Introduction to Computers and Programming

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Reading:

Lecture 13
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Recap

- Iteration versus Recursion
- Towers of Hanoi
- Computed time taken to solve towers of Hanoi

Divide and Conquer

- It is an algorithmic design paradigm that contains the following steps
  - **Divide**: Break the problem into smaller sub-problems
  - **Recur**: Solve each of the sub-problems recursively
  - **Conquer**: Combine the solutions of each of the sub-problems to form the solution of the problem

Represent the solution using a recurrence equation

Recurrence Equation

- A recurrence equation is of the form

  $T(n) = aT(m) + b, \quad n > 0, m < n$ (induction)

  and

  $T(0)$ = constant (base case)

Where:

- $aT(m)$: cost of solving a sub-problems of size m
- b: cost of pulling together the solutions

Solving Recurrence Equations

- Iteration
- Recurrence Trees
- Substitution
- Master Method

Towers of Hanoi

Given: $T(1) = 1$

$T(n) = 2\,T(n-1) + 1$

N	No.Moves
1	1
2	3
3	7
4	15
5	31

Using Iteration

$T(n) = 2\, T(n-1) + 1$
$T(n) = 2\, [\, 2\, T(n-2) + 1\,] + 1$
$T(n) = 2\, [\, 2\, [\, 2\, T(n-3) + 1\,] + 1\,] + 1$
$T(n) = 2\, [\, 2\, [\, 2\, [\, 2\, T(n-4) + 1\,] + 1\,] + 1\,] + 1$
$T(n) = 2^4\, T(n-4) + 15$

...

$T(n) = 2^k\, T(n-k) + 2^k - 1$

Since n is finite, $k \to n$.
Therefore,

$$\lim T(n)_{k \to n} = \mathbf{2^n - 1}$$

Greatest Common Divisor

Given two natural numbers a, b

- If b = 0, then GCD := a
- If b /= 0, then
  - c := a MOD b
  - a := b
  - b := c
  - GCD(a,b)

[The MOD function]

- Notation: $\boldsymbol{m}$ **mod** $\boldsymbol{n}$ = x
- x = integer remainder when m is divided by n

  $= m - \lfloor m/n \rfloor n$
- Examples:
  - 8 mod 3 = 2
  - 42 mod 6 = 0
  - 5 mod 7 = 5

Extended Euclid's Algorithm

$GCD(a,b) = a\mathbf{p}+b\mathbf{q}$

$38 \bmod 10 = 8 \qquad = 38 - 3 * 10$

$10 \bmod 8 = 2 \qquad = 10 - 1 * 8$

$\qquad = 10 - 1*(38 - 3 * 10)$

$\qquad = 4*10 - 1 *38$

$8 \bmod 2 = 0$

$GCD(2,0) = 2$

"2" can be expressed as linear combination of 10 and 38 – Solve Diophantine Equations

Exercise

- Write 6 as an integer combination of 10 and 38
  - –Find GCD (38,10)
  - –Express the GCD as a linear combination of 38 and 10
  - –Multiply that expression by (6/GCD)

$$6 = 3\,(4*10 - 1*38)$$

$$= 12 * 10 - 3 * 38$$

Multiplication

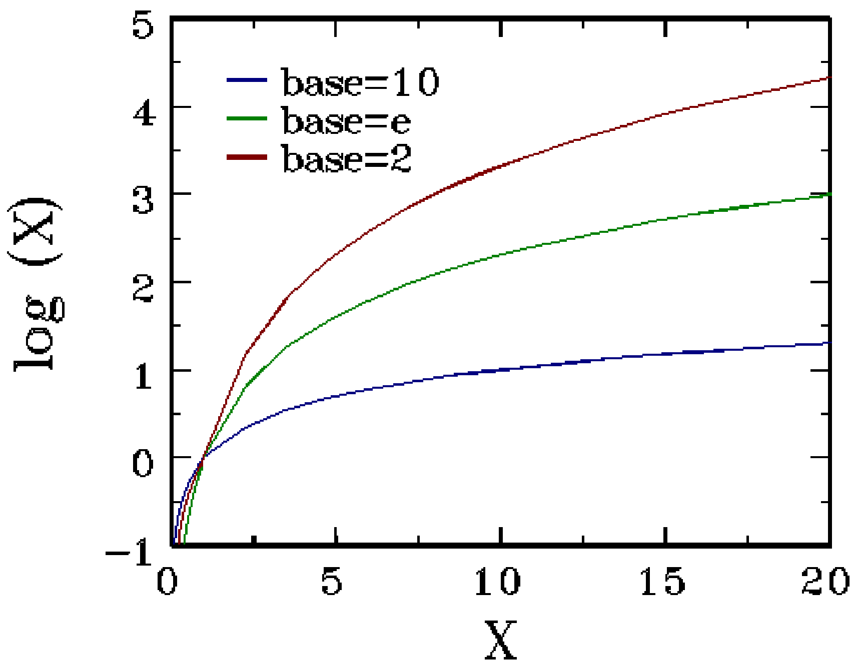
- Standard method for multiplying long numbers:
  $$(1000a+b)\times(1000c+d) = 1{,}000{,}000\ ac + 1000\ (ad + bc) + bd$$

