E, F, G, H, and I map to the point $af = 100a_o$ in the af plane. Finally, the test excursion shows that the interior of the contour in the s plane maps to the interior of the contour in the af plane. Clearly, for a large enough a_o, the points at $\pm 180°$ will be enclosed, and the system will be unstable.

The value of a_o required to reach the edge of instability can be solved for by finding the frequency at which $\measuredangle L(j\omega) = 180°$. Either directly from the Bode plot of Figure S7.2, or by iterating numerically on the expression for $\measuredangle L(j\omega)$

$$\measuredangle L(j\omega) = \frac{3\pi}{2} - \tan^{-1}\omega - 2\tan^{-1} 0.1\omega \qquad \textbf{(S7.9)}$$

we find that $\measuredangle L(j\omega) = 180°$ when $\omega = 2.2$ rad/sec. Then, the maximum a_o for which the system is stable is such that

$$|L(j\omega)| \Big|_{\omega=2.2} = 1 \qquad \textbf{(S7.10)}$$

Substituting in the expression for $|L(j\omega)|$ gives

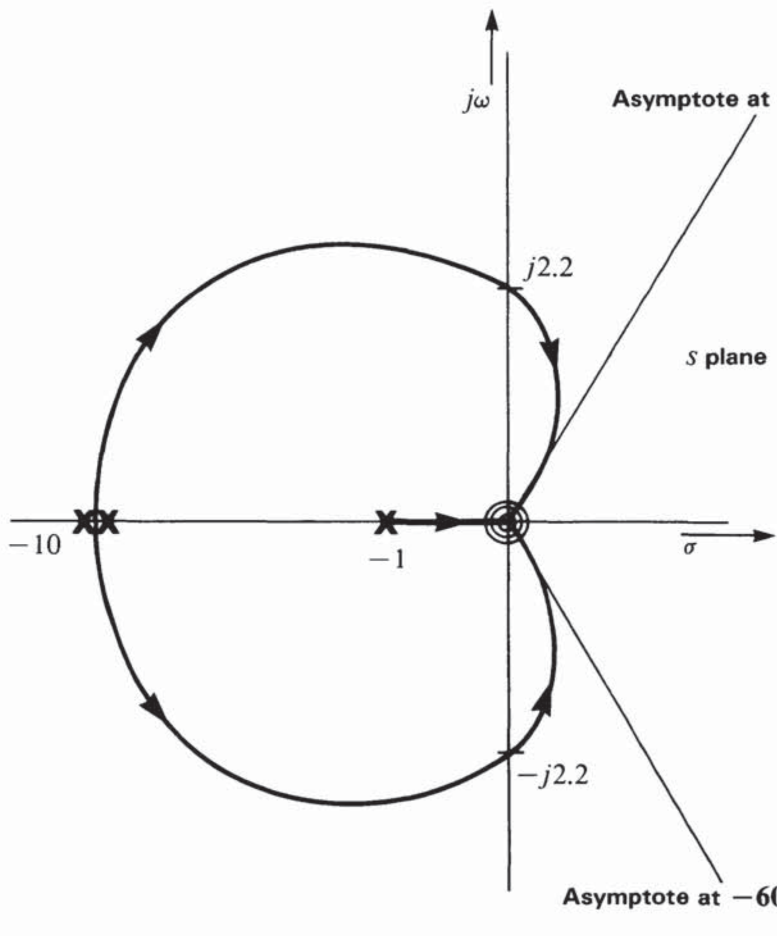
$$\frac{a_o\omega^3}{(\omega^2+1)^{1/2} \times ((0.1\omega)^2+1)} \Bigg|_{\omega=2.2} = 1 \qquad \textbf{(S7.11)}$$

or

$$a_o = \frac{((2.2)^2+1)^{1/2}((0.22)^2+1)}{2.2^3} = 0.24 \qquad \textbf{(S7.12)}$$

Thus, the system is stable for $a_o < 0.24$.

Figure S7.4 Root locus for Problem 7.2 (P4.10).



A root-locus construction also supports the conclusion that the system is unstable for large enough values of a_o, as shown in Figure S7.4. As previously calculated, the poles cross the imaginary axis at $\omega = 2.2$ and enter the right-half plane for $a_o > 0.24$. For large a_o, the two right-half-plane poles must approach the origin along asymptotes of $\pm 60°$, while the third pole approaches along the real axis. This must be so, because as the closed-loop poles approach the origin, the angle contribution from the pole at $s = -1$, and the two poles at $s = -10$, is essentially zero. Thus, the total angle from the three zeros to the closed-loop poles must be an odd multiple of 180°, which is satisfied by the asymptotes at $\pm 60°$.

Solution 7.3 (P4.11)

Poles that have a damping ratio of less than 0.707 lie to the right of a pair of lines at $\pm 45°$ from the negative real axis, because from Figure 3.7 of the textbook, $\theta = \cos^{-1}\zeta = \cos^{-1} 0.707 = 45°$. A contour that follows these lines, and encloses all poles with damping ratios less than 0.707 is shown in Figure S7.5.

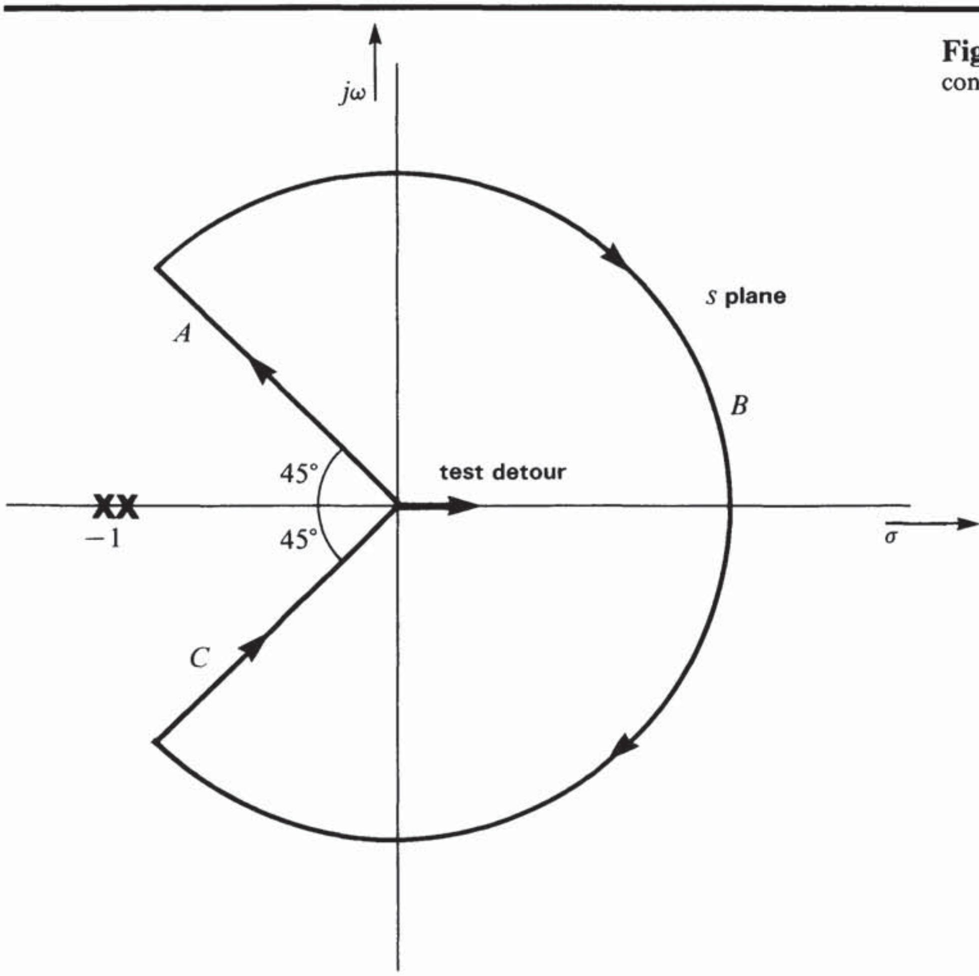


Figure S7.5 Modified Nyquist contour.

To make the modified Nyquist test we are interested in whether there are any poles within the contour of Figure S7.5, that is, whether there are any solutions of the characteristic equation $1 + a(s)f(s) = 0$ that occur for s within the contour of Figure S7.5. If there are such solutions, then the system has closed-loop poles with damping ratios less than 0.707. Thus the test in the af plane is unmodified. We look for points such that $a(s)f(s) = -1$ in exactly the same manner as the Nyquist test. Only the contour in the s plane needs to be changed to that shown in Figure S7.5. As in the Nyquist test, a test detour is used to determine where the interior of the contour in the s plane maps to in the af plane. The poles indicated at $s = -1$ are the poles of the transfer function

$$a(s)f(s) = \frac{a_o}{(s+1)^2} \tag{S7.13}$$

which we wish to evaluate using the modified Nyquist test. This test, then, is made by picking points in the s plane along the contour ABC, then plotting the value of $a(s)f(s)$ in the af plane for each of these points.

Applying this to the transfer function of Equation S7.13, at $s = 0$, $|a(s)f(s)| = a_o$, and $\measuredangle a(s)f(s) = 0°$. As $|s| \to \infty$ along contour A, $|a(s)f(s)| \to 0$ and $\measuredangle a(s)f(s) \to -270°$. Along the contour B, the magnitude of $a(s)f(s)$ remains small, and the angle changes from $-270°$ to $+270°$. The values of $a(s)f(s)$ resulting as the contour C is traversed are identical in magnitude and opposite in phase from the values generated along contour A. This preliminary analysis gives a general indication of the characteristics of the plot in the af plane. A more detailed analysis requires solving numerically. Along the contour A, $s = -\omega + j\omega$, thus

$$\begin{aligned} |a(s)f(s)| \Big|_{s=-\omega+j\omega} &= \frac{a_o}{|(-\omega + j\omega + 1)^2|} \\ &= \frac{a_o}{2\omega^2 - 2\omega + 1} \end{aligned} \tag{S7.14}$$

and

$$\begin{aligned} \measuredangle a(s)f(s) \Big|_{s=-\omega+j\omega} &= \measuredangle \frac{a_o}{(-\omega + j\omega + 1)^2} \\ &= -2\tan^{-1}\frac{\omega}{1-\omega} \end{aligned} \tag{S7.15}$$

The magnitude and angle of $a(s)f(s)$ can then be solved numerically as s takes on the values $s = -\omega + j\omega$, and ω is allowed to vary. (A programmable calculator is quite helpful here, as it is throughout the subject.) When solving for the angle, be careful because the arc tangent is not a single-valued function. The earlier preliminary analysis serves as a check on the numerical results. Some values are summarized in Table S7.1.

Table S7.1 Magnitude and angle of $\frac{a_o}{(s+1)^2}$, evaluated along the contour $s = -\omega + j\omega$.

ω	$\lvert a(-\omega + j\omega)f(-\omega + j\omega)\rvert$	$\sphericalangle a(-\omega + j\omega)f(-\omega + j\omega)$
0	$1.00a_o$	$0°$
0.01	$1.02a_o$	$-1.1°$
0.05	$1.10a_o$	$-6.0°$
0.10	$1.22a_o$	$-12.7°$
0.25	$1.60a_o$	$-36.9°$
0.50	$2.00a_o$	$-90.0°$
0.75	$1.60a_o$	$-143°$
1.00	$1.00a_o$	$-180°$
1.25	$0.62a_o$	$-203°$
1.50	$0.40a_o$	$-217°$
1.75	$0.28a_o$	$-226°$
2.50	$0.12a_o$	$-242°$
5.00	$0.02a_o$	$-257°$
10.00	$0.006a_o$	$-264°$
$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow -270°$

Using these values, the contour of Figure S7.6 is then drawn in the *af* plane. The test detour indicates that the interior of the contour in the *s* plane maps to the interior of the contour in the *af* plane. Then, there are closed-loop poles with a damping ratio of less than 0.707 when the *af* plot of Figure S7.6 encloses the points at unity magnitude and an angle of $\pm 180°$. There is a pair of poles with a damping ratio of exactly 0.707 when the *af* contour intersects the -1 point. From the numerical values of Table S7.1, or by examining Figure S7.6, this occurs for $a_o = 1$.

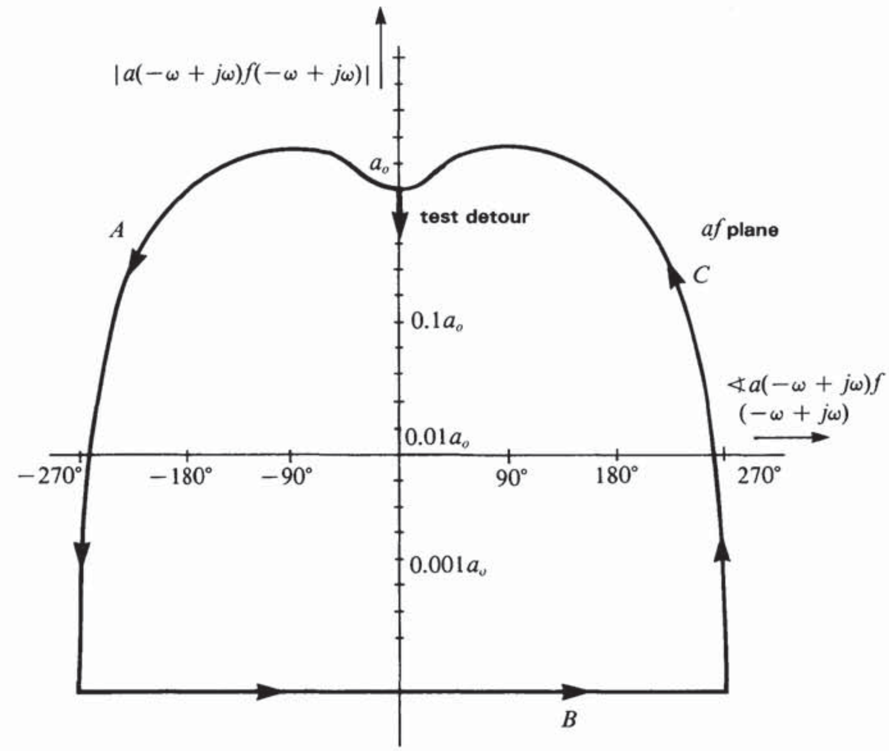


Figure S7.6 Modified Nyquist diagram for Problem 7.3 (P4.11).

We check this result by factoring the characteristic equation for $a_o = 1$. With $a_o = 1$, the characteristic equation is

$$1 + \frac{1}{(s+1)^2} = 0 \qquad \textbf{(S7.16)}$$

After clearing fractions, we have

$$s^2 + 2s + 2 = 0 \qquad \textbf{(S7.17)}$$

which has roots at

$$s = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j \qquad \textbf{(S7.18)}$$

These roots lie on lines at $\pm 45°$ from the negative real axis. Thus, as predicted, they have a damping ratio of 0.707.

Compensation 8

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 8.1 (P4.13)

As suggested in Lecture 8, to perform a Nyquist analysis, we first sketch the Bode plot. The transfer function of interest is the *af* product given by

$$a(s)f(s) = \frac{10^6(0.01s + 1)^2}{(s + 1)^3} \times f_o \tag{S8.1}$$

Using the methods of Section 3.4 of the textbook, the Bode plot is sketched in Figure S8.1. From this Bode plot, a gain-phase Nyquist plot is generated in Figure S8.2. From this figure, it is apparent that for some range of intermediate values of f_o, the -1 points will be enclosed within the contour, and the system will be unstable. However, for small enough *or* large enough values of f_o, the system will be stable. For instance, from Figure S8.2, the system is stable if $f_o = 1$, and it is certainly stable for any $f_o > 1$.

This same result can be obtained from a root-locus construction as shown in Figure S8.3. Because the two zeros are a factor of 100 farther from the origin than the three poles, the root-locus branches will initially follow asymptotes of $\pm 60°$ and $180°$ from the real axis, by Rules 7 and 5. The two branches that leave the real axis at $\pm 60°$ will enter the right half of the s plane at about $\omega = 1.7$ for large enough values of f_o. However, these two branches must rejoin the negative real axis at a point to the left of the two poles at $s = -100$, by Rules 2 and 3. Thus the branches cross back into the left half of the s plane and the system is stable for sufficiently large f_o. (Because only a qualitative analysis is required, the exact point at which the branches reenter the negative real axis will not be solved for.) Thus, both the Nyquist and root-locus analyses indicate that the system is stable for small values of f_o, unstable for intermediate values of f_o, and stable for large f_o.

Now, we can use a Routh analysis to determine the values of f_o that separate these regions of stability and instability. The characteristic equation is

$$1 + a(s)f_o = 1 + 10^6 f_o \frac{(0.01s + 1)^2}{(s + 1)^3} = 0 \tag{S8.2}$$

After clearing fractions and collecting terms, we have

Figure S8.1 Bode plot for Problem 8.1 (P4.13).

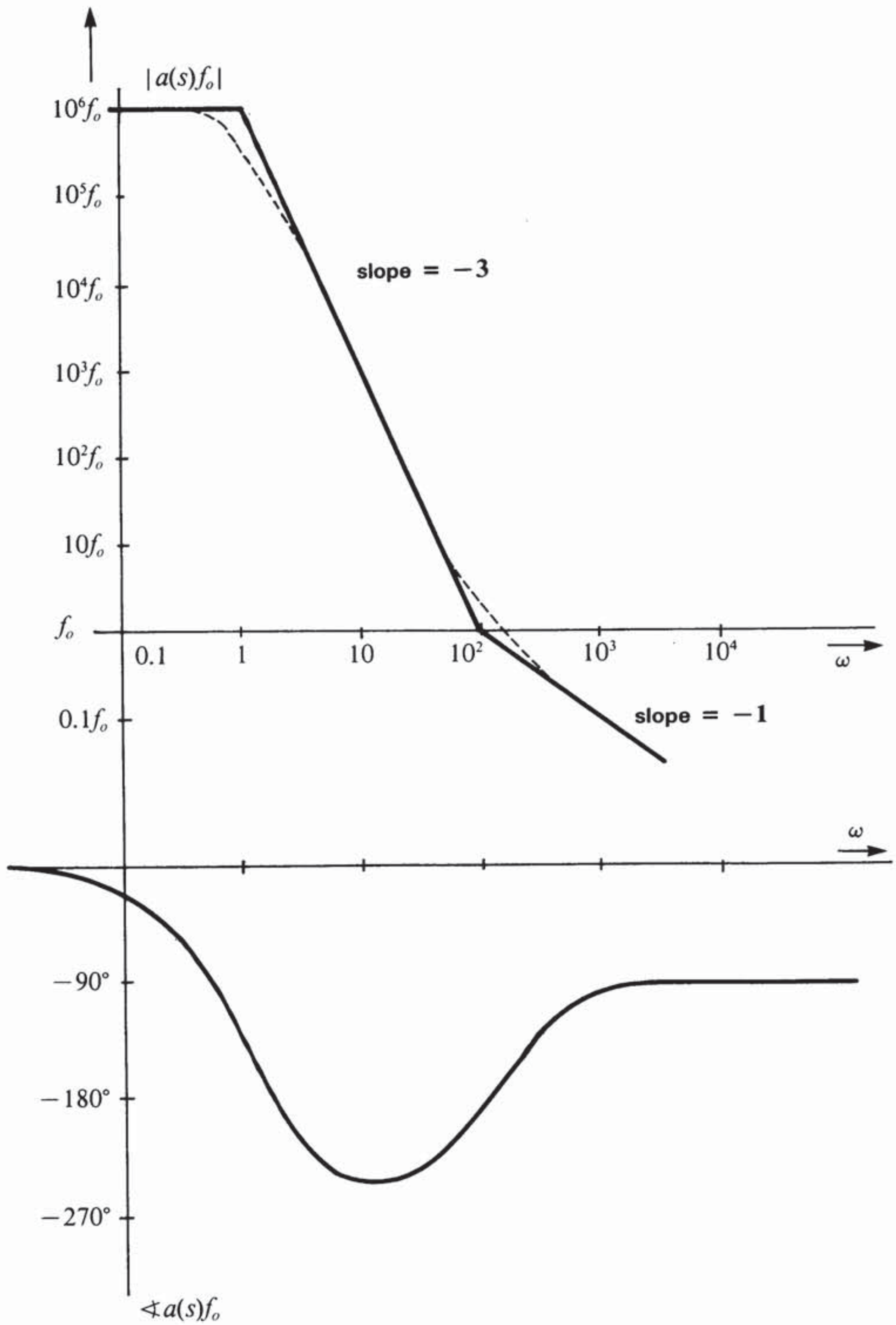


Figure S8.2 Nyquist plot for Problem 8.1 (P4.13).

