Unified Engineering II **Spring 2004**

Problem S10 (Signals and Systems) Solution

1. Because the numerator is the same order as the denominator, the partial fraction expansion will have a constant term:

$$\begin{aligned} G(s) &= \frac{3s^2+3s-10}{s^2-4} \\ &= \frac{3s^2+3s-10}{(s-2)(s+2)} \\ &= a + \frac{b}{s-2} + \frac{c}{s+2} \end{aligned}$$

   To find a, b, and c, use coverup method:

$$\begin{aligned} a &= G(s)\big|_{s=\infty} = 3 \\ b &= \left.\frac{3s^2+3s-10}{s+2}\right|_{s=2} = 2 \\ c &= \left.\frac{3s^2+3s-10}{s-2}\right|_{s=-2} = 1 \end{aligned}$$

   So

$$G(s) = 3 + \frac{2}{s-2} + \frac{1}{s+2}, \qquad \mathrm{Re}[s] > 2$$

   We can take the inverse LT by simple pattern matching. The result is that

$$g(t) = 3\delta(t) + \left(2e^{2t} + e^{-2t}\right)\sigma(t)$$

2.

$$\begin{aligned} G(s) &= \frac{6s^2+26s+26}{(s+1)(s+2)(s+3)} \\ &= \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{s+3} \end{aligned}$$

   Using partial fraction expansions,

$$\begin{aligned} a &= \left.\frac{6s^2+26s+26}{(s+2)(s+3)}\right|_{s=-1} = 3 \\ b &= \left.\frac{6s^2+26s+26}{(s+1)(s+3)}\right|_{s=-2} = 2 \\ c &= \left.\frac{6s^2+26s+26}{(s+1)(s+2)}\right|_{s=-3} = 1 \end{aligned}$$

So

$$G(s) = \frac{3}{s+1} + \frac{2}{s+2} + \frac{1}{s+3}, \qquad \text{Re}[s] > -1$$

The inverse LT is given by

$$\left(3e^{-t} + 2e^{-2t} + e^{-3t}\right)\sigma(t)$$

3. This one is a little tricky — there is a second order pole at $s = -1$. So the partial fraction expansion is

$$G(s) = \frac{4s^2 + 11s + 9}{(s+1)^2(s+2)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{c}{s+2}$$

We can find b and c by the coverup method:

$$\begin{aligned} b &= \left.\frac{4s^2 + 11s + 9}{s+2}\right|_{s=-1} = 2 \\ c &= \left.\frac{4s^2 + 11s + 9}{(s+1)^2}\right|_{s=-2} = 3 \end{aligned}$$

So

$$G(s) = \frac{a}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{s+2}$$

To find a, pick a value of s, and plug into the equation above. The easiest value to pick is $s = 0$. Then

$$G(0) = \frac{a}{1} + \frac{2}{(1)^2} + \frac{3}{2} = \frac{9}{2}$$

Solving, we have

$$a = 1$$

Therefore,

$$G(s) = \frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{3}{s+2}, \qquad \text{Re}[s] > -1$$

The inverse LT is then

$$g(t) = \left(e^{-t} + 2te^{-t} + 3e^{-2t}\right)\sigma(t)$$

4. This problem is similar to above. The partial fraction expansion is

$$G(s) = \frac{4s^3 + 11s^2 + 5s + 2}{s^2(s+1)^2} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+1} + \frac{d}{(s+1)^2}$$

We can find b and d by the coverup method

$$\begin{aligned} b &= \left.\frac{4s^3 + 11s^2 + 5s + 2}{(s+1)^2}\right|_{s=0} = 2 \\ d &= \left.\frac{4s^3 + 11s^2 + 5s + 2}{s^2}\right|_{s=-1} = 4 \end{aligned}$$

So

$$G(s) = \frac{4s^3 + 11s^2 + 5s + 2}{s^2(s+1)^2} = \frac{a}{s} + \frac{2}{s^2} + \frac{c}{s+1} + \frac{4}{(s+1)^2}$$

To find a and c, pick two values of s, say, $s = 1$ and $s = 2$. Then

$$\begin{aligned} G(1) &= \frac{4 + 11 + 5 + 2}{1^2(1+1)^2} = \frac{a}{1} + \frac{2}{1^2} + \frac{c}{1+1} + \frac{4}{(1+1)^2} \\ G(2) &= \frac{4 \cdot 2^3 + 11 \cdot 2^2 + 5 \cdot 2 + 2}{2^2(2+1)^2} = \frac{a}{2} + \frac{2}{2^2} + \frac{c}{2+1} + \frac{4}{(2+1)^2} \end{aligned}$$

Simplifying, we have that

$$\begin{aligned} a + \frac{c}{2} &= \frac{5}{2} \\ \frac{a}{2} + \frac{c}{3} &= \frac{3}{2} \end{aligned}$$

Solving for a and c, we have that

$$\begin{aligned} a &= 1 \\ c &= 3 \end{aligned}$$

So

$$G(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{3}{s+1} + \frac{4}{(s+1)^2}$$

and

$$g(t) = \left(1 + 2t + 3e^{-t} + 4te^{-t}\right)\sigma(t)$$

5. $G(s)$ can be expanded as

$$\begin{aligned} G(s) &= \frac{s^3 + 3s^2 + 9s + 12}{(s^2+4)(s^2+9)} \\ &= \frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s-2j)(s+3j)(s-3j)} \\ &= \frac{a}{s+2j} + \frac{b}{s-2j} + \frac{c}{s+3j} + \frac{d}{s-3j} \end{aligned}$$

The coefficients can be found by the coverup method:

$$\begin{aligned} a &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s-2j)(s+3j)(s-3j)}\right|_{s=-2j} = 0.5 \\ b &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s+3j)(s-3j)}\right|_{s=+2j} = 0.5 \\ c &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s-2j)(s-3j)}\right|_{s=-3j} = 0.5j \\ d &= \left.\frac{s^3 + 3s^2 + 9s + 12}{(s+2j)(s-2j)(s+3j)}\right|_{s=+3j} = -0.5j \end{aligned}$$

Therefore

$$G(s) = \frac{0.5}{s+2j} + \frac{0.5}{s-2j} + \frac{0.5j}{s+3j} + \frac{-0.5j}{s-3j}, \qquad \mathrm{Re}[s] > 0$$

and the inverse LT is

$$g(t) = 0.5\left(e^{-2jt} + e^{2jt} + je^{-3jt} - je^{3jt}\right)\sigma(t)$$

This can be expanded using Euler's formula, which states that

$$e^{ajt} = \cos at + j\sin at$$

Applying Euler's formula yields

$$g(t) = (\cos 2t + \sin 2t)\,\sigma(t)$$