Fluids – Lecture 17 Notes

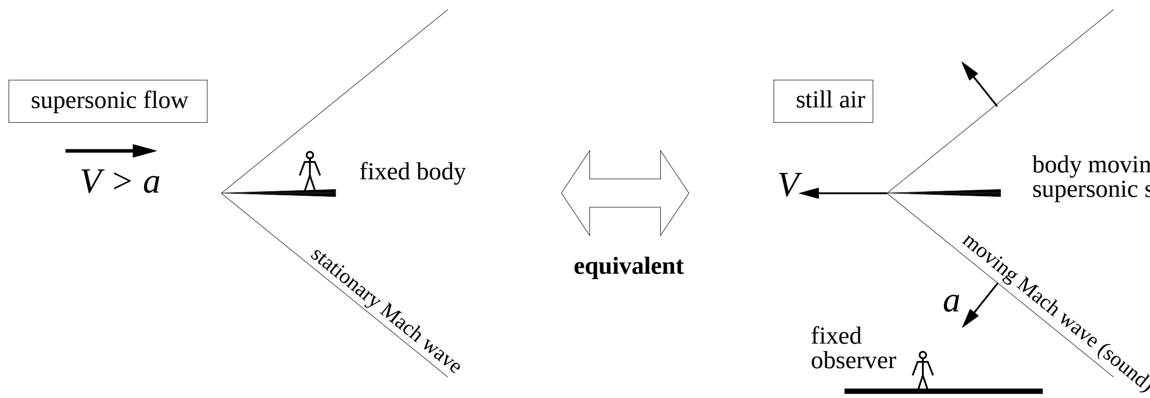
1. Oblique Waves

Reading: Anderson 9.1, 9.2

Oblique Waves

Mach waves

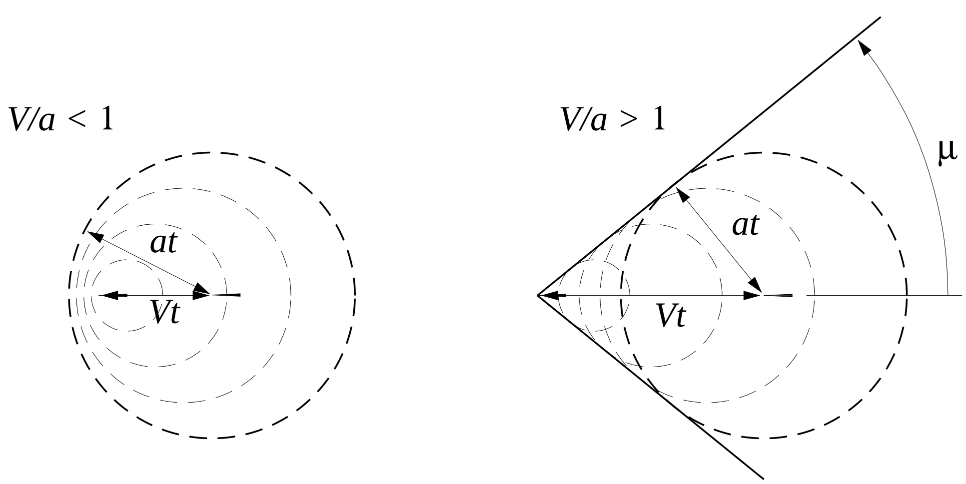
Small disturbances created by a slender body in a supersonic flow will propagate diagonally away as *Mach waves* These consist of small isentropic variations in ρ, V, p, and h, and are loosely analogous to the water waves sent out by a speedboat. Mach waves appear stationary with respect to the object generating them, but when viewed relative to the still air, they are in fact indistinguishable from sound waves, and their normal-direction speed of propagation is equal to a, the speed of sound.



The angle μ of a Mach wave relative to the flow direction is called the *Mach angle* It can be determined by considering the wave to be the superposition of many pulses emitted by the body, each one producing a disturbance circle (in 2-D) or sphere (in 3-D) which expands at the speed of sound a. At some time interval t after the pulse is emitted, the radius of the circle will be at, while the body will travel a distance Vt. The Mach angle is then seen to be

$$\mu \;=\; \arcsin \frac{at}{Vt} \;=\; \arcsin \frac{1}{M}$$

which can be defined at any point in the flow. In the subsonic flow case where $M = V/a < 1$ the expanding circles do not coalesce into a wave front, and the Mach angle is not defined.