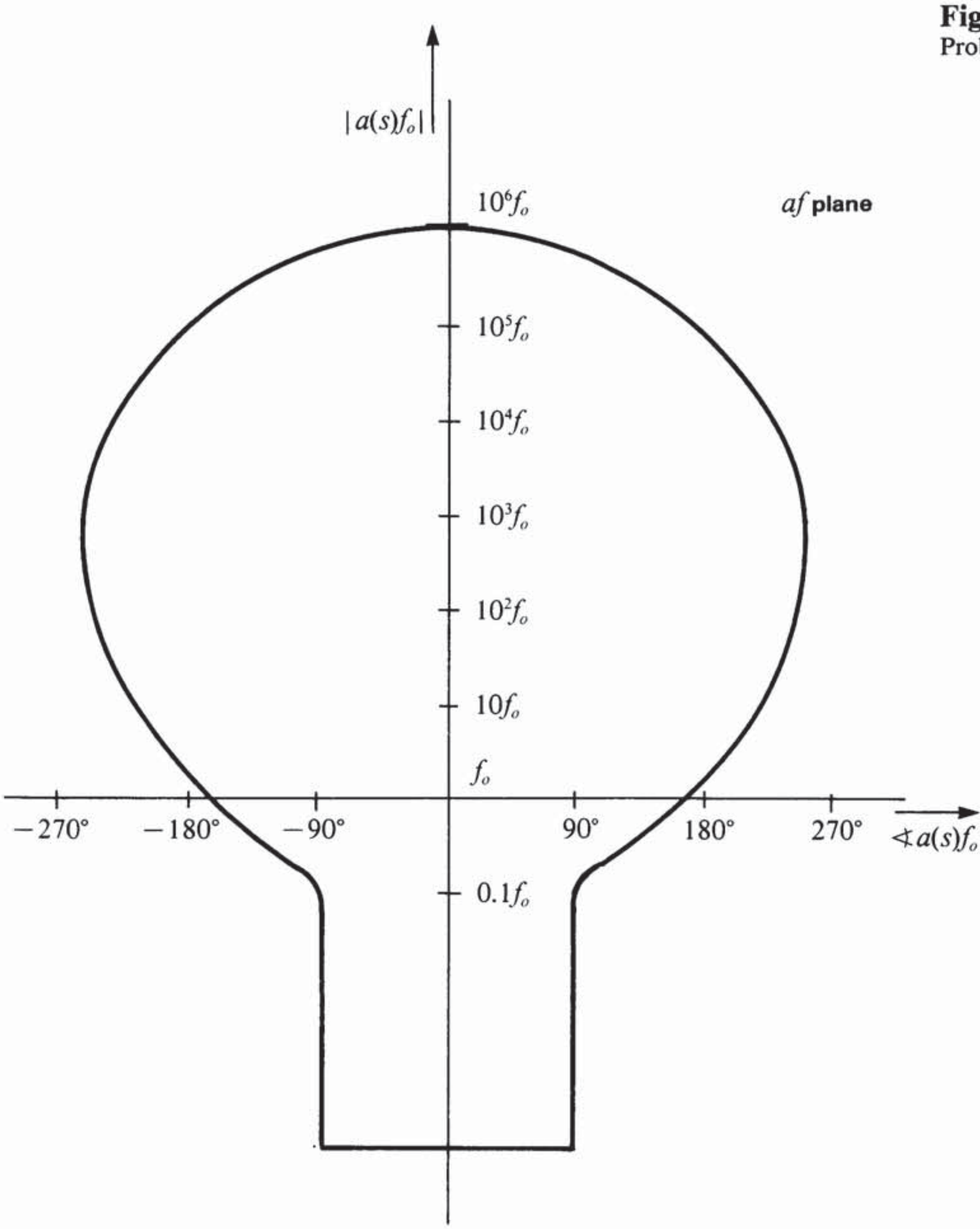
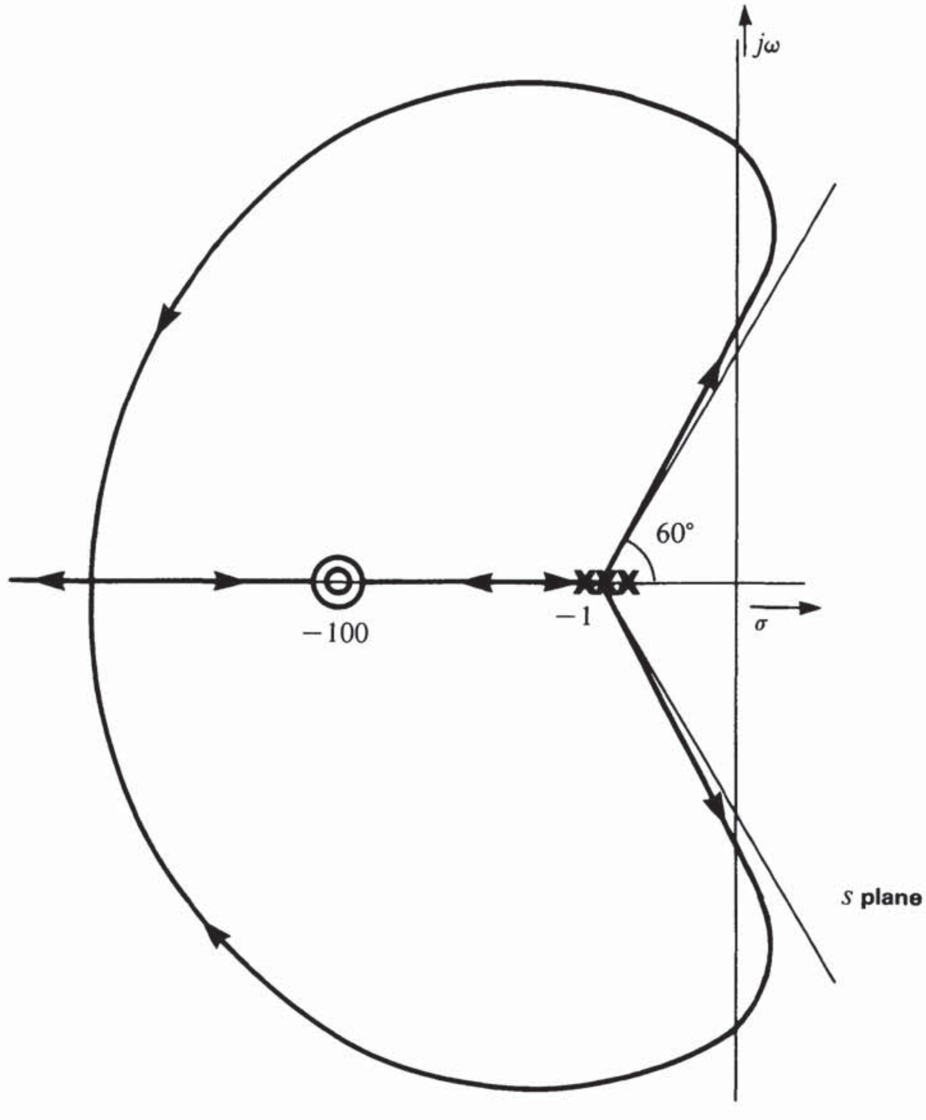


Figure S8.3 Root locus for Problem 8.1 (P4.13).



$$s^3 + (3 + 10^2 f_o)s^2 + (3 + 2 \times 10^4 f_o)s + 1 + 10^6 f_o = 0 \quad \textbf{(S8.3)}$$

From the polynomial, the Routh array is constructed as

$$\begin{array}{cc}
1 & 3 + 2 \times 10^4 f_o \\[1ex]
3 + 10^2 f_o & 1 + 10^6 f_o \\[1ex]
\dfrac{2 \times 10^6 f_o^2 - 0.94 \times 10^6 f_o + 8}{3 + 10^2 f_o} & 0 \\[2ex]
1 + 10^6 f_o & 0
\end{array} \quad \textbf{(S8.4)}$$

The third row becomes zero (indicating poles on the imaginary axis) when

$$2 \times 10^6 f_o^2 - 0.94 \times 10^6 f_o + 8 = 0 \quad \textbf{(S8.5)}$$

which is solved by

$$f_o = \frac{0.94 \times 10^6 \pm \sqrt{(0.94 \times 10^6)^2 - 64 \times 10^6}}{4 \times 10^6}$$

$$= 0.2350000 \pm 0.2349915 \quad \textbf{(S8.6)}$$

$$= 8.5 \times 10^{-6}, 0.47 \quad \textbf{(S8.7)}$$

Note that high numerical precision is required to extract the root at $f_o = 8.5 \times 10^{-6}$. This problem could be avoided by framing the Routh calculation in terms of $a_o f_o$. However, a scientific calculator can carry out this calculation with sufficient accuracy. We carry this precision only where necessary, and round the answers of Equation S8.7 to two significant figures. The third row is negative for

$$8.5 \times 10^{-6} < f_o < 0.47 \quad \textbf{(S8.8)}$$

Thus, the system is stable for

$$f_o < 8.5 \times 10^{-6}$$

and

$$f_o > 0.47 \quad \textbf{(S8.9)}$$

which are the two borderline values the problem statement asks for.

Solution 8.2 (P5.1)

The circuit of Figure S8.4 provides an ideal gain of -10, and allows lowering of the loop-transmission magnitude.

The block diagram for this connection is as in Figure S8.5.

Figure S8.4 Connection with gain of -10, which allows lowering of the loop-transmission magnitude.

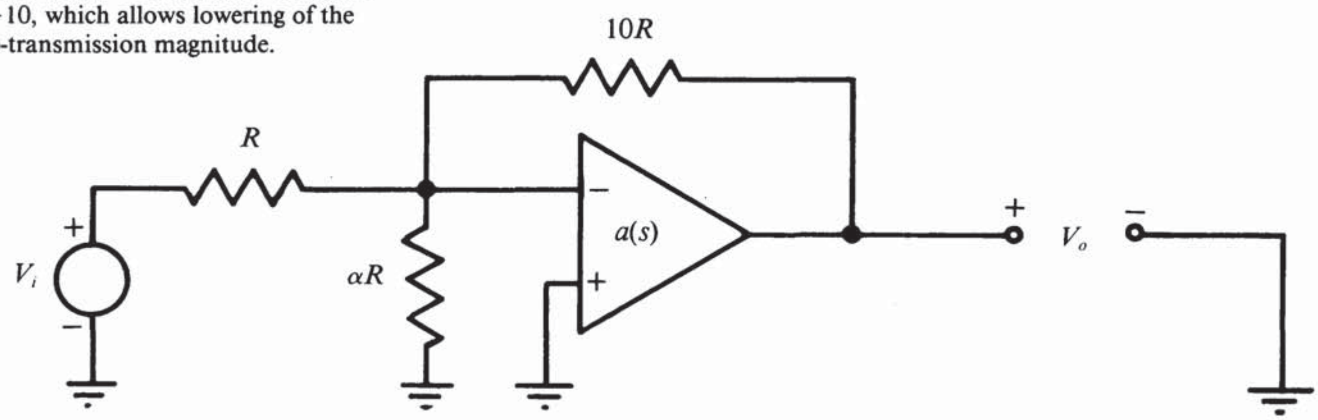
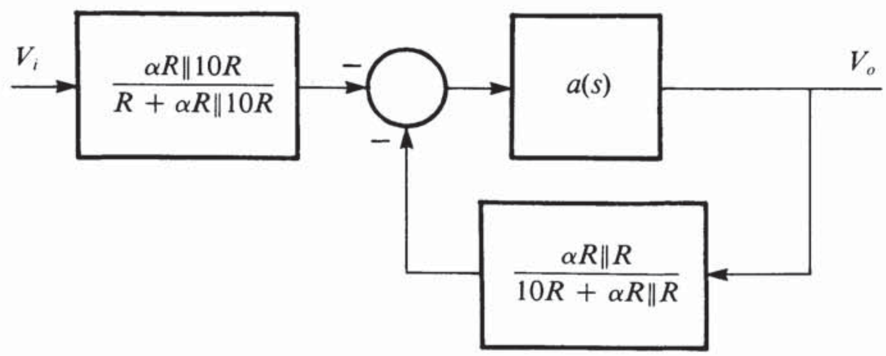


Figure S8.5 Block diagram for circuit of Figure S8.4.



The value of R cancels out of both blocks in which it appears. After algebraically reducing the expressions in these blocks, we have the diagram of Figure S8.6.

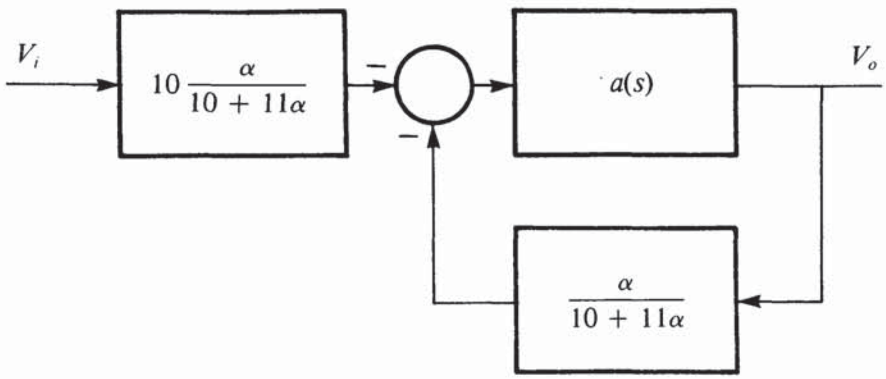


Figure S8.6 Reduced block diagram for Problem 8.2 (P5.1).

This can be further reduced as shown in Figure S8.7.

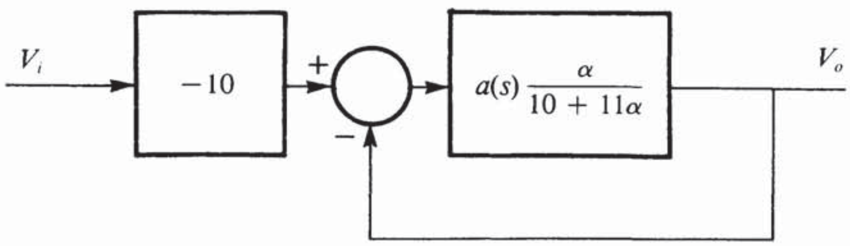


Figure S8.7 Reduced block diagram for Problem 8.2 (P5.1).

From the form of the block diagram, this system has an ideal gain of -10 as stated earlier. The negative of the loop transmission is

$$\begin{aligned} -\text{L.T.} &= a(s) \frac{\alpha}{10 + 11\alpha} \\ &= \frac{2 \times 10^5}{(0.1s + 1)(10^{-5}s + 1)^2} \times \frac{\alpha}{10 + 11\alpha} \end{aligned} \qquad \textbf{(S8.10)}$$

From Figure 4.26*b*, the required damping ratio for $P_o = 1.1$ is approximately 0.6. From Figure 4.26*a*, this implies a phase margin of about 58°. That is, the loop-transmission phase must be $-122°$ at the crossover frequency ω_c. The form of $a(s)$ allows us to readily solve for this frequency, because at frequencies where the two poles at $s = -10^5$ are contributing any significant phase shift, the pole at $s = -10$ is contributing $-90°$ of phase. Thus, at ω_c, the phase due to the two poles must be $-32°$. Applying Equation 3.47 from the textbook, we can write

$$-32° = -2 \tan^{-1} 10^{-5}\omega_c \qquad \textbf{(S8.11)}$$

which is solved for ω_c as

$$\omega_c = 10^5 \tan 16° = 2.87 \times 10^4 \text{ rad/sec} \tag{S8.12}$$

Now, we need to pick α to set the loop-transmission magnitude equal to unity at this frequency. Applying Equation 3.46 to the three poles gives

$$1 = \frac{2 \times 10^5}{\sqrt{0.01\omega_c^2 + 1}\,(10^{-10}\omega_c^2 + 1)} \times \frac{\alpha}{10 + 11\alpha} \tag{S8.13}$$

Substituting in $\omega_c = 2.87 \times 10^4$ gives

$$1 = 64.4 \frac{\alpha}{10 + 11\alpha} \tag{S8.14}$$

which is solved by

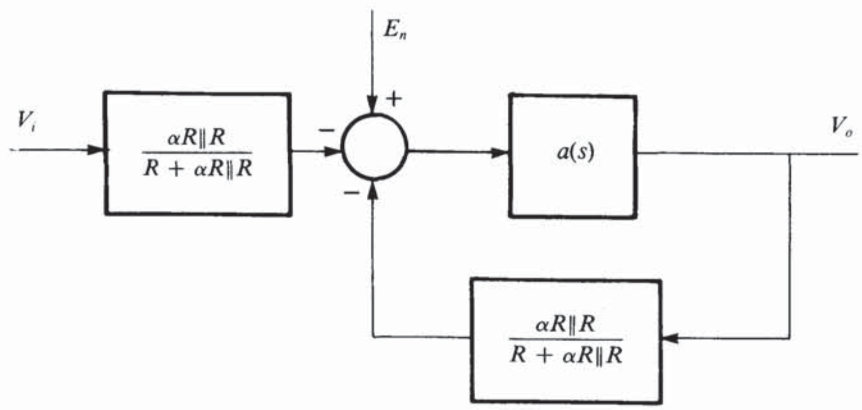
$$\alpha \simeq 0.19 \tag{S8.15}$$

Thus, the value of the attenuation resistor is $0.19R$.

Solution 8.3 (P5.2)

This is the same topology as in Problem 8.2 (P5.1). The only difference is that the noise voltage E_n adds directly to the error signal, and the ideal gain is -1 rather than -10. The appropriate modifications to the block diagram of Figure S8.5 give the block diagram of Figure S8.8, which represents the connection of Figure 5.23*a*.

Figure S8.8 Block diagram for circuit of Figure 5.23*a*.



By a block-diagram manipulation, Figure S8.8 reduces to Figure S8.9.

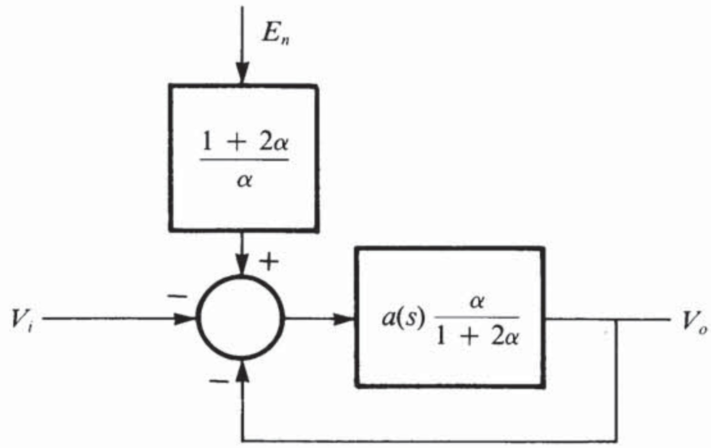


Figure S8.9 Reduced block diagram.