- Instead use:
  $$(1000a+b)\times(1000c+d) =$$
  $$1{,}000{,}000\ ac + 1000\ ((a+b)(c+d) - ac - bd) + bd$$

One length-k multiply = 3 length-k/2 multiplies and a bunch of additions and shifting

[Logarithms – $\log_b(x)$]

- A logarithm of base *b* for value *y* is the power to which *b* is raised to get *y*.
  - $\log_b y = x \leftrightarrow b^x = y \leftrightarrow b^{\log_b y} = y$
  - $\log_b 1 = 0$, $\log_b b = 1$ for all values of b



PRS - 1

- Given $\mathbf{n}$, $\mathbf{n \log n}$, $\mathbf{n^2}$, $\mathbf{n(\log n)^2}$, for large n:

1. **n** has the largest value
2. **n log n** has the largest value
3. $\mathbf{n^2}$ has the largest value
4. $\mathbf{n(\log n)^2}$ has the largest value

Relative size of n, $n \log n$, n^2, $n(\log n)^2$

- $(n\log n)/n = \log n \rightarrow \infty$
  n is more efficient than $\log n$

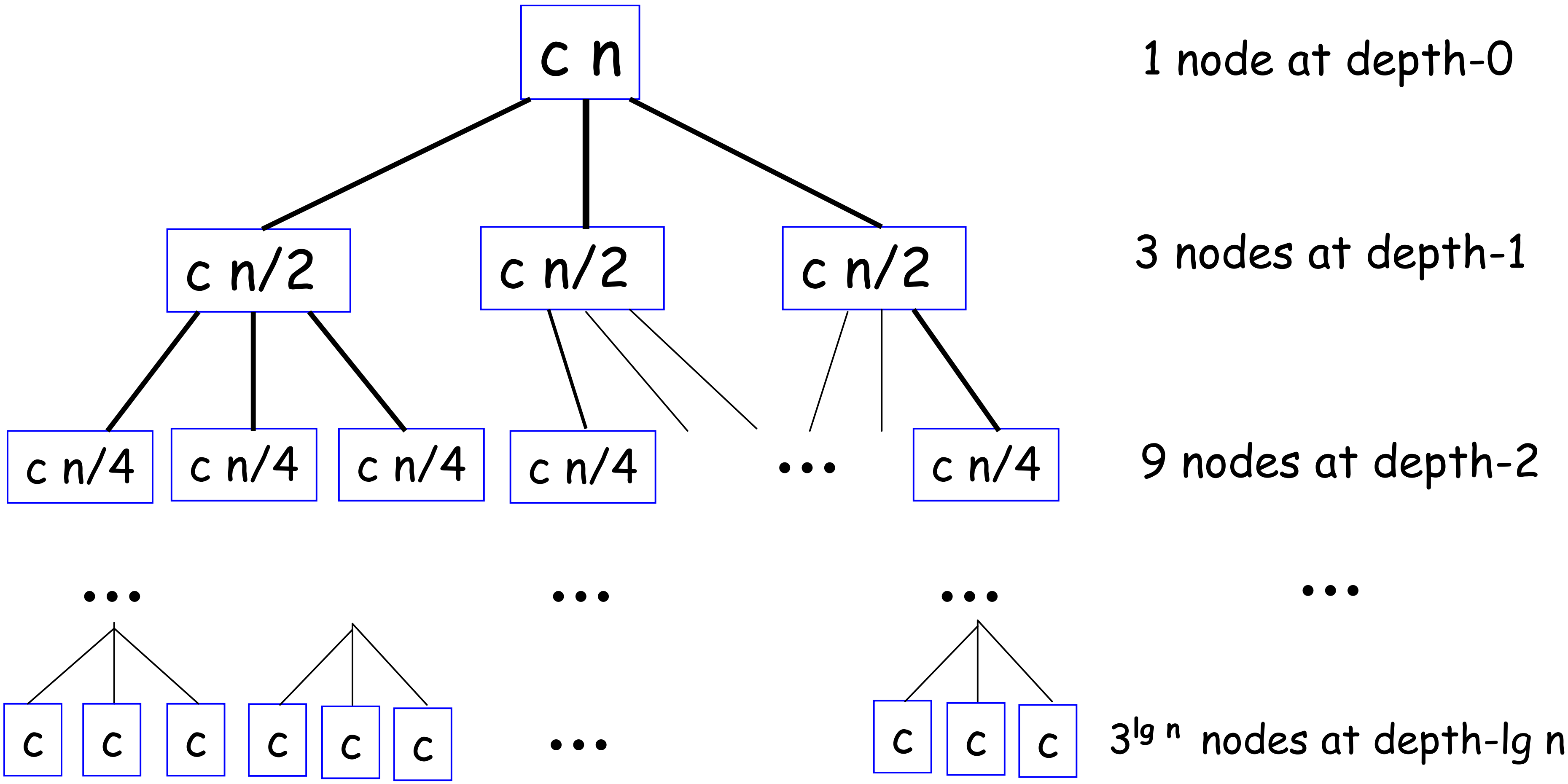
- $n(\log n)^2 / n\log n = \log n \rightarrow \infty$
  $n\log n$ is more efficient than $n(\log n)^2$