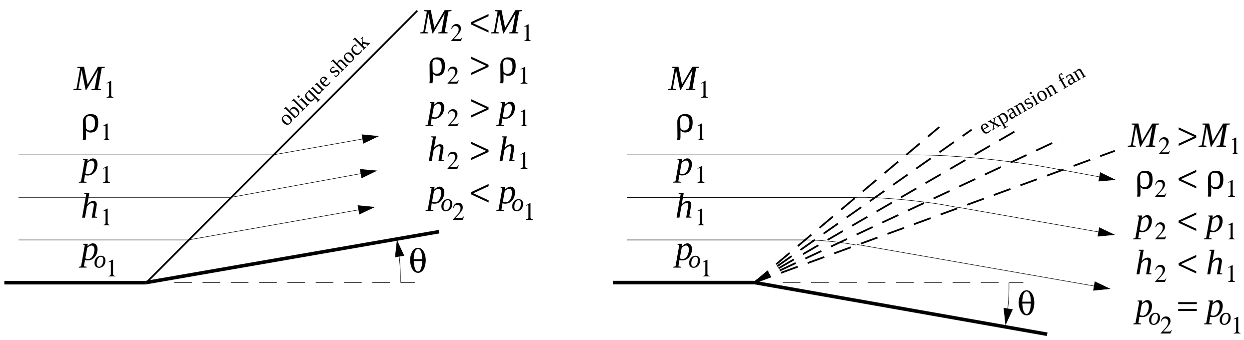


Oblique shock and expansion waves

Mach waves can be either compression waves ($p_2 > p_1$) or expansion waves ($p_2 < p_1$), but in either case their strength is by definition very small ($|p_2 - p_1| \ll p_1$). A body of finite thickness, however, will generate oblique waves of finite strength, and now we must distinguish between compression and expansion types. The simplest body shape for generating such waves is

– a concave corner, which generates an *oblique shock*(compression), or
– a convex corner, which generates an *expansion fan*

The flow quantity changes across an oblique shock are in the same direction as across a

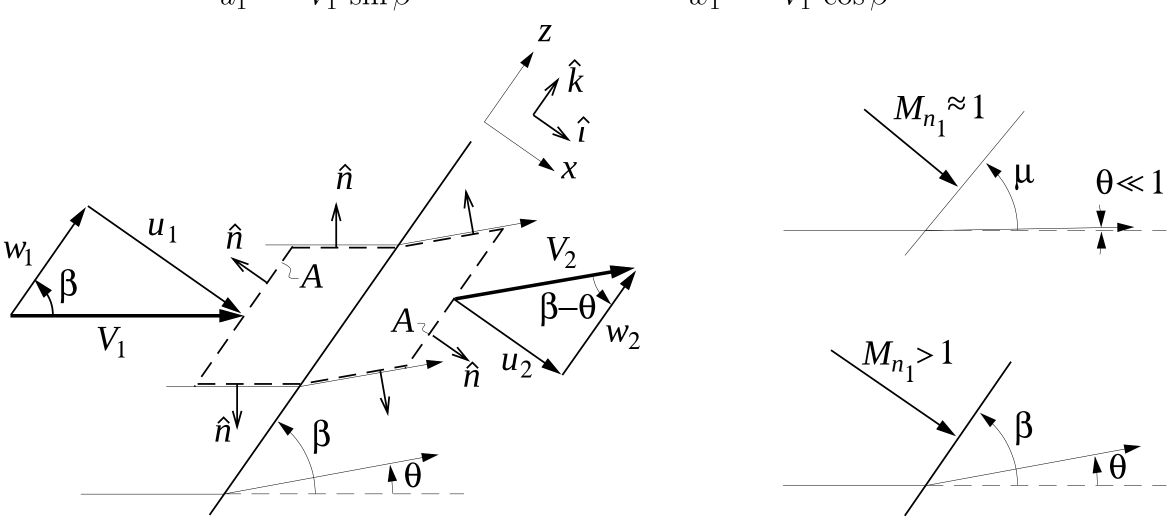


normal shock, and across an expansion fan they are in the opposite direction. One important difference is that p_o decreases across the shock, while the fan is isentropic, so that it has no loss of total pressure, and hence $p_{o_2} = p_{o_1}$.