From Figure S8.9, at frequencies where $\left| a(s)\,\frac{\alpha}{1 + 2\alpha} \right| \gg 1$

$$\frac{V_o(s)}{E_n(s)} \simeq \frac{1 + 2\alpha}{\alpha} \qquad \textbf{(S8.16)}$$

For $\alpha \gg 1$, this ratio is about 2. For $\alpha \ll 1$, the ratio $\frac{V_o(s)}{E_n(s)}$ becomes very large, verifying the assertion at the end of Section 5.2.1 that this type of attenuation increases voltage noise at the amplifier output.

S O L U T I O N S

More Compensation 9

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 9.1 (P5.3)

Because the operational amplifier is ideal, having infinite open-loop gain and contributing no dynamics, its connection forms an ideal integrator. Further, due to the infinite open-loop gain, the inverting input terminal is forced to ground potential, and thus the 10 kΩ input resistor has no effect on the system outer-loop transmission. Thus, the integrator connection contributes a term of $\frac{-1}{RCs}$ (where $R = 10\ \text{k}\Omega$) to the outer-loop transmission. Then, the negative of the outer-loop transmission is given by

$$-\text{L.T.} = \frac{1}{RCs} \times \frac{1}{(10^{-6}s + 1)(10^{-7}s + 1)} \qquad \textbf{(S9.1)}$$

To set the phase margin to 45°, this term must have a phase angle of $-135°$ at the loop-transmission crossover frequency. The integrator contributes $-90°$ at all frequencies. Thus, the crossover frequency, ω_c, must be such that

$$\measuredangle \left[\frac{1}{(10^{-6}s + 1)(10^{-7}s + 1)} \right] \bigg|_{s=j\omega_c} = -45° \qquad \textbf{(S9.2)}$$

A rough estimate of ω_c is 10^6 rad/sec, because at this frequency the pole at 10^6 rad/sec contributes $-45°$, while the pole at 10^7 rad/sec contributes little phase shift. This estimate may be refined by recognizing that the pole at 10^7 rad/sec will contribute $-5°$ of phase when ω is about 10^6 rad/sec (i.e., a decade below the upper pole). Thus, we look for the frequency where the pole at 10^6 rad/sec contributes $-40°$ of phase. That is,

$$\measuredangle \left(\frac{1}{10^{-6}s + 1} \right) \bigg|_{s=j\omega_c} = -40° \qquad \textbf{(S9.3)}$$

or

$$\tan^{-1} 10^{-6}\omega_c = 40° \qquad \textbf{(S9.4)}$$

which is solved by $\omega_c = 8.4 \times 10^5$ rad/sec. For a more exact answer, we can program the expression for the phase angle of Equation S9.2, and search numerically to find that the angle is $-45°$ at $\omega_c = 8.45 \times 10^5$ rad/sec, verifying the accuracy of the approximation of Equation S9.4. The ability to make useful approximations is important in this subject, both for efficiency of computation and for understanding the impact of individual terms on system performance. For instance, here the pole at 10^7 rad/sec has only a minor effect on system behavior. It is the pole at 10^6 rad/sec that is most significant.

Now, we pick the value of C to set crossover at 8.4×10^5 rad/sec. At this frequency, the poles at 10^6 and 10^7 rad/sec contribute a magnitude of 0.76. Thus, to set the crossover point, we must have

$$\frac{1}{RC\omega_c} \times 0.76 = 1 \tag{S9.5}$$

or

$$\begin{aligned} C = \frac{1}{R\omega_c} \times 0.76 &= \frac{0.76}{10^4 \times 8.4 \times 10^5} \\ &= 91 \text{ pF} \end{aligned} \tag{S9.6}$$

Solution 9.2 (P5.4)

(a) We begin by drawing the block diagram for the motor. Let the motor torque be T_m. Then, from the model of Figure 5.25*b*, we have

$$T_m = 0.1\, I_a = 0.1\, \frac{(V_a - 0.1\Omega_s)}{1} \tag{S9.7}$$

where V_a is the voltage at the output of the operational amplifier. Recall from physics that angular velocity is related to torque by integration, through the rotational equivalent of Newton's $F = ma$. That is,

$$\Omega_s = \frac{1}{J_L} \int T_\ell dt \tag{S9.8}$$

where T_ℓ is the torque applied to the load inertia. In the Laplace domain, this becomes

$$\Omega_s(s) = \frac{T_\ell}{J_L s} \tag{S9.9}$$

Now, because T_ℓ is the net torque applied to the load, it is the sum of the motor torque and the disturbance torque. That is,

$$T_\ell = T_m + T_d \tag{S9.10}$$

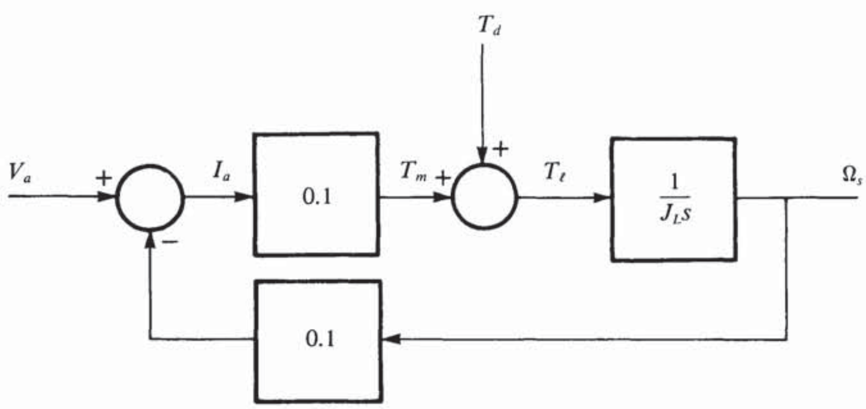


Figure S9.1 Block diagram for motor and load.

Combining Equations S9.7, S9.9, and S9.10, we draw the block diagram for the motor and load as shown in Figure S9.1.

Now, the remaining portion of the loop including the operational amplifier is shown in Figure S9.2.

Because the op amp is ideal, the inverting terminal is held by feedback at ground potential. The current through the series RC is then $\frac{(V_i + 0.01\Omega_s)}{10^5}$, and the output V_a is given by

$$V_a(s) = -\frac{(V_i + 0.01\Omega_s)}{10^5} \times \left(R + \frac{1}{Cs}\right) \tag{S9.11}$$

or

$$V_a(s) = -(V_i + 0.01\Omega_s)\left(\frac{RCs + 1}{10^5 Cs}\right) \tag{S9.12}$$

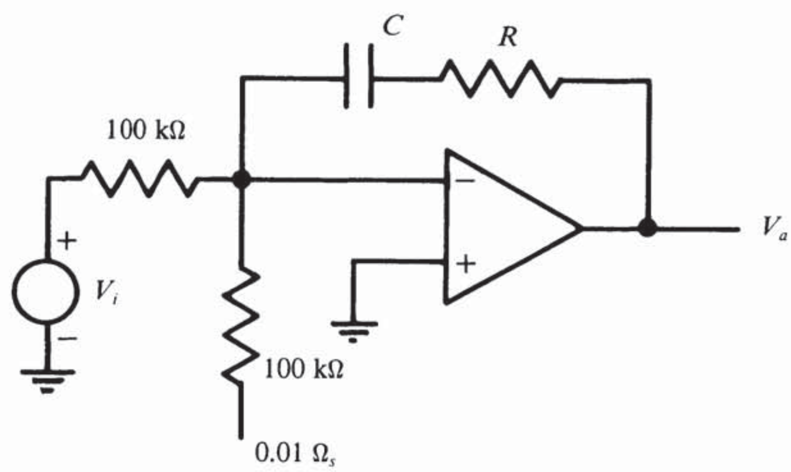


Figure S9.2 Amplifier connection from Figure 5.25.

The block-diagram representation of Equation S9.12 is added to the block diagram of Figure S9.1 to give the complete system block diagram as shown in Figure S9.3.

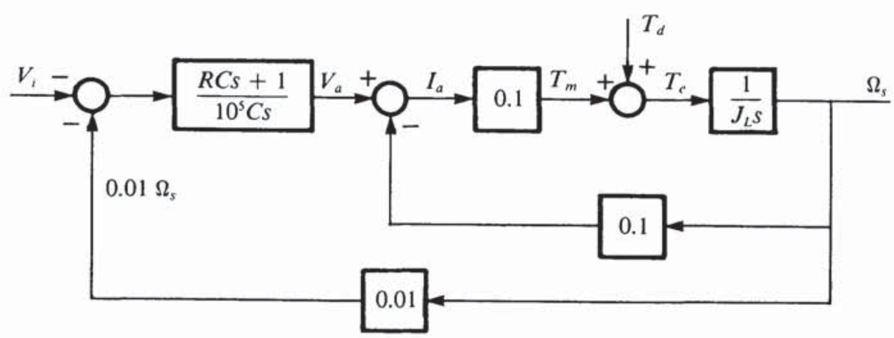


Figure S9.3 System block diagram for Problem 9.2 (P5.4).

(b) By inspection of Figure S9.3, for $T_d = 0$,

$$\frac{\Omega_s(s)}{V_a(s)} = \frac{\dfrac{0.1}{J_L s}}{1 + \dfrac{0.01}{J_L s}} = \frac{10}{100 J_L s + 1} \tag{S9.13}$$

Then, the system loop transmission is

$$\text{L.T.} = -0.01 \left(\frac{RCs + 1}{10^5 Cs}\right)\left(\frac{10}{100 J_L s + 1}\right) \tag{S9.14}$$

The problem requires that the loop transmission be $-100/s$. To achieve this, the zero due to the series RC is used to cancel the motor pole, and capacitor C is used to set the loop-transmission magnitude. That is, we require

$$RC = 100J_L \tag{S9.15}$$

and

$$10^6 C = 0.01 \tag{S9.16}$$

Thus, $C = 0.01\ \mu\text{F}$ and $R = 10^{10}J_L$.

(c) Here, we must examine the relation between T_d and Ω_s. It will be shown that this transfer function has a zero at the origin, thus the shaft velocity Ω_s is unaffected by any constant T_d.

For a system in standard form we recall that the transfer function is $\dfrac{a(s)}{1 + a(s)f(s)}$. Applying this to the block diagram of Figure S9.3, we identify

$$a(s) = \frac{1}{J_L s} \tag{S9.17}$$

and

$$f(s) = 0.1\left[0.1 + 0.01\left(\frac{RCs + 1}{10^5 Cs}\right)\right] \tag{S9.18}$$

But, from part b, $RC = 100J_L$ and $C = 0.01\ \mu\text{F}$, so

$$f(s) = 0.01 + \frac{100J_L s + 1}{s} \tag{S9.19}$$

Applying Equations S9.17 and S9.19 to the standard form, we have

$$\frac{\Omega_s(s)}{T_d(s)} = \frac{1/J_L s}{1 + \dfrac{1}{J_L s}\left(0.01 + \dfrac{100J_L s + 1}{s}\right)}$$

$$= \frac{s}{J_L s^2 + 0.01s + 100J_L s + 1} \tag{S9.20}$$

$$= \frac{s}{(100J_L s + 1)(0.01s + 1)}$$

The d-c response is given by

$$\lim_{s\to 0} \frac{\Omega_s(s)}{T_d(s)} = 0 \tag{S9.21}$$

Thus the response to a constant T_d is zero, as stated earlier.

Solution 9.3 (P5.5)

The transfer function for the network of Figure 5.26 is

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 + \dfrac{1}{C_2 s}}{R_2 + \dfrac{1}{C_2 s} + \dfrac{\dfrac{R_1}{C_1 s}}{R_1 + \dfrac{1}{C_1 s}}} \tag{S9.22}$$

After some algebraic manipulation, this becomes

$$\frac{V_o(s)}{V_i(s)} = \frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1} \tag{S9.23}$$

At this point, attempting to find the pole locations by factoring the denominator of Equation S9.23 simply leads to an algebraic mess. We can avoid this difficulty by examining the general characteristics of the lag-lead network. Such a network will have a pole-zero pair and a zero-pole pair at higher frequencies, forming the lag and lead components respectively. Furthermore, the network of Figure 5.26 imposes the constraint that the α of the lag and lead pairs must be identical. (See Section 5.2.3 in the textbook for the definition of α.) This is required because the magnitude of $\dfrac{V_o(s)}{V_i(s)}$ is unity both in the limit as $s \to 0$ and in the limit as $s \to \infty$. (Consider the limiting cases where the capacitors are open-circuited [$s \to 0$], or short-circuited [$s \to \infty$].) If the lag and lead α parameters were not equal, this would not be possible. Following this argument, the singularities will be placed as shown in Figure S9.4, where τ_1 and τ_2 are the time constants associated with the two zeros.

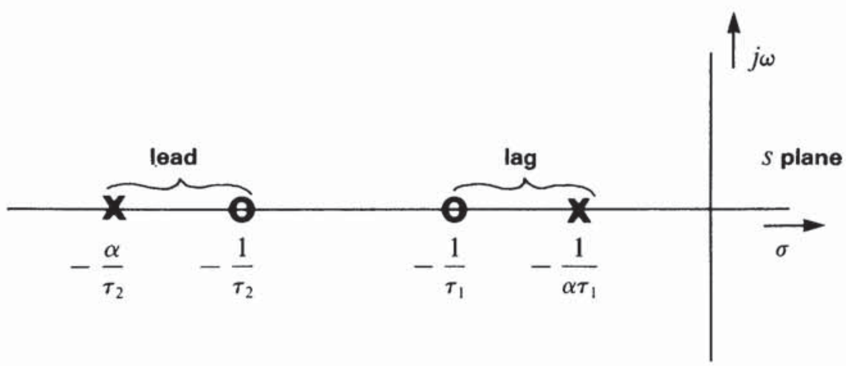


Figure S9.4 Singularities for circuit of Figure 5.26.

The transfer function that realizes the singularity pattern of Figure S9.4 is

$$H(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{(\alpha\tau_1 s + 1)\left(\frac{\tau_2}{\alpha} s + 1\right)} \tag{S9.24}$$

This reduces to

$$H(s) = \frac{(\tau_1 s + 1)(\tau_2 s + 1)}{\tau_1\tau_2 s^2 + \left(\alpha\tau_1 + \frac{\tau_2}{\alpha}\right) s + 1} \tag{S9.25}$$

By means of the above argument, we have restricted the form of the transfer function we are trying to realize, thereby simplifying the design process. At this point, we match coefficients in the denominators of Equations S9.23 and S9.25 to find that

$$R_1C_1R_2C_2 = \tau_1\tau_2 \tag{S9.26}$$

and

$$R_1C_1 + R_2C_2 + R_1C_2 = \alpha\tau_1 + \frac{\tau_2}{\alpha} \tag{S9.27}$$

Further, matching numerators requires that either $R_1C_1 = \tau_1$ and $R_2C_2 = \tau_2$, or $R_1C_1 = \tau_2$ and $R_2C_2 = \tau_1$. The choice between these two solutions is arbitrary. That is, for a given transfer function of the form of Equation S9.25, there are two sets of network element values that will realize the desired transfer function. We arbitrarily choose the solution where

$$R_1C_1 = \tau_1 \tag{S9.28a}$$

and

$$R_2C_2 = \tau_2 \tag{S9.28b}$$

Equation S9.26 is identically satisfied, and substitution into Equation S9.27 yields:

$$\tau_1 + \tau_2 + R_1C_2 = \alpha\tau_1 + \frac{\tau_2}{\alpha} \tag{S9.29}$$

which reduces to

$$R_1C_2 = (\alpha - 1)\tau_1 + \left(\frac{1}{\alpha} - 1\right)\tau_2 \tag{S9.30}$$

Note that the equivalence of Equations S9.23 and S9.25 has imposed only three independent constraints, while the network of Figure 5.26 has four elements. This means that the choice of one network element value is arbitrary. Once this element value is picked, the other three values are determined by Equations S9.28 and S9.30. Thus, it is shown that the network of Figure 5.26 may be used to realize an arbitrary lag-lead network.

Now, we turn to the question of realizing the given transfer function

$$\frac{V_o(s)}{V_i(s)} = \frac{(0.1s + 1)(10^{-2}s + 1)}{(s + 1)(10^{-3}s + 1)} \tag{S9.31}$$

Comparing this with Equation S9.24 gives

$$\tau_1 = 0.1 \tag{S9.31a}$$

$$\tau_2 = 10^{-2} \tag{S9.31b}$$

and

$$\alpha = 10 \tag{S9.31c}$$

Combining these with Equations S9.28 and S9.30 gives

$$R_1C_1 = \tau_1 = 0.1 \tag{S9.32a}$$

and

$$R_2C_2 = \tau_2 = 10^{-2} \tag{S9.32b}$$

and

$$\begin{aligned} R_1C_2 &= (\alpha - 1)\tau_1 + \left(\frac{1}{\alpha} - 1\right)\tau_2 \\ &= 9 \times 0.1 - 0.9 \times 10^{-2} = 0.891 \end{aligned} \tag{S9.32c}$$

Use the one arbitrary component value choice to let $R_1 = 10\ \text{k}\Omega$.

Then, by S9.32*a*,

$$C_1 = 10\ \mu\text{F} \tag{S9.33a}$$

by S9.32*c*,

$$C_2 = 89.1\ \mu\text{F} \tag{S9.33b}$$

and by S9.32*b*,

$$R_2 = 112\ \Omega \tag{S9.33c}$$

This is one set of possible component values. Due to the several arbitrary choices involved in this design process, there are an infinite number of other sets of component values that will give the desired transfer function.

Solution 9.4 (P5.6)

Because $L(s) = -\frac{10^6}{s^2}$, it contributes a constant phase shift of $-180°$ to the *af* product. Thus, to maximize phase margin, we must place the loop-transmission crossover frequency at the point where the positive phase shift of the lead network is maximum. Recall from Section 5.2.3, that the point of maximum positive phase shift occurs at the geometric mean of the pole and zero locations. For the lead network, let the pole location be ω_p. Then, given that $\alpha = 10$, the zero is at $\frac{\omega_p}{10}$, and the maximum positive phase shift occurs at $\omega_{max} = \frac{\omega_p}{\sqrt{10}}$, the geometric mean of these locations. At this frequency, the lead-network gain is $\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{10}}$. As stated earlier, we must set crossover frequency ω_c at ω_{max}, to achieve maximum phase margin. Because the lead network contributes a gain of $\frac{1}{\sqrt{10}}$ at ω_{max}, we require

$$|L(j\omega_{max})| = \frac{10^6}{\omega_{max}^2} = \sqrt{10} \tag{S9.34}$$

to set the loop-transmission magnitude to unity at ω_{max}. This is solved by $\omega_{max} = 10^{2.75} = 562$ rad/sec, which will be the system crossover frequency. From Equation 5.6 in the textbook, at this frequency, with $\alpha = 10$, the lead network contributes a positive phase shift of

$$\phi_{max} = \sin^{-1}\left(\frac{9}{11}\right) = 54.9° \qquad \textbf{(S9.35)}$$

Because, as stated earlier, the rest of the loop contributes a phase of $-180°$, the loop phase margin is 54.9°.