- $n(\log n)^2 / n^2 = (\log n)^2 / n \rightarrow 0$
  $n(\log n)^2$ is more efficient than n^2

- Order of efficiency is
  $\boldsymbol{n, n\log n, n(\log n)^2, n^2}$

Recurrence Tree

Recurrence Equation : T(n) < 3T(n/2) + c n



Solving using Recurrence Tree

$$
\begin{aligned}
T(n) \; & < cn\,(1 + 3(1/2) + 9(1/4) + \ldots + 3^{\lg n}(1/\,2^{\lg n})) \\
& < cn\,(1 + 3/2 + (3/2)^{2} + \ldots + (3/2)^{\lg n}\,). \\
& \\
& < cn\,(\,(3/2)^{(\lg n\,+\,1)} - 1)\,/\,((3/2)\text{-}1) \\
& < cn\,(\,(3/2)^{\lg n}\,(3/2) - 1)\,/\,(1/2) \\
& < ((c\,n\,(3/2)^{\lg n}\,(3/2))/(1/2)\,) - 2cn \\
& < c\,n\,(3/2)^{\lg n} - 2cn\,.
\end{aligned}
$$

$T(n) < 3\,c\,n\,(3/2)^{\lg n}$ --approximation

$3cn\,(n^{\lg(3/2)}) = 3c\,n^{1+\lg(3/2)}$

Important Theorems

Arithmetic Series

For $n \geq 1$, $1 + 2 + \ldots + n = n(n+1)/2$

Geometric Series

For $a \geq 1$, $a^{k} + a^{k-1} + \ldots 1 = (a^{k+1} - 1) / (a-1)$

Logarithmic Behavior

$a^{\lg b} = b^{\lg a}$

Recurrence Examples

$$s(n) = \begin{cases} 0 & n = 0 \\ c + s(n-1) & n > 0 \end{cases}$$

$$s(n) = \begin{cases} 0 & n = 0 \\ n + s(n-1) & n > 0 \end{cases}$$

$$T(n) = \begin{cases} c & n = 1 \\ 2T\left(\frac{n}{2}\right) + c & n > 1 \end{cases}$$

$$T(n) = \begin{cases} c & n = 1 \\ aT\left(\frac{n}{b}\right) + cn & n > 1 \end{cases}$$