Oblique geometry and analysis

As with the normal shock case, a control volume analysis is applied to the oblique shock flow, using the control volume shown in the figure. The top and bottom boundaries are chosen to lie along streamlines so that only the boundaries parallel to the shock, with area A, have mass flow across them. Velocity components are taken in the x-z coordinates normal and tangential to the shock, as shown. The tangential z axis is tilted from the upstream flow direction by the *wave angle*β, which is the same as the Mach angle μ only if the shock is extremely weak. For a finite-strength shock, $\beta > \mu$. The upstream flow velocity components are

$$u_1 \;=\; V_1 \,\sin\beta \qquad\qquad w_1 \;=\; V_1 \,\cos\beta$$



All the integral conservation equations are now applied to the control volume.

Mass continuity

$$\oiint \rho \vec{V}\cdot\hat{n}\, dA = 0$$
$$-\rho_1 u_1 A + \rho_2 u_2 A = 0$$
$$\rho_1 u_1 = \rho_2 u_2 \tag{1}$$

x-Momentum

$$\oiint \rho \vec{V}\cdot\hat{n}\, u\, dA + \oiint p\hat{n}\cdot\hat{\imath}\, dA = 0$$
$$-\rho_1 u_1^2 A + \rho_2 u_2^2 A - p_1 A + p_2 A = 0$$
$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{2}$$

z-Momentum

$$\oiint \rho \vec{V}\cdot\hat{n}\, w\, dA + \oiint p\hat{n}\cdot\hat{k}\, dA = 0$$
$$-\rho_1 u_1 A\, w_1 + \rho_2 u_2 A\, w_2 = 0$$
$$w_1 = w_2 \tag{3}$$

Energy

$$\oiint \rho \vec{V}\cdot\hat{n}\, h_o\, dA = 0$$
$$-\rho_1 u_1 h_{o_1} A + \rho_2 u_2 h_{o_2} A = 0$$
$$h_{o_1} = h_{o_2}$$
$$h_1 + \frac{1}{2}\left(u_1^2 + w_1^2\right) = h_2 + \frac{1}{2}\left(u_2^2 + w_2^2\right)$$
$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 \tag{4}$$

Equation of State

$$p_2 = \frac{\gamma - 1}{\gamma}\rho_2 h_2 \tag{5}$$

Simplification of equation (3) makes use of (1) to eliminate $\rho u A$ from both sides. Simplification of equation (4) makes use of (1) to eliminate $\rho u A$ and then (3) to eliminate w from both sides.

Oblique/normal shock equivalence

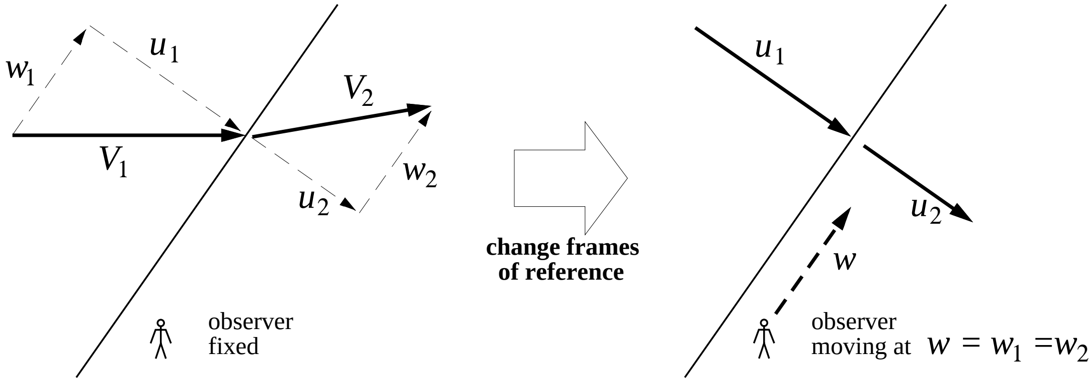
It is apparent that equations (1), (2), (4), (5) are in fact identical to the normal-shock equations derived earlier. The one addition z-momentum equation (3) simply states that the tangential velocity component doesn't change across a shock. This can be physically interpreted if we examine the oblique shock from the viewpoint of an observer moving with the everywhere-constant tangential velocity $w = w_1 = w_2$. As shown in the figure, the moving observer sees a normal shock with velocities u_1, and u_2. The static fluid properties p, ρ, h, a are of course the same in both frames.