S O L U T I O N S

Compensation Example 10

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 10.1 (P5.8)

(a) The solution of this problem is outlined in the discussion on p. 183 of the textbook. Associated with this discussion are the circuit and block diagrams of Figure 5.13, which are applicable to this problem. In the textbook, the block diagram of Figure 5.13*b* is presented without derivation. Here, we fill in the details of this derivation.

When faced with deriving a block diagram for the circuit of Figure 5.13*a*, one may proceed by writing network equations in terms of V_i and V_o. Then, after some algebraic manipulation, these equations are used to draw the block diagram. The disadvantage of this approach is that it is algebra intensive and tends to obscure physical insight. What is perhaps a more illuminating approach is detailed below.

We start by constructing the Thevenin equivalent circuit for the *R*-9*R* feedback network as shown in Figure S10.1*a* and *b*.

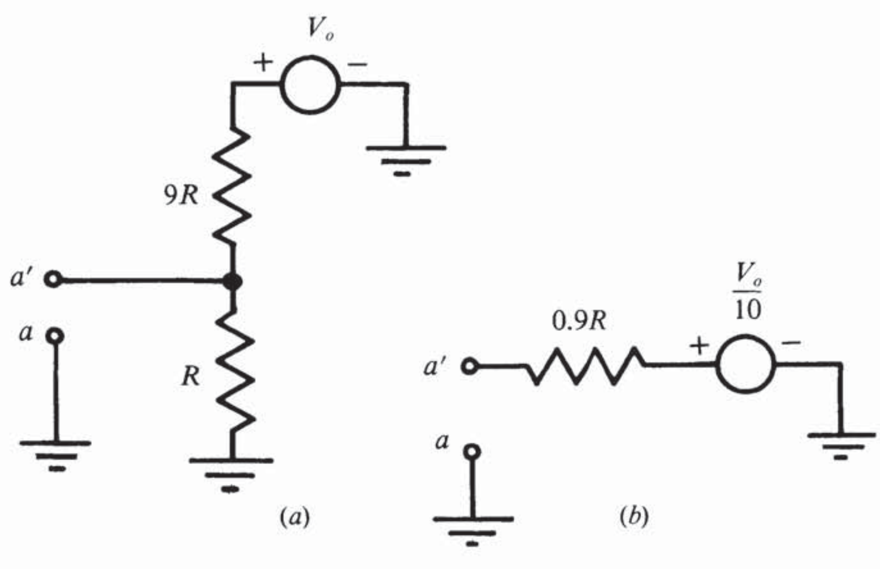
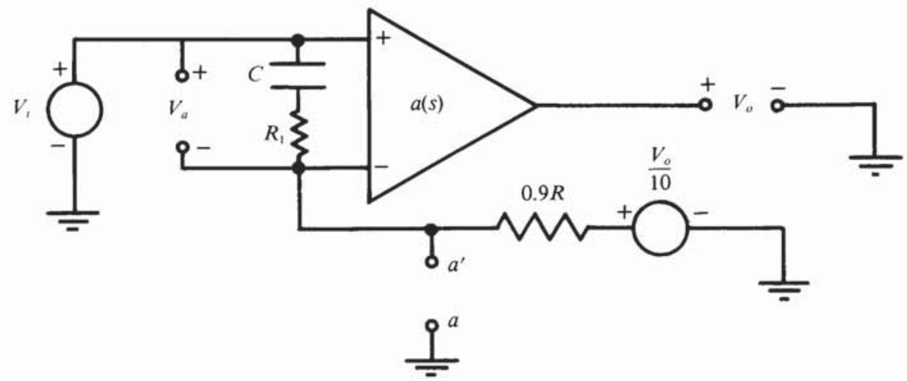


Figure S10.1 Analysis of Problem 10.1 (P5.8) through the use of a Thevenin equivalent circuit. (*a*) *R*-9*R* feedback network. (*b*) Thevenin equivalent as seen at terminal pair *aa'*.

With this manipulation, the circuit diagram is as shown in Figure S10.2,

Figure S10.2 Modified circuit diagram for Problem 10.1 (P5.8).



where V_a is the differential input voltage. That is, $V_o(s) = a(s)V_a(s)$. Then, by superposition and the voltage divider relationship,

$$\begin{aligned} V_a(s) &= \frac{R_1 + \dfrac{1}{Cs}}{R_1 + \dfrac{1}{Cs} + 0.9R}\left[V_i(s) - \frac{V_o(s)}{10}\right] \\ &= \frac{\tau s + 1}{\alpha\tau s + 1}\left[V_i(s) - \frac{V_o(s)}{10}\right] \end{aligned} \tag{S10.1}$$

where $\alpha = \dfrac{R_1 + 0.9R}{R_1}$ and $\tau = R_1C$ as defined on p. 181 of the textbook. The block diagram of Figure 5.13*c* follows directly from Equation S10.1 and our definition of V_a.

The negative of the loop transmission for this system is then as given by Equation 5.14.

$$a''(s)f''(s) = 0.1\,\frac{\tau s + 1}{\alpha\tau s + 1}\,a(s) \tag{S10.2}$$

If we short out the capacitor (i.e., let $C \rightarrow \infty$, and thus $\tau \rightarrow \infty$),

$$a'''(s)f'''(s) = \frac{0.1}{\alpha}\,a(s) \tag{S10.3}$$

To lower the loop transmission by 6.2 at all frequencies then requires that

$$\alpha = 6.2 = \frac{R_1 + 0.9R}{R_1} \qquad \textbf{(S10.4)}$$

This is solved by $R_1 = 0.173R$, as suggested in the textbook. Thus the circuit with $a_o f_o$ reduced by a factor of 6.2 is as shown in Figure S10.3.

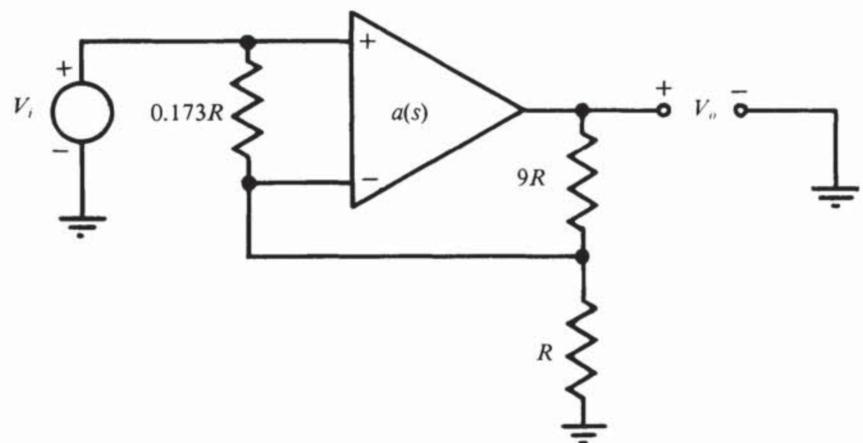


Figure S10.3 Circuit with $a_o f_o$ reduced by a factor of 6.2.

To lower the lowest-frequency loop-transmission pole by a factor of 6.2, we use the lag-network zero to cancel the pole of $a(s)$ at $s = -1$. Then, the lag network α is set at 6.2, so that the lag-network pole is at $s = \dfrac{-1}{6.2}$. That is, given that the negative of the loop transmission for Figure 5.13c is

$$a''(s)f''(s) = 0.1 \frac{\tau s + 1}{\alpha\tau s + 1} \frac{5 \times 10^5}{(s + 1)(10^{-4}s + 1)(10^{-5}s + 1)} \qquad \textbf{(S10.5a)}$$

if we let $\tau = 1$ and $\alpha = 6.2$ this becomes

$$a''(s)f''(s) = 0.1 \frac{5 \times 10^5}{(6.2s + 1)(10^{-4}s + 1)(10^{-5}s + 1)} \qquad \textbf{(S10.5b)}$$

and the lowest-frequency pole has been effectively moved down by a factor of 6.2. From earlier results, for $\alpha = 6.2$, $R_1 = 0.173R$. Then because $\tau = R_1C$, for $\tau = 1$ we have $R_1C = 1$, which is solved by $C = \dfrac{5.78}{R}$. Thus, the circuit that implements this pole lowering is as shown in Figure S10.4.

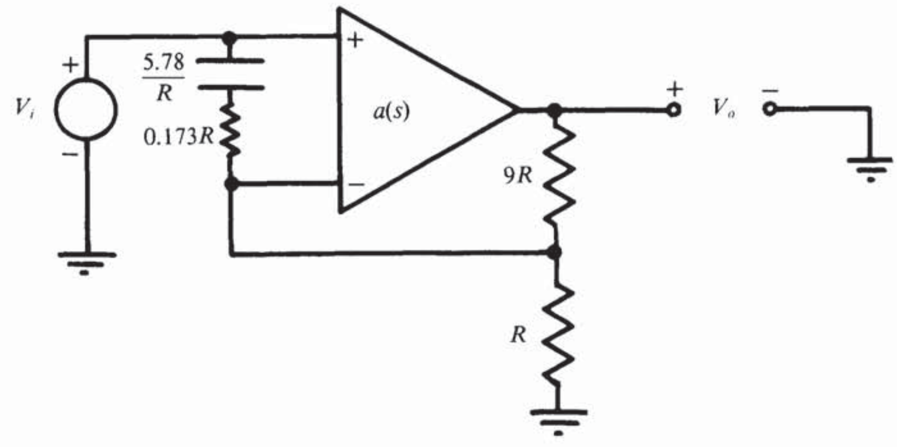


Figure S10.4 Circuit with lowest-frequency pole moved down by a factor of 6.2.

(b) The loop-transmission magnitude Bode plots for the two compensation schemes are shown in the textbook in Figure 5.16. The corresponding angle curves are not difficult to sketch, and thus are not included here.

(c) The phase curves for reduced $a_o f_o$, and the lowered first pole differ only for frequencies below about 10 rad/sec, due to the difference in the low-frequency pole location. They are identical in the vicinity of crossover. At the crossover frequency of 6.7×10^3 rad/sec, the phase for both compensation schemes is $-128°$. Thus, the phase margin is 52°, which is better than the lag compensation by about 5°. This improvement is due to the fact that the lag network has a residual phase of $-5°$ at the crossover frequency. By moving the lag network to a lower frequency, the phase margin may be slightly improved (up to 5°) at the expense of midband desensitivity.

Solution 10.2 (P5.12)

As a matter of cultural interest, the factor

$$\frac{(s^2/12) - (s/2) + 1}{(s^2/12) + (s/2) + 1} \tag{S10.6}$$

is the second-order Pade approximation to a 1-second time delay. See p. 530 of the textbook for further discussion of this topic. This factor has a pole-zero pattern as shown in Figure 12.26 and a phase characteristic as shown in Figure 12.27.

To compensate this system, a lead network alone is not useful, because its transfer function magnitude increases with frequency, and the Pade approximation magnitude is constant. This is discussed in greater detail in Section 5.2.6 of the textbook. To force the loop to crossover, we must introduce a compensating element that provides attenuation with increasing frequency. At the same time, the compensating element should introduce a minimum of negative phase shift. A single pole satisfies these criteria, so we will compensate the loop by using a dominant pole.

Let's design the loop compensation for 45° of phase margin. Near crossover, the dominant pole will contribute a phase of $-90°$ because it is to be located well below crossover. Thus, to have 45° of phase margin, crossover should be set at the frequency where the Pade approximation has a phase of $-45°$. From Figure 12.27, then, crossover must occur at a frequency somewhat below 1 rad/sec. The exact expression for the phase of the Pade approximation is given in Equation 12.65. However, as explained on p. 531 of the textbook, for ω less than 2 rad/sec, the phase is well approximated by an angle of $-57.3°\omega$. Using this approximation, the frequency at which the phase is $-45°$ is $\omega_c = \frac{45}{57.3} = 0.79$ rad/sec. Because the Pade approximation has unity magnitude, then, the compensating transfer function must pass through unity magnitude at $\omega_c = 0.79$ rad/sec. For a dominant pole located at ω_p, where $\omega_p \ll \omega_c$, and with a d-c gain of a_o, the compensating transfer function is

$$H(j\omega) = \frac{a_o}{j\frac{\omega}{\omega_p} + 1} \qquad \textbf{(S10.7)}$$

To have unity magnitude at ω_c then requires that $a_o = \frac{\omega_c}{\omega_p} = \frac{0.79}{\omega_p}$. Any compensation satisfying the above conditions will yield a phase margin of 45°.

To achieve maximum desensitivity, a_o should be made as large as possible. In the limit, let $\omega_p \rightarrow 0$ and $a_o \rightarrow \infty$, while maintaining the relationship $a_o = \frac{0.79}{\omega_p}$. The result is

$$H(j\omega) = \frac{0.79}{j\omega} \qquad \textbf{(S10.8)}$$

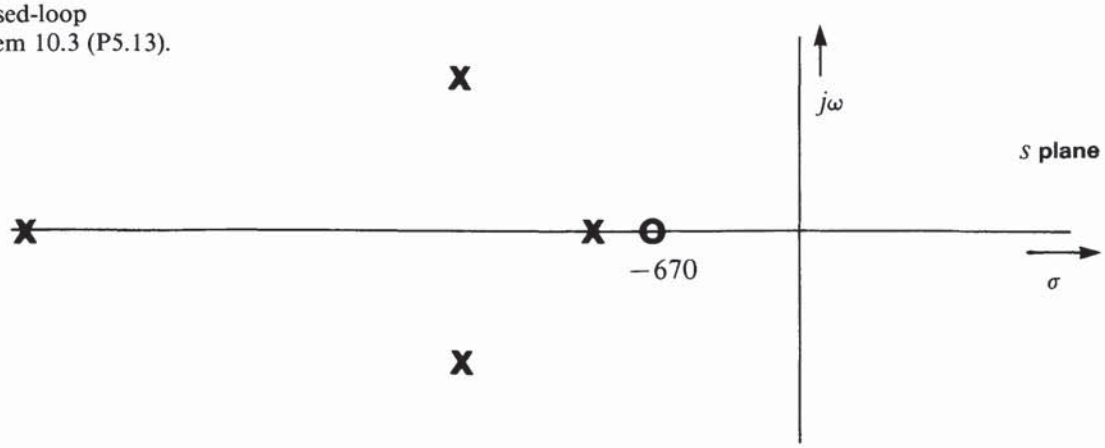
That is, the compensation is an integrator, which has infinite gain at d-c, and thus infinite desensitivity at d-c. As expected, this has unity magnitude at $\omega_c = 0.79$ rad/sec, and an angle of $-90°$ at all frequencies, thus the loop has 45° of phase margin.

More complex schemes, such as using a double integration ($1/s^2$) with lead compensation, are also possible. This would offer higher desensitivity at midband frequencies. However, when implemented with real hardware, such a scheme is more sensitive to component variations than the single-pole compensation, which is quite robust. Therefore, except under special circumstances, the single-pole compensation is the most reasonable solution.

Solution 10.3 (P5.13)

For the lag-compensated system described by Equation 5.15, the root locus is as shown in Figure 5.15*c*. For the given compensation, and value of $a_o f_o$, the closed-loop singularities are located approximately as shown in Figure S10.5.

Figure S10.5 Closed-loop singularities for Problem 10.3 (P5.13).



The pole-zero doublet near $s = -670\ \text{sec}^{-1}$ is responsible for the long-time constant tail associated with the lag-compensated system.

As the lag-compensated system is fourth order, an exact solution of the closed-loop pole locations will require solving for the roots of the fourth-order equation $1 + a''(s)f''(s) = 0$. This is feasible with machine computation. However, as suggested in the problem assignment, a simplifying approximation is possible. That is, we ignore the poles at $s = -10^4$ and $s = -10^5$, and assume that $a(s)$ is given by

$$a(s) = \frac{5 \times 10^5(1.5 \times 10^{-3}s + 1)}{(s + 1)(9.3 \times 10^{-3}s + 1)} \tag{S10.9}$$

The root locus for this simplified system is sketched in Figure S10.6.

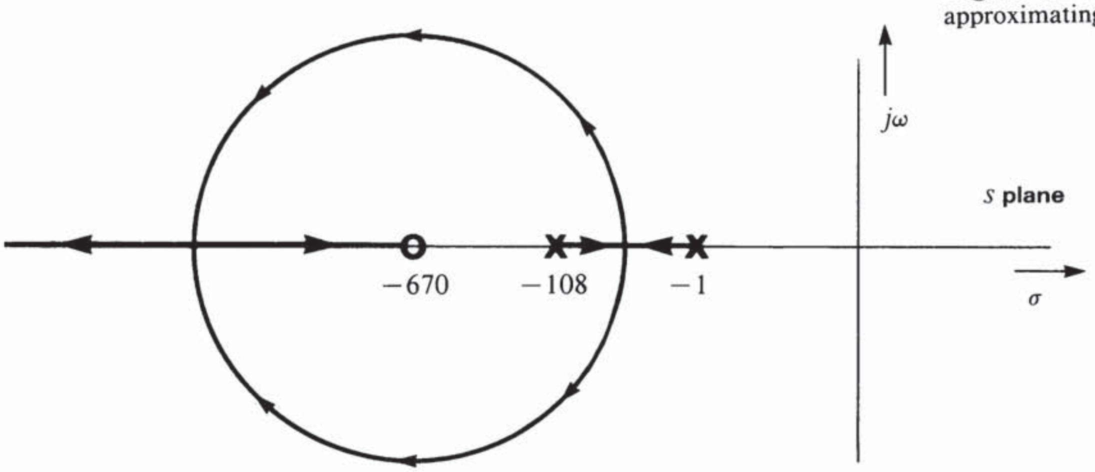


Figure S10.6 Root locus of approximating second-order system.

Note that this locus is very similar to a section of the root locus of Figure 5.15*c* in the textbook. Further, since the closed-loop complex pair of the full fourth-order representation is located a decade up in frequency from the zero at $s = -670$, it is reasonable to expect that the complex pair has only a slight influence on the locus in the vicinity of the zero.

For the above reasons, we expect that the second-order approximation of Equation S10.9 will be acceptably accurate. Using this approximation, the closed-loop transfer function is

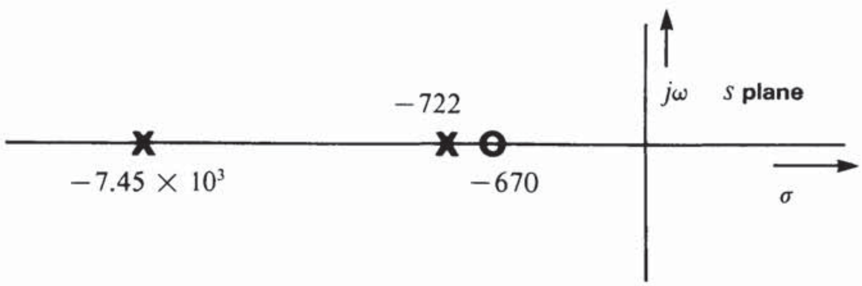
$$A(s) = \frac{a(s)}{1 + a(s)f(s)}$$

$$= \frac{5 \times 10^5(1.5 \times 10^{-3}s + 1)}{(s + 1)(9.3 \times 10^{-3}s + 1) + 5 \times 10^4(1.5 \times 10^{-3}s + 1)}$$

$$= 10\,\frac{1.5 \times 10^{-3}s + 1}{1.86 \times 10^{-7}s^2 + 1.52 \times 10^{-3}s + 1} \qquad \textbf{(S10.10)}$$

The poles of $A(s)$ are at $s = -722$ and $s = -7.45 \times 10^3$. That is, the closed-loop pole configuration is as sketched in Figure S10.7.

Figure S10.7 Closed-loop singularities for the approximate system.



An exact analysis of the fourth-order case indicates that the pole of the pole-zero doublet is actually at $s = -717$. The close agreement with the approximate value of $s = -722$ verifies the approximation. The settling time will be dominated by the pole-zero doublet. The step response of the doublet alone is given by

$$v(t) = 1 + 0.08e^{-722t} \qquad \textbf{(S10.11)}$$

This will settle to within 1% when

$$v(t_o) = 1.01 = 1 + 0.08e^{-722t_o} \qquad \textbf{(S10.12)}$$

or

$$0.01 = 0.08e^{-722t_o} \qquad \textbf{(S10.13)}$$

which is solved by $t_o = 2.9$ msec. The solution is arrived at by recognizing that the initial value theorem requires that the step response at $t \rightarrow 0+$ is $\frac{722}{670} = 1.08$. Further, as $t \rightarrow \infty$ the final value theorem requires the step response to approach unity. The time constant of the exponential connecting these two values is $\frac{1}{722} =$ 1.39 msec.

For the first-order system, with a crossover at $\omega = 6.7 \times 10^3$ rad/sec, the step response will be given by

$$v(t) = 1 - e^{-6.7\times 10^3 t} \qquad \textbf{(S10.14)}$$

This will settle to 1% when $e^{-6.7\times 10^3 t_1} = 0.01$. This is solved by $t_1 =$ 0.69 msec. That is, the first-order system is about a factor of 4 faster than the lag-compensated system. In many instances, such as analog-to-digital conversion, settling time is quite important. The lesson of this problem is that a pole-zero doublet can have a very significant effect on an amplifier settling time.

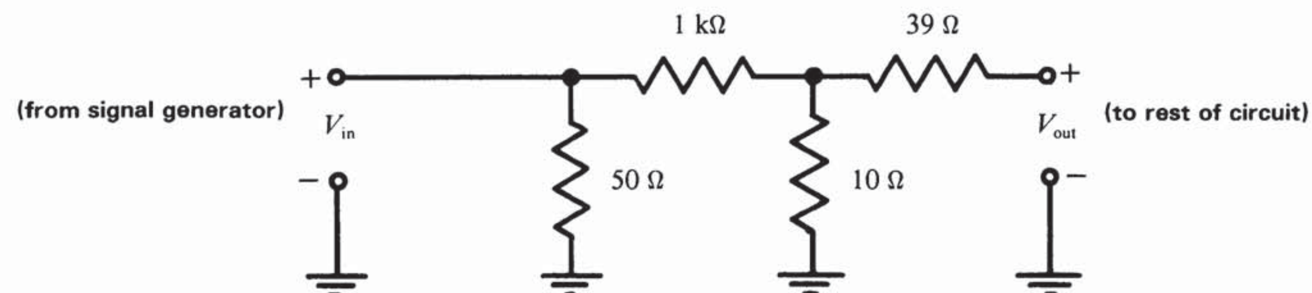
Solution 10.4 (P5.15)

The principal objective of this problem is to provide the student with the experience of applying analytical results in the laboratory. As the laboratory portion of the problem is essential, and the hands-on experience more important than the actual answers, extensive solutions are not provided here. Furthermore, as with most design problems, there is no one correct answer, so it is expected that each student's solution will differ in some respects from other students' solutions.

Included here are some general guidelines and suggestions for approaching the problem, as well as answers to some of the questions posed in the problem statement. Appropriate topologies for each of the three compensation techniques are also given. If the analytical and experimental portions of the problem are properly solved, the student should gain confidence that the analytical approaches we have studied thus far are useful design techniques and can be applied with accuracy to real circuit problems.

The problem statement suggests the use of a resistive attenuation at the amplifier input. A possible topology is shown in Figure S10.8.

Figure S10.8 Attenuator.



This attenuator provides a 100:1 attenuation ratio, with 50Ω input and output resistances. Thus, it is also useful for high-frequency applications, where a 50Ω impedance must be maintained. For the purposes of this lab, because only low frequencies are of interest, both the 50Ω and 39Ω resistor may be omitted.

The capacitor C adjusts the location of the low-frequency pole associated with the LM301A. Therefore, because this pole is at a frequency much lower than 10^3 rad/sec, C may be used to adjust the loop-transmission crossover frequency without significantly affecting the phase at crossover. By adjusting C to bring the configuration to the verge of instability, the crossover frequency is set near the point where the negative phase shift of the loop transmission is slightly less than 180°. Stability can easily be ascertained by examining the amplifier step response. Adjust C to create a lightly damped step response. The longer the step response rings, the closer the poles are to the $j\omega$ axis. A ring time of 200 to 500 msec is sufficient. Note that smaller values of C will result in longer ring times. Also make certain that the circuit is stable, that is, the ringing step response must decay with time.

In order to achieve good numerical agreement between theoretical and experimental results, use resistors that match the values indicated in Figures 5.28, 5.29, and 5.30 within $\pm 1\%$, and capacitors that match the indicated values within $\pm 5\%$. This capacitor tolerance does not apply to the two 0.01 μF decoupling capacitors. Do make sure to include these decoupling capacitors located close to the LM301A in order to avoid instabilities caused by power-supply lead inductance. In general, a bit of care in circuit construction will pay off in reliable circuit operation.

For the purposes of analysis, use the approximate transfer function for $a(s)$ as given in the problem statement. Because the pole at $s = -1$ is providing $-90°$ of phase shift in the vicinity of crossover, standardization by adjusting C places crossover near the point where the poles at 10^3 and 10^4 rad/sec are providing an additional $-90°$ of phase shift. This occurs at $\omega = 3.16 \times 10^3$ rad/sec.

The Bode plot for the inverting gain-of-ten configuration of Figure 5.30 should indicate a crossover frequency of 2×10^3 rad/sec, with a phase margin of 15.3°, and a gain margin of 2.4. With this phase margin, we expect $M_p = \dfrac{1}{\sin 15.3°} = 3.8$.

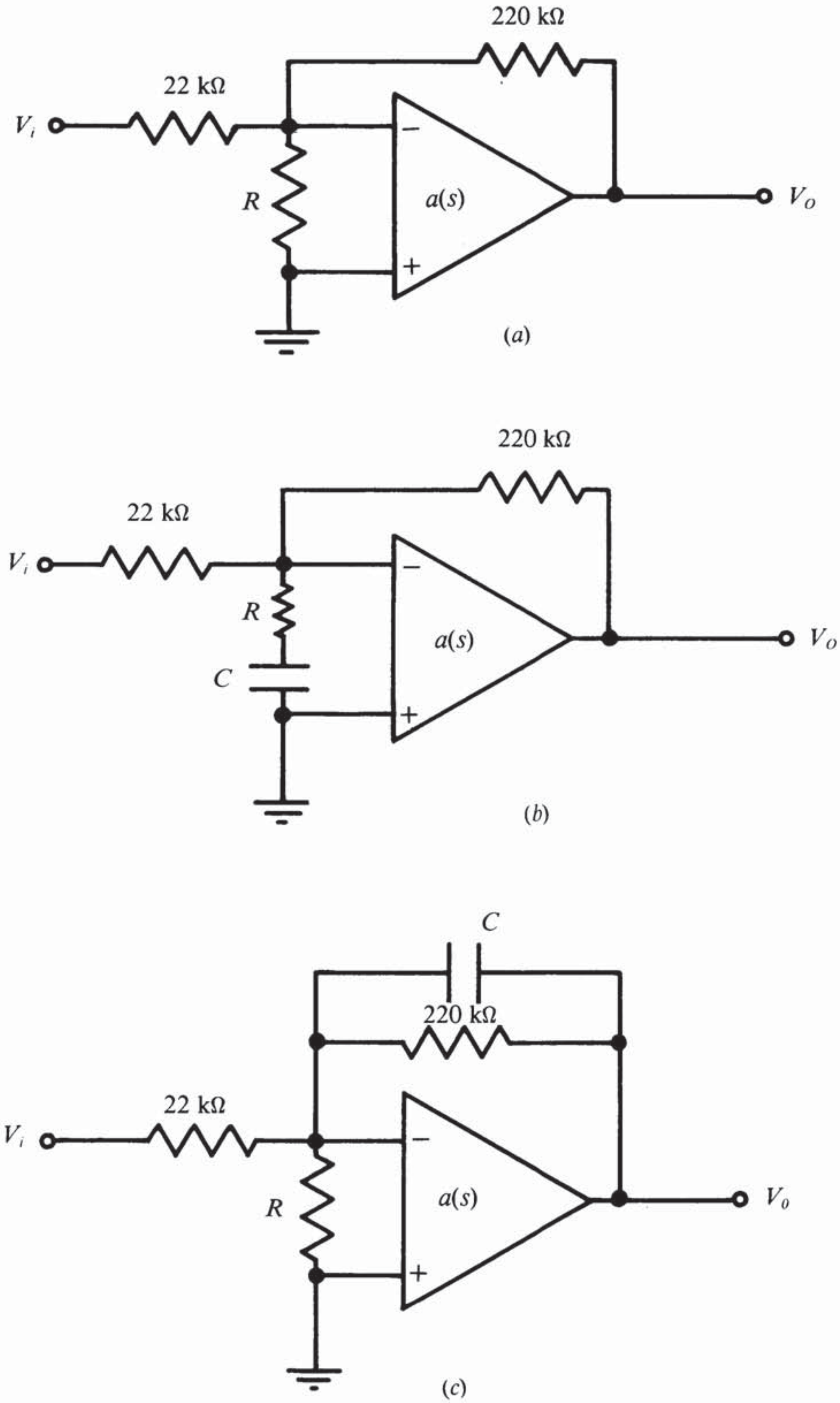


Figure S10.9 Suggested topologies for the three compensation schemes. (*a*) Reduced $a_o f_o$. (*b*) Lag. (*c*) Lead with reduced $a_o f_o$.

The three types of compensation have been covered in detail in this chapter, and thus are not solved for explicitly here. During the design and analysis of each compensation scheme, it is useful to have some form of computation that can provide results of transfer-function magnitude and phase versus frequency. A short program written on a computer or hand-held calculator will suffice. Appropriate topologies for the three forms of compensation are shown in Figure S10.9. Others are certainly possible.

S O L U T I O N S

11 Feedback Compensation

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 11.5 (P5.14)

(a) If the major loop crosses over at $\omega = 10^3$ rad/sec, then it is very likely that we can choose b and τ such that the minor-loop transmission crosses over well above this frequency. For $\omega \gg 1$, the minor-loop transmission is given approximately by

$$\text{L.T.} \simeq -10^{10}\frac{b}{\tau s + 1} \tag{S11.1}$$

At $\omega = 10^3$ rad/sec the minor-loop transmission magnitude is approximately $10^7\dfrac{b}{\tau}$, which will be large when $10^7 b \gg \tau$. We proceed under this assumption, and check its validity later in the solution. Also note that the phase shift of the negative of the minor-loop transmission never exceeds $-90°$. Thus, stability of the minor loop is guaranteed for all positive values of b and τ.

Assuming the minor-loop transmission magnitude is large, and following the development in Section 5.3, the major-loop transmission is given approximately as

$$a(s) = 3 \times 10^{-3}\left(\frac{\tau s + 1}{bs^2}\right) \tag{S11.2}$$

To achieve 55° of phase margin, the zero must supply 55° of positive phase shift at the crossover frequency of 10^3 rad/sec. Thus, we require

$$\tan^{-1} 10^3\tau = 55°$$

or

$$\tau = 10^{-3}\tan 55° = 1.43 \times 10^{-3} \tag{S11.3}$$

That is, the zero should be located at $\omega = 700$ rad/sec. At crossover, with this value of τ, the major-loop transmission magnitude is given by

$$\begin{aligned} |a(s)| &= \frac{3 \times 10^{-3}}{b \times 10^6}\sqrt{(1.43 \times 10^{-3})^2(10^3)^2 + 1} \\ &= \frac{1}{b} \times 5.2 \times 10^{-9} \end{aligned} \tag{S11.4}$$

Thus, to set the magnitude equal to unity, we must have $b = 5.2 \times 10^{-9}$. Now to check the original assumption. At $\omega = 10^3$ rad/sec, the minor-loop transmission magnitude is about 30, which is sufficiently greater than 1 to satisfy the conditions of our analysis.

Now, we sketch the open-loop Bode plot for the amplifier. An approximate analysis follows. For frequencies well below the zero location, the feedback path of the minor loop is approximately bs^2, and the open loop is approximately given by

$$\frac{V_o(s)}{V_i(s)} \simeq 3 \times 10^{-3} \times \frac{-10^{10}}{(s+1)^2 + 10^{10}bs^2} \qquad \textbf{(S11.5)}$$
$$= 3 \times 10^7 \frac{1}{53.3s^2 + 2s + 1}, \qquad |s| \ll 700$$

which has a complex pair of poles at $s = -1.8 \times 10^{-2} \pm j0.14$, which is lightly damped, but stable. From earlier results, we know that there is an open-loop zero at $s = -700$ rad/sec. That is, the pole $\frac{1}{\tau s + 1}$ in the minor-loop feedback path is an open-loop zero of the amplifier. Finally, for frequencies well above the zero location, the minor-loop feedback path is approximately $\frac{bs}{\tau}$, and the open-loop transfer function is approximately given by

$$\frac{V_o(s)}{V_i(s)} \simeq 3 \times 10^{-3} \frac{-10^{10}}{s^2 + \dfrac{10^{10}bs}{\tau}} \qquad \textbf{(S11.6)}$$
$$= \frac{-8.2 \times 10^2}{s\left(\dfrac{s}{3.7 \times 10^4} + 1\right)}, \quad |s| \gg 700$$

This has a pole at the origin, which represents the net effect of the two poles and the zero as seen at frequencies much greater than 700 rad/sec. The higher frequency pole at $s = -3.7 \times 10^4$ rad/sec is due to the minor-loop transmission crossover. Thus, the open-loop transfer function has a complex pole pair at $s = -1.8 \times 10^{-2} \pm j0.14$, a zero at $s = -700$, and a pole at $s = -3.7 \times 10^4$. An exact numerical solution of the full third-order open-loop transfer function confirms these approximate results. Given the above singularity locations, we can sketch the Bode plot as shown in Figure S11.1.

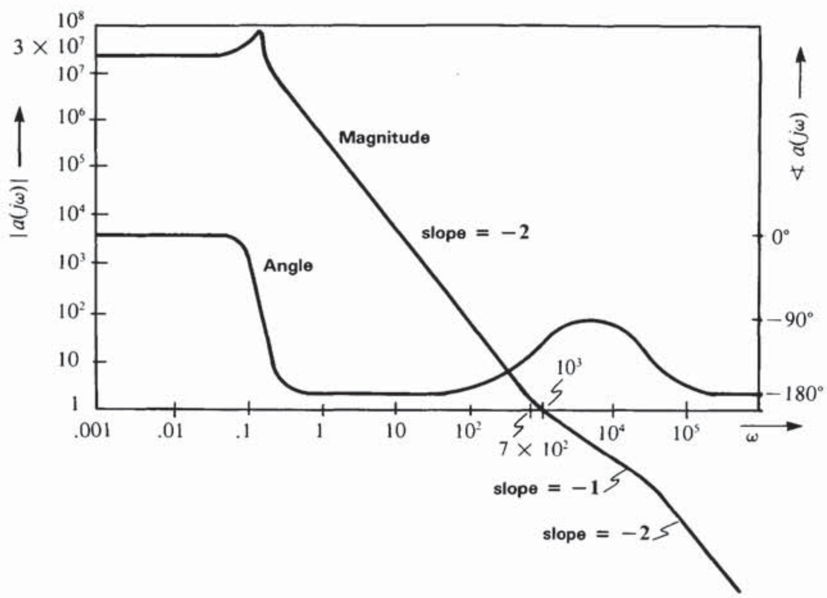


Figure S11.1 Open-loop magnitude and phase versus frequency.

(b) It is not possible to match the resonant peak at $\omega = 0.135$ rad/sec with any $|a_c(j\omega)| \leq 1$, however, we are only asked to match the magnitude characteristics asymptotically. This is possible by placing two poles at $s = -0.135$, two zeros at $s = -1$ to cancel the poles at $s = -1$, one zero at $s = -700$, and one pole at $s = -3.7 \times 10^4$. This will give a transfer function of

$$a_c(s) = \frac{(s+1)^2\left(\dfrac{s}{700}+1\right)}{\left(\dfrac{s}{0.135}+1\right)^2\left(\dfrac{s}{3.7\times 10^4}+1\right)} \tag{S11.7}$$

This has $|a_c(j\omega)| \leq 1$ for all ω, although it is not physically realizable, because at high frequencies $|a_c(j\omega)| \simeq 0.96$, implying infinite frequency response, which is of course impossible for any real circuit.

Feedback Compensation of an Operational Amplifier 12

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 12.1 (P13.5)

This approximate $a(s)$ is identical to the approximate $a(s)$ given by Equation 13.20 of the textbook with $K = 2 \times 10^{-4}$ mho. If a single capacitor is used for the compensating element, and, as is the case here, the crossover frequency is low relative to higher-order singularities, then Equation 13.26 applies. Here, $f_o = 1$, thus the closed-loop crossover frequency is given by $\omega_h = \dfrac{K}{C_c}$. This can be set to 10^6 rad/sec by choosing $C_c = \dfrac{2 \times 10^{-4}}{10^6} = 200$ pF.

Solution 12.2 (P13.6)

The loop transmission for the log circuit of Figure 13.9a is given by Equation 13.19 of the textbook. At room temperature, $\dfrac{q}{kT} \simeq 40$, thus the loop transmission is $L(s) = -40\ a(s)v_I$. The loop transmission varies from 0 to $-400\ a(s)$ as the input varies from 0 to $+10$ volts. Thus, we need to maintain adequate phase margin over a wide range of frequencies. This requires single-pole compensation.

With single-pole compensation, if the system is stable for the largest loop-transmission magnitude, then it will be stable for all smaller loop-transmission magnitudes. Thus, we force crossover at 1 MHz when $v_I = +10$ volts, and the crossover will occur at lower frequencies with adequate phase margin for all $0 \leq v_I < 10$ volts. Letting $Y_c(s) = Cs$, and evaluating the magnitude at 1 MHz, we have

$$
\begin{aligned}
|L(j\omega)|\Big|_{\omega=6.28\times10^6\ \text{rad/sec}} &= 400 \times \frac{2 \times 10^{-4}}{C \times 6.28 \times 10^6} \\
&= \frac{1.27 \times 10^{-8}}{C}
\end{aligned}
\tag{S12.1}
$$

Unity-gain crossover at 1 MHz is set by choosing $C = 1.27 \times 10^{-8} \simeq 0.013\ \mu F$.

Now, with $V_I = 0.1$ volt, the loop-transmission magnitude is reduced by a factor of 100 from the value when $V_I = 10$ volts, and crossover occurs at 10 kHz. The circuit step response will be first order with a time constant $\tau = \dfrac{1}{2\pi \times 10^4} = 16\ \mu\text{sec}$. Settling to within 1% of final value requires that $e^{-t/\tau} = 0.01$, which is solved by $t = 4.6\tau = 73\ \mu\text{sec}$.

Solution 12.3 (P13.7)

The closed-loop response of a unity-gain inverting amplifier is given by $A(s) = -\frac{1}{2}\dfrac{a(s)}{1 + \frac{1}{2}a(s)}$. As is apparent from this expression, the closed-loop steady-state response to a sinusoid at 10 kHz is determined entirely by the magnitude and phase of $a(s)$ at 10 kHz. We shall see that the two-pole compensation yields better phase accuracy, with slightly less closed-loop gain accuracy than the single-pole compensation, due to the differing magnitude and phase of $a(s)$ under the two compensation schemes.