Oblique shock relations

The effective equivalence between an oblique and a normal shock allows re-use of the already derived normal shock jump relations. We only need to construct the necessary transformation from one frame to the other.

First we define the *normal Mach number components* seen by the moving observer.

$$M_{n_1} \equiv \frac{u_1}{a_1} = \frac{V_1 \sin\beta}{a_1} = M_1 \sin\beta \tag{6}$$
$$M_{n_2} \equiv \frac{u_2}{a_2} = \frac{V_2 \sin(\beta-\theta)}{a_2} = M_2 \sin(\beta-\theta)$$



These are then related via our previous normal-shock $M_2 = f(M_1)$ relation, if we make the substitutions $M_1 \rightarrow M_{n_1}$, $M_2 \rightarrow M_{n_2}$. The fixed-frame M_2 then follows from geometry.

$$M_{n_2}^2 = \frac{1 + \frac{\gamma-1}{2} M_{n_1}^2}{\gamma M_{n_1}^2 - \frac{\gamma-1}{2}} \tag{7}$$

$$M_2 = \frac{M_{n_2}}{\sin(\beta - \theta)} \tag{8}$$

The static property ratios are likewise obtained using the previous normal-shock relations.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_{n_1}^2}{2 + (\gamma-1) M_{n_1}^2} \tag{9}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1} \left(M_{n_1}^2 - 1 \right) \tag{10}$$

$$\frac{h_2}{h_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2} \tag{11}$$

$$\frac{p_{o_2}}{p_{o_1}} = \frac{p_2}{p_1} \left(\frac{h_1}{h_2} \right)^{\gamma/(\gamma-1)} \tag{12}$$

To allow application of the above relations, we still require the wave angle β. Using the result $w_1 = w_2$, the velocity triangles on the two sides of the shock can be related by

$$\frac{\tan(\beta - \theta)}{\tan \beta} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{(\gamma+1) M_1^2 \sin^2 \beta}{2 + (\gamma-1) M_1^2 \sin^2 \beta}$$

Solving this for θ gives

$$\tan \theta = \frac{2}{\tan \beta} \frac{M_1^2 \sin^2 \beta - 1}{M_1^2 (\gamma + \cos 2\beta) + 2} \tag{13}$$

which is an implicit definition of the function $\beta(\theta, M_1)$.

Oblique-shock analysis: Summary

Starting from the known upstream Mach number M_1 and the flow deflection angle (body surface angle) θ, the oblique-shock analysis proceeds as follows.

$$\theta\,,\; M_1 \quad \xrightarrow{Eq.(13)} \quad \beta \quad \xrightarrow{Eq.(6)} \quad M_{n_1} \quad \xrightarrow{Eqs.(7)--(12)} \quad M_{n_2}\,,\; M_2\,,\; \frac{\rho_2}{\rho_1}\,,\; \frac{p_2}{p_1}\,,\; \frac{h_2}{h_1}\,,\; \frac{p_{o_2}}{p_{o_1}}$$

Use of equation (13) in the first step can be problematic, since it must be numerically solved to obtain the $\beta(\theta, M_1)$ result. A convenient alternative is to obtain this result graphically, from an oblique shock chart, illustrated in the figure below. The chart also reveals a number of important features:

1. There is a maximum turning angle $\theta_{\max}$ for any given upstream Mach number M_1. If the wall angle exceeds this, or $\theta > \theta_{\max}$, no oblique shock is possible. Instead, a *detached shock* forms ahead of the concave corner. Such a detached shock is in fact the same as a bow shock discussed earlier.

2. If $\theta < \theta_{\max}$, two distinct oblique shocks with two different β angles are physically possible. The smaller β case is called a *weak shock*, and is the one most likely to occur in a typical supersonic flow. The larger β case is called a *strong shock*, and is unlikely to form over a straight-wall wedge. The strong shock has a subsonic flow behind it.

3. The strong-shock case in the limit $\theta \to 0$ and $\beta \to 90°$, in the upper-left corner of the oblique shock chart, corresponds to the normal-shock case.