The loop transmission is $-\frac{1}{2}a(s)$. Thus, for single-pole compensation, to cross over at 1 MHz, we must choose $a'(s) = \dfrac{10^6}{\dfrac{0.5s}{2\pi} + 1}$. At 10 kHz, $a'(s)$ is

$$a'(j2\pi 10^4) = \frac{10^6}{j5 \times 10^3 + 1} = 199.99999\ e^{-j1.5705963} \tag{S12.2}$$

Thus, the single-pole compensated closed-loop response at 10 kHz is

$$\begin{aligned} A'(j2\pi 10^4) &= -\frac{1}{2}\frac{199.99999\ e^{-j1.5705963}}{1 + \frac{1}{2} \times 199.99999\ e^{-j1.5705963}} \\ &= 0.99994800 \measuredangle 179.427^\circ \end{aligned} \tag{S12.3}$$

where phasor notation has been used to indicate the angle in degrees.

For two-pole compensation, many choices are possible; however, for simplicity, we choose to place the compensating zero at 100 kHz, a factor of 10 below crossover. The double poles must then be placed at 447 Hz to set crossover at 1 MHz. That is, $a''(s)$ is given by

$$a''(s) = \frac{10^6\left(\dfrac{s}{2\pi \times 10^5} + 1\right)}{\left(\dfrac{s}{2\pi \times 447} + 1\right)^2} \qquad \textbf{(S12.4)}$$

At 10 kHz, $a''(s)$ is

$$a''(j2\pi 10^4) = \frac{10^6(0.1j + 1)}{\left(\dfrac{j10^4}{447} + 1\right)^2} = 2004.0513\, e^{-j2.9525834} \qquad \textbf{(S12.5)}$$

Thus, the two-pole compensated closed-loop response is

$$\begin{aligned} A''(j2\pi 10^4) &= -\frac{1}{2}\frac{2004.0513\, e^{-j2.9525834}}{1 + \frac{1}{2} \times 2004.0513\, e^{-j2.9525834}} \\ &= 1.0009811 \measuredangle 179.989° \end{aligned} \qquad \textbf{(S12.6)}$$

again in phasor notation.

Thus, for the single-pole compensation the closed-loop gain is accurate to within about 0.005%, with a phase error of about 0.57°. For the two-pole compensation, the closed-loop gain is accurate to within about 0.1%, with a phase error of about 0.01°. So in terms of phase accuracy, the two-pole compensation is far superior. For gain accuracy, the single-pole scheme is better.

A good approximation to the above results can be derived with far less computational effort by using the asymptotic values for $a'(s)$ and $a''(s)$. That is, assume that at 10 kHz, $a'(j2\pi 10^4) \simeq 200\, e^{-j\pi/2}$ and $a''(j2\pi 10^4) \simeq 2000\, e^{-j\pi}$. Then, plug these approximations into the expressions for $A'(j2\pi 10^4)$ and $A''(j2\pi 10^4)$. The results will be essentially the same as the detailed analysis above.

Solution 12.4 (P13.8)

(a) With the given input, the output of the amplifier will be a ramp with a slope of 10^5 volts per second. The input to the amplifier is 10 mV. Following the discussion of Section 13.3.3, we assume that the amplifier functions as an integrator on an open-loop basis, that is, $a(s) \simeq \dfrac{k}{s}$. The constant k is in volts per second per volt and is given by $k = \dfrac{10^5 \text{ V/sec}}{10 \text{ mV}} = 10^7$. Thus $a(s) \simeq \dfrac{10^7}{s}$.

(b) Following the discussion of Sections 5.3, 9.2.3, and 13.3, the magnitude of the open-loop response of the amplifier will follow the lower of the single-pole approximation and the uncompensated transfer function. Thus, we can use the transfer function given in the problem statement to refine the transfer-function estimate. The two magnitude curves are shown in Figure S12.1. We have omitted the corresponding phase curves because they are unnecessary for this problem. The more accurate approximation is shown as the darkened line indicating the lower of the two curves.

(c) For an LM301A, from Equation 13.20, $a(s) \simeq \dfrac{K}{Y_c(s)}$, where $K = 2 \times 10^{-4}$ mho. From part a, we have $a(s) \simeq \dfrac{10^7}{s}$. Thus, $Y_c(s) = \dfrac{Ks}{10^7} = 2 \times 10^{-11}s$. This is the admittance for a 20 pF capacitance, which is therefore the compensating element.

(d) As described in Section 13.3.3, for essentially zero steady-state ramp error, we select two-pole compensation. From Equation 13.36, for two-pole compensation,

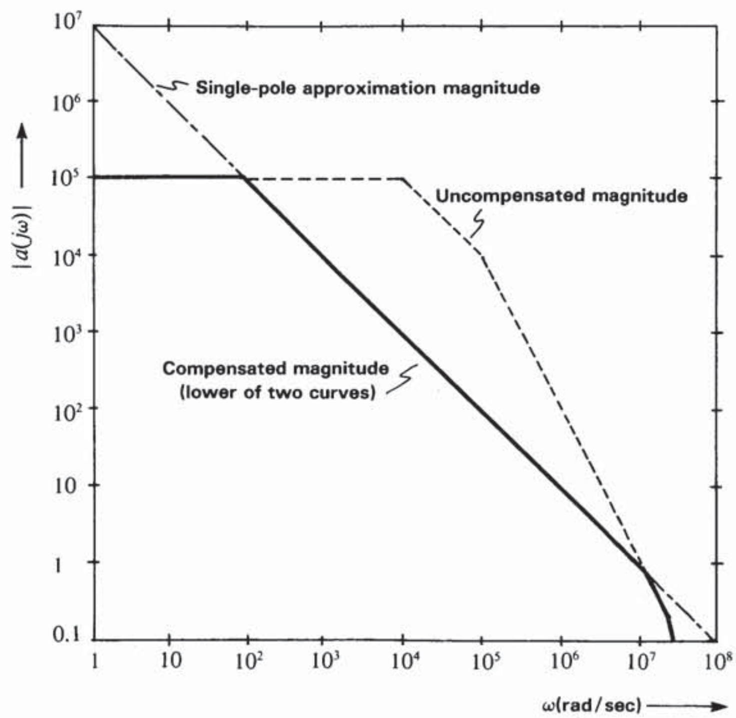


Figure S12.1 Open-loop transfer functions for amplifier of Problem 12.4 (P13.8).

$$a(s) \simeq \frac{K'(\tau s + 1)}{s^2} \qquad \textbf{(S12.7)}$$

where $\tau = R(C_1 + C_2)$ and $K' = K/RC_1C_2$. As usual, $K = 2 \times 10^{-4}$ mho. The compensating network topology is shown in Figure 13.19 of the text. The results of part b indicate that the single pole compensated gain-of-ten amplifier loop transmission will cross over at 10^6 rad/sec, with about 90° of phase margin. We now design the two-pole compensator to have the same crossover frequency. We locate the zero a decade below crossover to guarantee adequate phase margin. Thus, $\tau = 10^{-5}$ sec. Then, to set crossover at 10^6 rad/sec, we must have $K'\tau = 10^7$, which gives $K' = 10^{12}$. One more constraint is required to solve for the compensating element values. This represents an extraneous degree of freedom, which we eliminate by choosing $C_1 = C_2$. Then, the above equations in τ and K' allow us to solve for $C_1 = C_2 = 40$ pF, and $R = 125$ kΩ. The resulting phase margin is about 84°.

S O L U T I O N S

Operational Amplifier Compensation (cont.) 13

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 13.1 (P13.10)

With minor-loop compensation, including the capacitive loading,

$$a(s) \simeq \frac{K}{Y_c(s)(10^{-6}s + 1)} \tag{S13.1}$$

The desired open-loop transfer function is

$$a(s) \simeq \frac{2 \times 10^{11}(5 \times 10^{-6}s + 1)}{s^2} \tag{S13.2}$$

Equating the above two expressions allows us to solve for $Y_c(s)$ as

$$Y_c(s) = \frac{Ks^2}{2 \times 10^{11}(5 \times 10^{-6}s + 1)(10^{-6}s + 1)} \tag{S13.3}$$

This is a second-order transfer function; thus, we can realize it with two energy storage elements. We choose to use two capacitors.

At high frequencies $(|s| \gg 10^6)$,
$Y_c(s) = \dfrac{K}{2 \times 10^{11}(5 \times 10^{-6})(10^{-6})} = 2 \times 10^{-4}$. That is, at high frequencies the network is resistive, with $R = 5\ \text{k}\Omega$. Thus, the compensating network is of the form shown in Figure S13.1, consisting of a second-order network yet to be determined, in series with a 5 kΩ resistor. Further, the two-port network must appear as a short circuit at high frequencies.

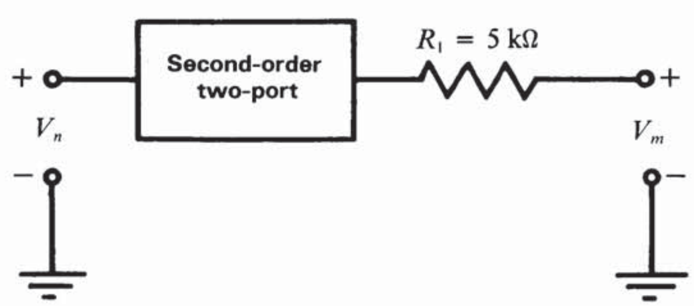
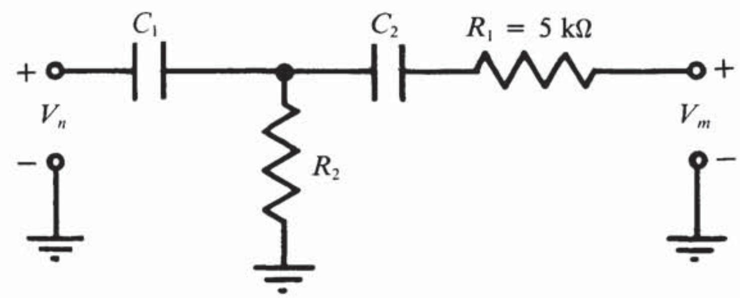


Figure S13.1 Compensating network for Problem 13.1 (P13.10).

Using the above discussion for guidance, we pick the compensating network topology shown in Figure S13.2.

Figure S13.2 Compensating network for Problem 13.1 (P13.10).



This network has a short-circuit transfer admittance of

$$Y(s) = \frac{R_2C_1C_2s^2}{R_1R_2C_1C_2s^2 + [R_1C_2 + R_2(C_1 + C_2)]s + 1} \quad \textbf{(S13.4)}$$

Now, we can solve for element values by equating S13.4 and S13.3. Because we've already found the value of $R_1 = 5\ \text{k}\Omega$, this gives only two independent equations, that is,

$$R_2C_1C_2 = 10^{-15} \quad \textbf{(S13.5)}$$

$$R_1C_2 + R_2(C_1 + C_2) = 6 \times 10^{-6} \quad \textbf{(S13.6)}$$

These two equations are in three unknowns, implying an extraneous degree of freedom. This degree of freedom is eliminated by choosing $R_2 = 1\text{k}\Omega$. (There is no real solution if $R_1 = R_2$.) Even with this constraint, as the equations are quadratic in form, there are still two sets of solutions. They are

Solution A	*Solution B*
$C_1 = 1269$ pF	$C_1 = 4719$ pF
$C_2 = 788$ pF	$C_2 = 212$ pF

Either solution set is acceptable. There are also many other parameter sets that will yield the same transfer admittance, as well as other possible network topologies.

Solution 13.2 (P13.11)

As described in Section 13.3.4, the feedback network of a differentiator adds a loop-transmission pole. The solution is to apply minor-loop compensation to create an open-loop zero that will partially offset the negative phase shift of the pole in the vicinity of crossover. From Equation 13.45, with a series $R_c - C_c$ compensating network,

$$a(s) \simeq \frac{K(R_cC_cs + 1)}{C_cs} \quad \textbf{(S13.7)}$$

The pole resulting from the feedback network is at $s = -1/RC = -10\text{ sec}^{-1}$. Thus, the approximate loop transmission is

$$L(s) \simeq -\frac{K(R_cC_cs + 1)}{C_cs(0.1s + 1)} \quad \textbf{(S13.8)}$$

Crossover is specified at 10^4 rad/sec. To achieve 60° of phase margin, we set the zero at $\omega_z = 5.8 \times 10^3$ rad/sec. Thus, $R_cC_c = 1/\omega_z = 1.7 \times 10^{-4}$ sec. (Because no phase margin requirement is specified in the problem, other solutions are also acceptable.) Now, we can use C_c to set $\omega_c = 10^4$ rad/sec by requiring that $|L(j10^4)| = 1$. Because crossover occurs well above the pole at 10 rad/sec, the resulting equation is

$$|L(j\omega)|\bigg|_{\omega=10^4} \simeq \left|\frac{K(1.7 \times 10^{-4}j\omega + 1)}{0.1\, C_c\omega^2}\right|\Bigg|_{\omega=10^4} = 1 \quad \textbf{(S13.9)}$$

This is solved by $C_c = 40$ pF. Then, because $R_cC_c = 1.7 \times 10^{-4}$ sec, we find that $R_c = 4.3\text{ M}\Omega$.

The 3 pF shunt guarantees that the compensating network is capacitive in the vicinity of minor-loop crossover. Note that this topology is slightly different than that of Figure 13.29 in the textbook. That is, for this problem, the compensating network is as shown in Figure S13.3, with 3 pF shunting the entire compensating network. With this network, the loop transmission becomes

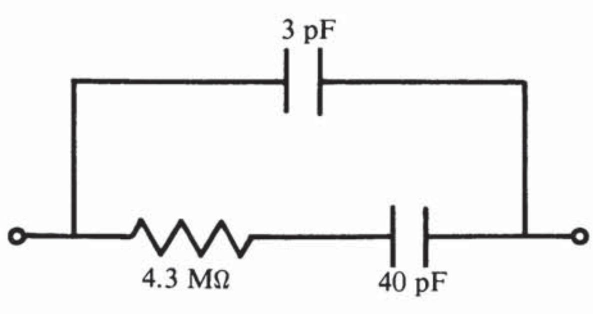


Figure S13.3 Modified compensation network for Problem 13.2 (P13.11).

$$L(s) \simeq -\frac{2 \times 10^{-4}(1.7 \times 10^{-4}s + 1)}{4.3 \times 10^{-11}s(0.1s + 1)(1.2 \times 10^{-5}s + 1)} \quad \textbf{(S13.10)}$$

There are two effects of the shunt. First the loop-transmission magnitude is lowered by about 8% across all frequencies. This is due to the additional admittance of the 3 pF cap. Secondly, a loop-transmission pole is introduced at 8.3×10^4 rad/sec. These two effects combine to reduce the loop-transmission crossover to $\omega_c = 9.3 \times 10^3$ rad/sec with a phase margin of about 51°. This will have only a minor effect on closed-loop performance.

S O L U T I O N S

Linearized Analysis of Nonlinear Systems 14

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 14.1 (P6.1)

When the elements are cascaded in the order *ab,* the overall output is zero for all inputs, as shown in Figure S14.1*a*. When they are cascaded in the order *ba,* the transfer characteristic shown in Figure S14.1*b* results.

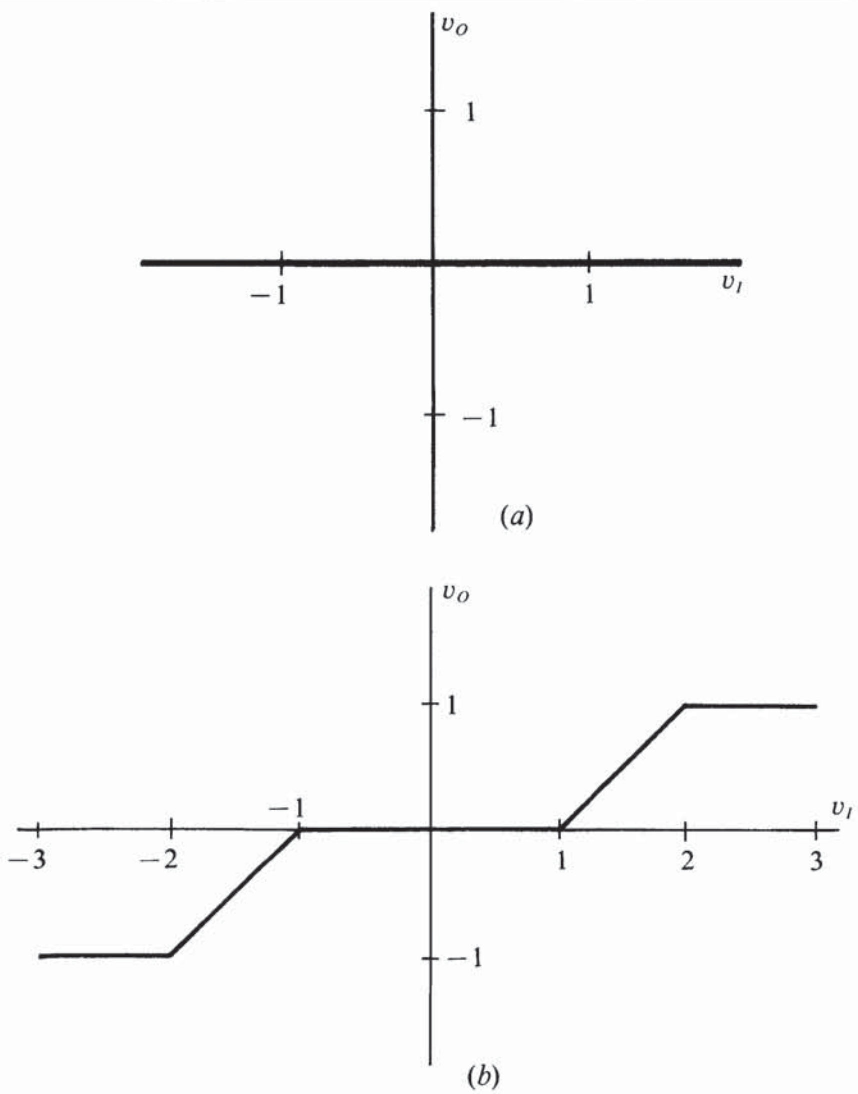


Figure S14.1 Transfer characteristics for system of Problem 14.1 (P6.1). (*a*) Cascaded in order *ab*. (*b*) Cascaded in order *ba*.

Solution 14.2 (P6.2)

(a) At each of the equilibria, $\theta_E = n\pi$. The incremental gain at each equilibrium is $\frac{dv_E}{d\theta_E}$, and is given by

$$\frac{dv_E}{d\theta_E} = \begin{cases} 1 & \theta_E = 2n\pi \\ -1 & \theta_E = (2n+1)\pi \end{cases} \quad n = 0, 1, \ldots \tag{S14.1}$$

At equilibrium, then, the linearized loop transmission is

$$\text{L.T.} = \begin{cases} \dfrac{-10}{s(0.1s+1)}, & \theta_E = 2n\pi \\ \dfrac{10}{s(0.1s+1)}, & \theta_E = (2n+1)\pi \end{cases} \quad n = 0, 1, \ldots \tag{S14.2}$$

We see that the loop transmission is positive for odd multiples of π, which leads one to suspect that the loop will be unstable in this case. From a root locus perspective, for any positive d-c loop-transmission magnitude, the open-loop pole at $s = 0$ will move into the right-half plane, resulting in an unstable system. Solving for the closed-loop poles as the roots of $1 - \text{L.T.}$ confirms this, as there is a right-half-plane pole at $s = 6.2\ \text{sec}^{-1}$ for the equilibria at $\theta_E = (2n + 1)\pi$, $n = 0, 1, \ldots$, whereas both poles are in the left-half plane for $\theta_E = n\pi$, $n = 0, 1, \ldots$. Thus, the equilibria are unstable when $\theta_E = (2n + 1)\pi$ and stable when $\theta_E = n\pi$, for all integers n.

(b) In the steady state, the output will also ramp at 7 rad/sec. That is, we will have $\dot{\theta}_O = 7$ rad/sec. From the block diagram, $\dot{\theta}_O$ is related to v_E as

$$\dot{\theta}_O = \frac{10}{0.1s + 1} v_E \tag{S14.3}$$

Steady-state conditions are evaluated by setting $s = 0$, to find $v_E = 0.1\,\dot{\theta}_O = 0.7$. Then, the steady-state θ_E is $\theta_E = \sin^{-1} v_E = 0.78$ radians. Note that this is an exact value for the operating point, but because θ_E is small, it is close to the error predicted by the model linearized about zero, which would be 0.7 radians. However, the slope of the sine function at 0.78 radians is quite different from unity, so we should linearize about $\theta_E = 0.78$ radians to maintain accuracy in the incremental analysis.

(c) The incremental gain of the resolver at this operating point is $\left.\frac{dv_E}{d\theta_E}\right|_{0.78} = 0.71$. Using this incremental gain, we solve for the closed-loop poles, which are located at $s = -5.0 \pm j6.7$. Thus, the angle error θ_E will take a positive step of about 0.01 radians, and the output angle will settle down to a ramp of 7.1 rad/sec. Both of these changes occur with an initial second-order transient characterized by $\omega_n = 8.4$ rad/sec, and $\zeta = 0.6$.

Solution 14.3 (P6.3)

Applying Equation 6.10 from the textbook, the closed-loop poles are located at the roots of $1 + V_B a(s)/20$. Here we are given $a(s) = \frac{3 \times 10^5}{(s+1)(10^{-5}s+1)^2}$, and look for the range of V_B for which the closed-loop poles remain in the left-half plane. A Routh analysis indicates that two closed-loop poles lie in the right-half plane for $V_B > 13.3$, and a single pole lies in the right-half plane for $V_B < -6.7 \times 10^{-5} \approx 0$. Between these values, all poles are in the left-half plane. Thus, the loop *is* stable for the specified input ranges.

For the square-root circuit, the ideal input–output relationship is found by applying the virtual ground method, as in Section 6.2.2. This yields

$$v_I + v_B = 0 \tag{S14.4}$$

and

$$v_B = \frac{v_A^2}{10} = \frac{v_O^2}{10} \tag{S14.5}$$

Solving Equations S14.4 and S14.5 for v_O in terms of v_I yields the ideal relationship

$$v_O = \sqrt{-10v_I}, \quad v_I < 0 \tag{S14.6}$$

Note that v_I must be negative for a real solution to exist. Applying Equation 6.3 to Equation S14.5 shows that

$$V_B + v_b = \frac{V_O^2}{10} + \frac{V_O}{5} v_o \tag{S14.7}$$

The incremental portion of this equation is

$$v_b = \frac{V_O}{5} v_o \tag{S14.8}$$

Then, the incremental dependence of V_o on V_i is given by

$$\frac{V_o(s)}{V_i(s)} = \frac{-\frac{1}{2}a(s)}{1 + \dfrac{V_O a(s)}{10}} \tag{S14.9}$$

With the given $a(s)$, a Routh analysis indicates that the poles of this expression are in the left-half plane for $0 < V_O < 6.67$. Thus, because by Equation S14.6, $v_I = -\dfrac{v_O^2}{10}$, the system will be stable for $-4.44 < V_I < 0$.

Describing Functions 15

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 15.1 (P6.6)

This problem is most readily solved by recognizing that the given nonlinearity can be represented by combining two elements as shown in Figure S15.1.

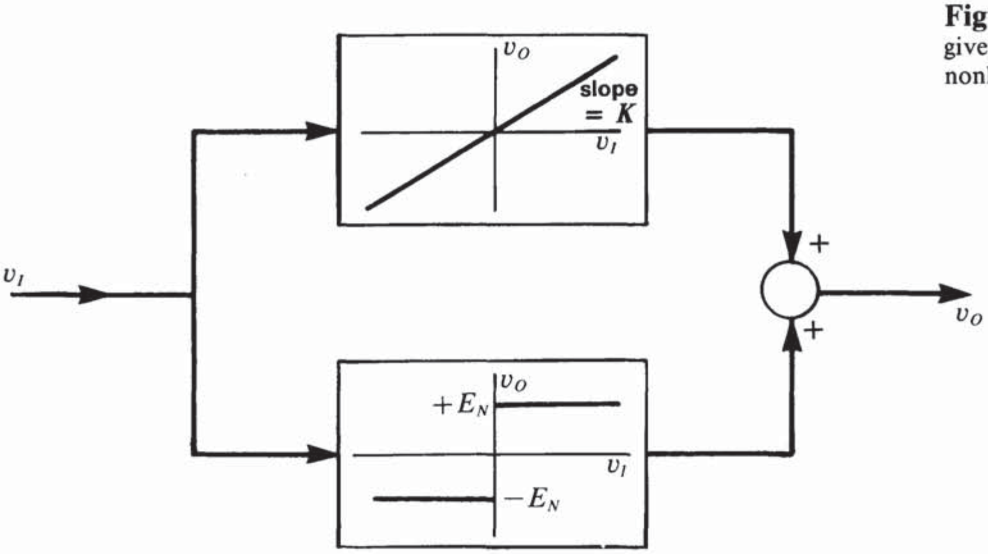


Figure S15.1 Decomposition of given nonlinearity into two nonlinearities.

The describing function for the upper element of Figure S15.1 is simply its linear gain K with zero phase. The describing function for the lower element of Figure S15.1 is tabulated in Table 6.1 of the textbook as $\frac{4E_N}{\pi E} \measuredangle 0°$. Then, the overall describing function is the sum of the two individual describing functions, and is given by

$$G_D(E) = \left(K + \frac{4E_N}{\pi E}\right) \measuredangle 0° \qquad \textbf{(S15.1)}$$

Solution 15.2 (P6.7)

Following the analysis of Chapter 6 in the textbook, stable oscillations may exist where $a(j\omega) = \frac{-1}{G_D(E)}$ when the system is in the form shown in Figure 6.9. The nonlinear oscillator of Figure 6.26 is in the appropriate form, with $a(s) = \frac{10}{(s+1)(0.1s+1)(0.01s+1)}$. For the given nonlinearity, $G_D(E) = \frac{4}{\pi E} \measuredangle 0°$, as tabulated in Table 6.1. Thus, oscillations of frequency ω can exist where

$$\frac{10}{(j\omega+1)(0.1j\omega+1)(0.01j\omega+1)} = -\frac{\pi E}{4} \qquad \textbf{(S15.2)}$$

Notice that the phase of the right-hand side of Equation S15.2 is $-180°$ for all ω.

A rough sketch of the Bode plot of $a(s)$ shows that the phase of the left-hand side is $-180°$ at about the geometric mean of the breakpoints at 10 and 100 rad/sec, which is $\omega \simeq 32$ rad/sec. An exact solution indicates that equality in S15.2 occurs for $\omega = 33$ rad/sec. At this frequency, the magnitude of $a(s)$ is 8.3×10^{-2}. Thus to satisfy S15.2, we must have $8.3 \times 10^{-2} = \frac{\pi E}{4}$, which is solved by $E = 0.11$ volts. This is the amplitude of the signal into the nonlinearity. (Note that the peak-to-peak value of the signal into the nonlinearity is $2E$ or 0.22 volts.)

The above analysis can also be carried out graphically in the gain-phase plane. Such a graphical analysis will appear very similar to the plot in Figure 6.13 of the textbook. This analysis will also verify that the oscillation at 33 rad/sec is stable, because following the discussion in Section 6.3.2, increasing E moves the point on the $-\frac{1}{G_D(E)}$ curve upwards, and thus to the left of the $a(j\omega)$ curve.

We have seen that the signal v_A is a 33 rad/sec sinusoid with an amplitude of 0.11 volts. Thus, v_B is a square wave in phase with v_A and with an amplitude of 1 volt (2 volts peak to peak). Now, consider the level of the third harmonic at the output of the nonlinearity. By the usual Fourier series calculations we find that the amplitude of the third harmonic is ⅓ that of the fundamental. Thus, because the fundamental of v_B has an amplitude of $4/\pi = 1.27$ volts, the third harmonic of v_B has an amplitude of 0.42 volts. Of course, this third harmonic is at a frequency of $3 \times 33 \simeq 100$ rad/sec, and is thus attenuated by a factor of about 0.007 by the third-order transfer function that filters v_B. Thus, the amplitude of the third harmonic in v_A is 0.42 volts $\times$ 0.007 = 0.0029 volts. The ratio in v_A of the amplitude of the third harmonic to the fundamental, then, is given by $\dfrac{0.0029 \text{ volts}}{0.11 \text{ volts}} = 0.027$.

Describing Functions (cont.) 16

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 16.1 (P6.8)

Here, the describing function for the nonlinear element can be constructed as the sum of a linear gain of unity, and a nonlinear element of the form of the third entry in Table 6.1, with $E_M = 1$, and $K = 1$. Thus, for the overall nonlinear element, the describing function is

$$G_D(E) = 1 \measuredangle 0^\circ \quad E < 1 \qquad \textbf{(S16.1}\boldsymbol{a}\textbf{)}$$

$$G_D(E) = 1 + [1 - \frac{2}{\pi}(\sin^{-1}R + R\sqrt{1 - R^2})] \measuredangle 0^\circ \quad E > 1 \qquad \textbf{(S16.1}\boldsymbol{b}\textbf{)}$$

where $R = \dfrac{1}{E}$.

The loop of Figure 6.27 is of the form in Figure 6.9, with $a(s) = \dfrac{5}{(\tau s + 1)^3}$. Thus, oscillations may be possible if particular values E_1 and ω_1 exist such that

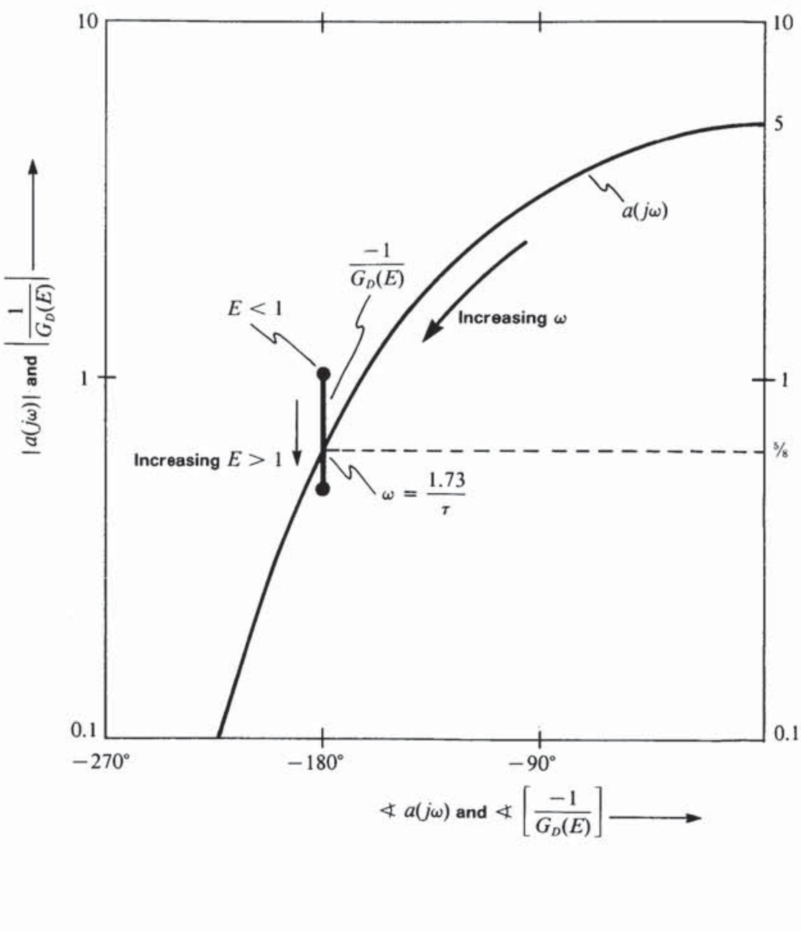
$$a(j\omega_1) = -\frac{1}{G_D(E_1)} \qquad \textbf{(S16.2)}$$

Because the phase of $G_D(E)$ is zero for all E, the only solution of Equation S16.2 occurs where $\measuredangle\, a(j\omega_1) = -180^\circ$. For the given $a(s)$, this requires that $-3 \tan^{-1}\tau\omega_1 = -180^\circ$, which is satisfied by $\omega_1 = \dfrac{1.73}{\tau}$. At this frequency, the magnitude of $a(s)$ is given by $|a(j\omega_1)| = \dfrac{5}{(\sqrt{\tau^2\omega_1^2 + 1})^3} = 5/8$. Thus, to satisfy Equation S16.2, we must have $G_D(E_1) = 8/5$ for oscillations to exist. Note that in a describing function sense, the gain of the nonlinearity is 1 for signals less than 1 volt in amplitude. For signals of greater than 1 volt amplitude the gain varies monotonically from 1 to 2 as the signal amplitude varies from 1 to infinity. The above statement fits our intuitive view of the behavior of the nonlinearity and is stated mathematically in Equation S16.1.

With the above information, we can plot $-\dfrac{1}{G_D(E)}$ and $a(j\omega)$ on a gain-phase plane, without solving explicitly for numerical values of $G_D(E)$. This plot is shown in Figure S16.1, and the intersection of the two curves indicates that oscillations may exist. However, they are not stable, as the following analysis shows. Consider the system to be oscillating with frequency $\omega = \dfrac{1.73}{\tau}$ and an amplitude E such that $G_D(E) = 8/5$. A slight increase in E will move the $\dfrac{-1}{G_E(E)}$ point to the right of the $a(j\omega)$ curve implying growing amplitude oscillations. Similarly, if E decreases, decreasing amplitude oscillations result. Thus, a stable amplitude limit cycle is not possible, even though the curves intersect.

This result can be verified by performing a linearized analysis about operating points V_A. For $|V_A| < 1$, the incremental gain of the nonlinearity is 1, and the linearized system can be shown to be stable. For all operating points with $|V_A| > 1$, the incremental gain of the nonlinearity is 2 and the system can be shown to be unstable. Thus, there is no operating point that is marginally stable (poles on $j\omega$ axis), and no stable amplitude oscillations can exist.

Figure S16.1 Describing function analysis for Problem 16.1 (P6.8).



S O L U T I O N S

Conditional Stability 17

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 17.1 (P6.9)

The limiter characteristics are shown in the first entry of Table 6.1. For the purposes of this problem, let $K = E_M = 1$. Then, from Table 6.1,

$$G_D(E) = 1 \measuredangle 0° \quad E < 1 \qquad \textbf{(S17.1}\textbf{\textit{a}}\textbf{)}$$

$$G_D(E) = \frac{2}{\pi}(\sin^{-1}R + R\sqrt{1 - R^2}) \measuredangle 0° \quad E > 1 \qquad \textbf{(S17.1}\textbf{\textit{b}}\textbf{)}$$

where $R = \dfrac{1}{E}$.

We want to combine this nonlinearity with a transfer function $a(s)$ in a loop of the form shown in Figure 6.9. Given this topology, and the $G_D(E)$ in Equation S17.1, a gain-phase plot of $-\dfrac{1}{G_D(E)}$ and $a(j\omega)$ that will yield stable amplitude limit cycles at two different frequencies is shown in Figure S17.1. The two intersections with positive slope of the $a(j\omega)$ curve represent stable oscillation points, as shown.

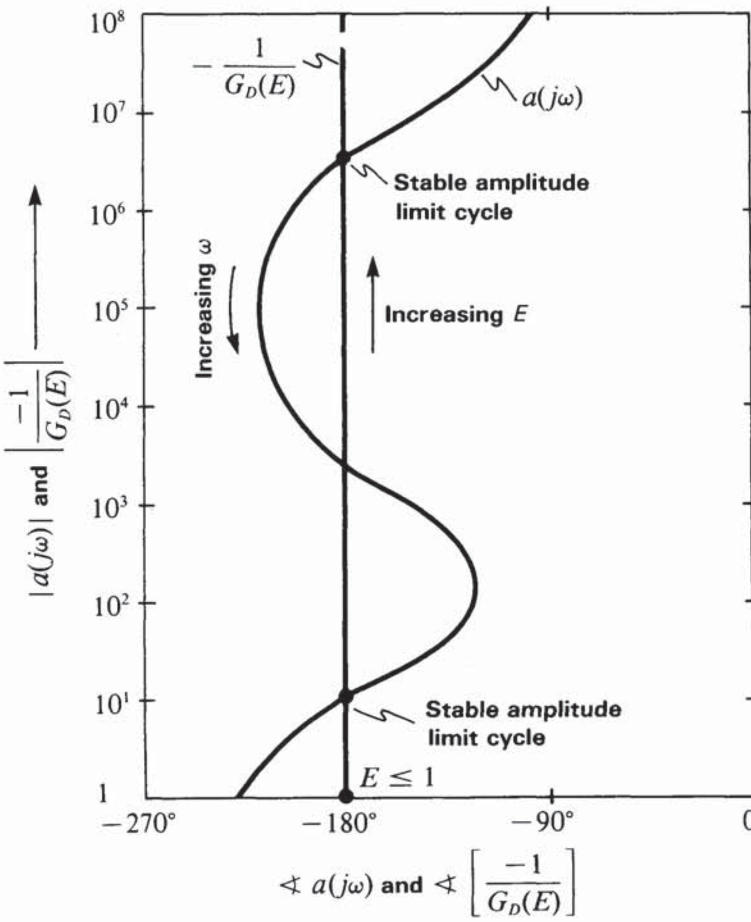
An $a(j\omega)$ that realizes the curve indicated in Figure S17.1 is

$$a(s) = \frac{K(0.05s + 1)^2}{s(s + 1)^2(10^{-3}s + 1)^2} \qquad \textbf{(S17.2)}$$

There are many possible $a(j\omega)$ that will result in two stable oscillation points, and the $a(j\omega)$ indicated in Equation S17.2 is just one of them. They all have the common characteristic of two distinct regions of the $a(j\omega)$ curve where $\measuredangle\, a(j\omega) < -180°$. These regions are separated by a region where $\measuredangle\, a(j\omega) > -180°$. This is one of those problems where there will be as many different solutions as there are students.

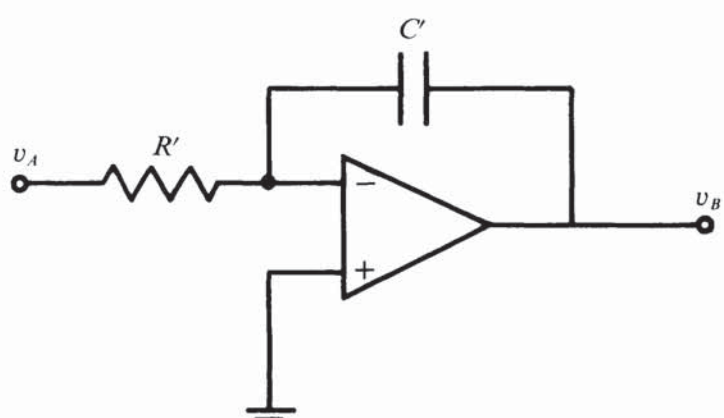
Given the $a(s)$ of Equation S17.2, we notice that it combines an integrator, two identical lag networks with $\alpha = 20$ formed by the poles at $s = -1$ and the zeros at $s = -20$, and a pair of poles at $s = -10^3$. The constant K is used to adjust the amplitude of the oscillations.

Figure S17.1 Describing function analysis for system of Problem 17.1 (P6.9) with two stable amplitude limit cycles.



The circuit shown in Figure S17.2 realizes the integration and provides a gain constant $G = \frac{-1}{R'C'}$. The negative sign of G supplies the inversion indicated in Figure 6.9.

Figure S17.2 Integration for Problem 17.1 (P6.9).



Using the results of Section 5.2.3, we can design a lag network with its pole at $s = -1$ and its zero at $s = -20$. That is, $\alpha\tau = 1$, and $\tau = 0.05$, $\alpha = 20$. This network appears as shown in Figure S17.3.

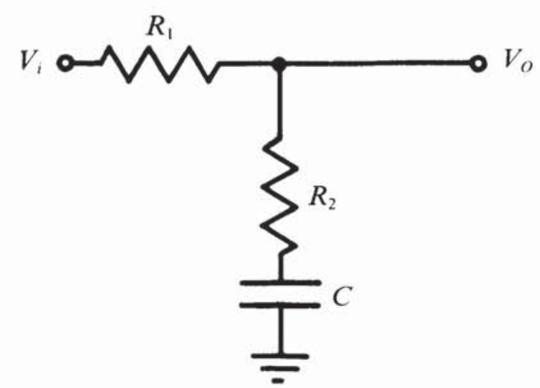


Figure S17.3 Lag network for Problem 17.1 (P6.9).

Now, if we cascade two such networks, and arrange to have the input impedance of the second network much larger than the output impedance of the first network, loading will be insignificant. Thus, for the first network, because $\alpha = \dfrac{R_1 + R_2}{R_2}$ and $\tau = R_2C$, let $R_1 = 100$ kΩ. Then to set $\alpha = 20$, $R_2 = 5.26$ kΩ. Then, because $\tau = 0.05$, we have $C = 9.5$ μF. For the second network, to reduce loading, we multiply the resistances of the first network by 20 and divide the capacitance by 20, to maintain the same α and τ. (An operational amplifier acting as a buffer between the two sections could be used to reduce loading further.) With this impedance scaling, the second lag network uses $R_1 = 2$ MΩ, $R_2 = 105$ kΩ, and $C = 0.48$ μF. The cascaded lag networks form the center section of the circuit shown in Figure S17.4.

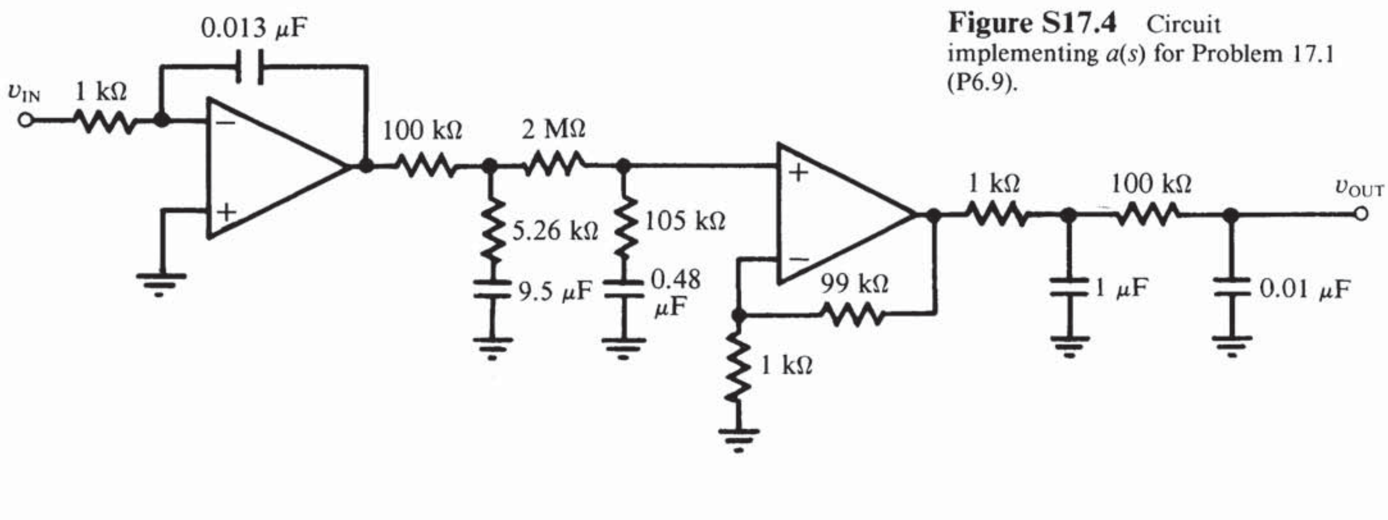


Figure S17.4 Circuit implementing $a(s)$ for Problem 17.1 (P6.9).

We follow the lag networks with a gain of 100 connected operational amplifier to act as a buffer and provide gain. Finally, the two poles at $s = -10^3$ must be implemented. As shown in Figure S17.4, we choose to implement these poles with the cascade of two first-order sections with $RC = 10^{-3}$. Again, the impedance of the second section is scaled up relative to the first section (in this case by a factor of 100) to minimize loading, and make our calculations easier.

Finally, the integrator gain constant G must be adjusted to ensure that the $|a(j\omega)| > 1$ when $\measuredangle\, a(j\omega)$ crosses through $-180°$ for the last time. Calculations show that this occurs for $\omega = 10^3$ rad/sec, and if G is chosen as $G = -8 \times 10^4$ (recall that we also have a gain of 100 from the second amp, thus $K = 100\ G$), this crossing will occur with $|a(j\omega)| = 10$. This ensures an intersection with the $-\dfrac{1}{G_D(E)}$ curve, which does not continue below $\left|\dfrac{1}{G_D(E)}\right| = 1$. Then, because $G = \dfrac{-1}{R'C'}$, we have $R'C' = 1.25 \times 10^{-5}$, which can be satisfied by $R' = 1\ \text{k}\Omega$, and $C' = 0.013\ \mu\text{F}$. The complete circuit is shown in Figure S17.4.

S O L U T I O N S

Oscillators (Intentional) 18

Note: All references to Figures and Equations whose numbers are *not* preceded by an "S" refer to the textbook.

Solution 18.1 (P12.1)

From Equation 12.1, with $\omega = \frac{1}{RC}$,

$$V_a\left(\frac{j}{RC}\right) = \frac{j}{j^2 + 3j + 1} = \frac{1}{3} \qquad \textbf{(S18.1)}$$

Thus, at $\omega = \frac{1}{RC}$, the feedback path to the noninverting terminal has the same transfer function as the feedback path to the inverting terminal. Thus, the voltages at both terminals are equal.

The modified topology will not function as an oscillator because in this case, the resistive positive feedback makes the op-amp connection unstable.

Solution 18.2 (P12.2)

To make the example specific let the parallel leg resistance increase by 5% to 1.05 R. Then,

$$\frac{V_a(s)}{V_o(s)} = \frac{1.05\ RCs}{1.05\ R^2C^2s^2 + 3.1\ RCs + 1} \qquad \textbf{(S18.2)}$$

Now, let the closed-loop gain of the noninverting connection equal k. (In Section 12.1.1, $k = 3$.) Then, the characteristic equation is:

$$\begin{aligned} 1 - L(s) &= 1 - \frac{1.05\ kRCs}{1.05\ R^2C^2s^2 + 3.1\ RCs + 1} \\ &= \frac{1.05\ R^2C^2s^2 + (3.1 - 1.05k)\ RCs + 1}{1.05\ R^2C^2s^2 + 3.1\ RCs + 1} \end{aligned} \qquad \textbf{(S18.3)}$$

This has imaginary zeros at $s = \frac{\pm j}{\sqrt{1.05}RC}$ when $k = 2.95$. Thus, the component values in the resistive network must satisfy $\frac{R_1 + R_2}{R_2} = 2.95$ where R_1 connects the amplifier output to the inverting input, and R_2 is connected between the inverting input and ground. This is satisfied by $R_1 = 1.95\ R_2$, a 2.5% change in R_1. (In Section 12.1.1, $R_1 = 2R_2$.)

Solution 18.3 (P12.4)

Consider the double integrator of Figure 11.12 with the $R/2$ valued resistor replaced by a resistance of $\frac{R}{2}(1 + \Delta)$, where Δ is the fractional change in resistance. Then, Equation 11.21 becomes

$$I_f(s) = \frac{(1 + \Delta)RC^2s^2}{2((1 + \Delta)RCs + 1)} V_o(s) \qquad \textbf{(S18.4)}$$

Then, using Equation 11.20, and applying the constraint $I_f = -I_i$ yields

$$\frac{V_o(s)}{V_i(s)} = -\frac{(1 + \Delta)RCs + 1}{(RCs + 1)(1 + \Delta)(RCs)^2} \qquad \textbf{(S18.5)}$$

Note that if $\Delta = 0$ this reduces to the original Equation 11.22, as expected.

Then, with the output of the double integrator connected back to its input, the loop transmission $L(s)$ is given by Equation S18.5. The characteristic equation then is

$$\begin{aligned} 1 - L(s) &= 1 + \frac{(1 + \Delta)RCs + 1}{(RCs + 1)(1 + \Delta)(RCs)^2} \\ &= \frac{R^3C^3(1 + \Delta)s^3 + R^2C^2(1+ \Delta)s^2 + RC(1 + \Delta)s + 1}{(RCs + 1)(1 + \Delta)(\mathrm{RCs})^2} \end{aligned} \qquad \textbf{(S18.6)}$$

This is identical with Equation 12.9, and thus for small Δ, Equation 12.10 applies. Therefore, the performance of the oscillator is dominated by a complex-conjugate root pair with $\omega_n \simeq \frac{1}{RC}$, and $\zeta \simeq \Delta/4$. The closed-loop poles can be made to lie in either the left half or the right half of the s plane according to the sign of Δ. Thus, by adjusting Δ, the envelope of the sinusoidal output may be made exponentially increasing ($\Delta < 0$) or exponentially decreasing ($\Delta > 0$). By this mechanism the amplitude can be controlled.

Now, because Equation 12.10 applies, following the analysis on pp. 490–91 of the textbook, the linearized transfer function relating envelope amplitude to Δ is

$$\frac{E_a(s)}{\Delta(s)} = -\frac{E_A}{4\,RCs} \qquad \textbf{(S18.7)}$$

as given by Equation 12.17. Note that we are letting $v_A(t)$ be equal to the double integrator output voltage $v_o(t)$ in order to conform to the notation of Section 12.1.4.

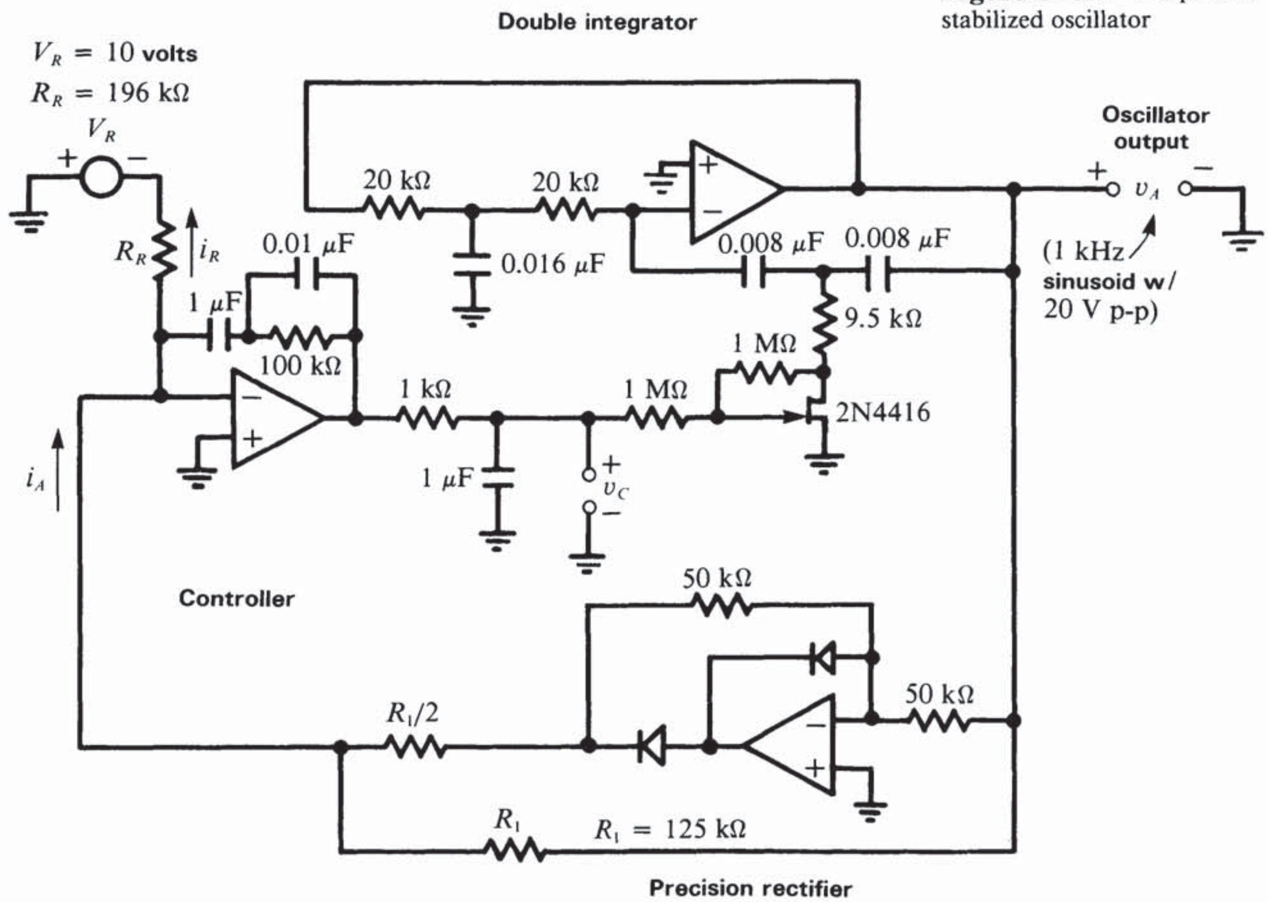


Figure S18.1 Amplitude stabilized oscillator

Now, as shown in Figure S18.1, we use the same FET circuit as in Figure 12.4, including the 9.5 kΩ resistor, so that nominally $R/2$ = 10 kΩ. (Thus, R = 20 kΩ.) Then, all the equations describing the FET in Section 12.1.4 apply. Specifically, from Equation 12.23

$$\left.\frac{\delta\Delta}{\delta v_C}\right|_{v_C=-4\text{V}} = -0.0125 \text{ V}^{-1} \quad \textbf{(S18.8)}$$

Now, to set the operating frequency to 1 kHz, we require that

$$\omega_n = \frac{1}{RC} = 2\pi \times 10^3 \quad \textbf{(S18.9)}$$

Because we have already determined that R = 20 kΩ, this is solved by C ≃ 0.008 μF. Thus, the components of the double-integrator loop are chosen as shown in Figure S18.1.

Also, for 20 volts peak-to-peak output, $E_A = 10$. With these values, Equation S18.7 becomes

$$\frac{E_a(s)}{\Delta(s)} = -\frac{1.57 \times 10^4}{s} \tag{S18.10}$$

Combining this with Equation S18.8 yields the linearized incremental relation between $E_a(s)$ and $V_c(s)$.

$$\frac{E_a(s)}{V_c(s)} = \frac{196}{s} \tag{S18.11}$$

Now, we turn our attention to the amplitude-measuring circuit and the controller circuit. As shown in Figure S18.1, we use a precision full-wave rectifier to provide an amplitude measurement. The controller amplifier provides a ground potential current-summing node, so only one additional amplifier is required to realize the precision rectifier. This is the same connection as is used in the precision phase shifter of Figure 12.32. The precision rectifier is discussed in more detail in Section 11.5.1.

The controller circuit has the same topology as in Figure 12.6. Note, however, that the R_1 and $R_1/2$ valued resistors of the precision rectifier replace the 312.5 kΩ input resistor of Figure 12.6. Because the loop in Figure 12.4 is very similar to the loop in Figure S18.1, we can use the same controller. That is, because they are oscillating at similar frequencies we can set crossover for both loops at 100 rad/sec. Further, the incremental relations between $V_c(s)$ and $E_a(s)$ (Equations 12.24 and S18.11) differ only in the gain constant. Thus, by simply scaling the gain constant of $a(s)$ (i.e., by varying the controller input resistor), the two loops can be made to have identical amplitude-control loop transmissions.

The only remaining subtlety is in determining the incremental relation between $E_a(s)$ and $V_c(s)$. With ω_c set to 100 rad/sec, the controller $a(s)$ effectively filters out all signal components at the oscillator frequency (1 kHz) and above. Thus, we can effectively ignore these harmonics, and focus on the propagation of the amplitude signal around the loop. That is, the amplitude-control loop is really feeding back on the amplitude parameter $e_A(t)$, and not on the detailed waveform $v_A(t)$.

Because it is full-wave rectified, the current into the controller summing junction $i_A(t)$ is always greater than or equal to zero and is given by $\frac{1}{R_1}\left| e_A(t) \sin \omega t \right|$. For slowly varying $e_A(t)$, the low frequency portion of this signal is given by finding the d-c Fourier component under the assumption that $e_A(t)$ is fixed. That is, the low-frequency current $i_{ALF}(t)$ into the summing junction is

$$
\begin{aligned}
i_{ALF}(t) &\simeq \frac{\omega}{R_1 \pi} \int_0^{\pi/\omega_0} e_A(t) \sin \omega t \, dt \\
&\simeq \frac{2}{R_1 \pi} e_A(t)
\end{aligned}
\qquad \textbf{(S18.12)}
$$

Now, the transfer function from this low-frequency current to voltage v_c is given by

$$
\frac{V_c(s)}{I_{alf}(s)} = -\frac{(0.1s + 1)}{10^{-6}s(10^{-3}s + 1)^2} \qquad \textbf{(S18.13)}
$$

Then, combining Equations S18.11, S18.12, and S18.13 yields the amplitude-control loop transmission

$$
\begin{aligned}
L(s) &= -\left(\frac{196}{s}\right)\left(\frac{2}{R_1 \pi}\right)\left[\frac{0.1s + 1}{10^{-6}s(10^{-3}s + 1)^2}\right] \\
&= \frac{-1.25 \times 10^8(0.1s + 1)}{R_1 s^2(10^{-3}s + 1)^2}
\end{aligned}
\qquad \textbf{(S18.14)}
$$

Equating this with the negative of Equation 12.26, so the loop transmissions are identical, yields $R_1 = 125$ kΩ. The system crossover frequency is 100 rad/sec and the phase margin exceeds 70°.

One consequence of the chosen topology is that the amplitude reference signal is negative. That is, when the loop is in equilibrium, the average current drawn by the reference $\left(i_R = \frac{V_R}{R_R}\right)$ will be equal to the average current i_{ALF} supplied by the precision rectifier. We choose R_R so that the steady-state amplitude of $v_A(t)$ will be equal to V_R. That is, equating i_{ALF} with i_R, and setting $V_R = e_A$ gives

$$
\frac{2}{R_1 \pi} e_A = \frac{V_R}{R_R} = \frac{e_A}{R_R} \qquad \textbf{(S18.15)}
$$

Given $R_1 = 125$ kΩ, this is solved by $R_R = 196$ kΩ. Then, for an output amplitude of 10 volts, $V_R = 10$ volts.