M14 Transformation of Stress (continued). Introduction to Strain

Principal Stresses/Axes

There is a set of axes (in 2 or 3-D) into which any state of stress can be resolved such that there are no shear stresses. These are known as the principal axes of stress.

$\sigma_I, \sigma_{II}, \sigma_{III}$ (By convention, σ_I is most tensile, σ_{III} is most compressive)

Can also see from Mohr's circle that two of these are the largest (and smallest) or extreme values of stress.

Side Note: if we thought about stress as a matrix:

$$d\underline{F} = \underline{\sigma} d\underline{A} \qquad \underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{21} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$
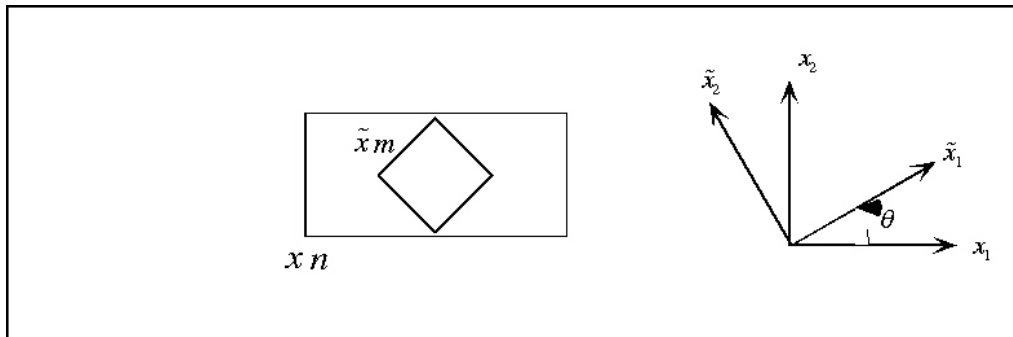
$$d\underline{F} = \underline{\sigma} d\underline{A}$$

The principal stresses are the eigenvalues of $\underline{\sigma}$ and the principal directions the eigenvectors.
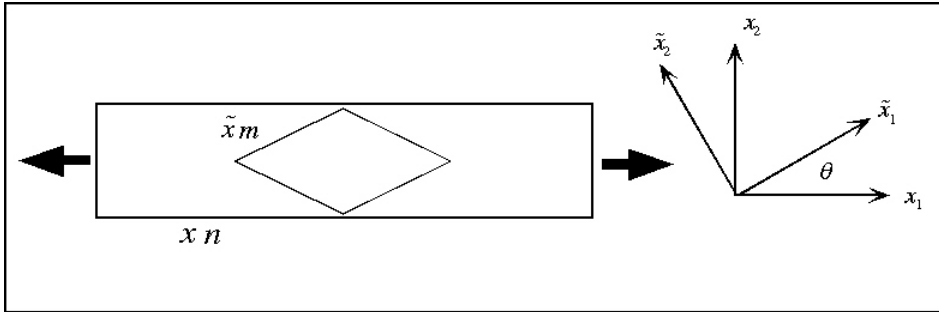
Important:

We are transforming the axis system, not the stress state.

Consider a bar with two grid squares (rectangle) on the surface, one rotated through 45°:

the bar now undergoes a tensile loading, and this generates a tensile strain

x_n rectangle is stretched/elongated (angles remain the same = 90°.)

$\tilde{x}_m$ is sheared (i.e angles changed)

But both grid squares (rectangles) are experiencing the same stress state (and strain state)

Stress Invariants

- actually the basis for Mohr's Circle

Note that in 2-D, $\sigma_{11} + \sigma_{22} = \sigma_I + \sigma_{II} = constant$

In 3-D, $\sigma_{11} + \sigma_{22} + \sigma_{33} = constant$ ($Invariant$)

i.e., the diameter Of the Mohr's circle (will meet other invariants elsewhere in solid mechanics).

We have examined stress, the continuum generalization of forces, now let's look at the continuum generalization of deformations:
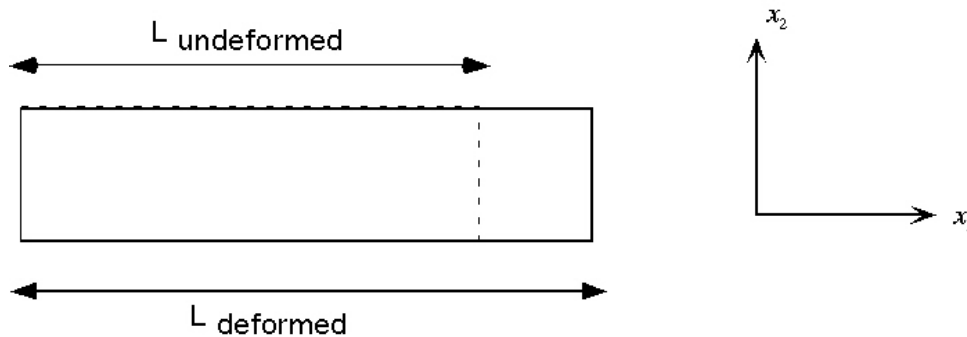
Definition of Strain

Strain is the deformation of the continuum at a point. Or, the relative deformation of an infinitessimal element.

Two ways that bodies deform: By elongation and shear:

Elongation (extension, tensile strain)

Tensile strain can be thought of as the change in length relative to the original length

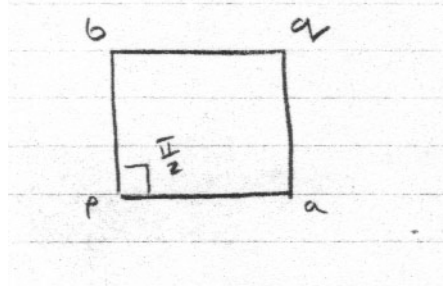$$tensile\ strain = \frac{L_{deformed} - L_{undeformed}}{L_{deformed}}$$

but body can also deform in shear – angles change
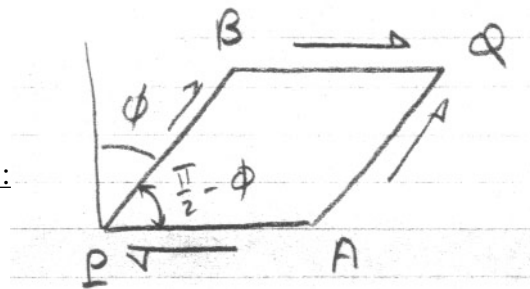
Shear

This produces an angle change in the body (with no rotation for pure shear)

Consider infinitesimal element.

Undeformed:

Deformed   :

Consider change in angle BPA = ϕ

$$\phi \quad = \angle\ undeformed - \angle\ deformed$$

NOTE: By convention positive strain is a reduction in angle (consistent with positive shear stresses)

$$\Delta \angle = \left[\frac{\Pi}{2} - \left(\frac{\Pi}{2} - \phi\right)\right] = \phi \quad \underline{radians}$$

NOTE: Strain is a non-dimensional quantity

(Just as for stress, strain varies with position and direction. Use tensor formulation to describe)

Formally Define ε_{mn} = strain tensor

For complete definition see:

Bisplinghoff, Mar and Pian, "Statistics of Deformable Solids", Chapter 5

This is the general definition of strain. But usually we are concerned with small strains:
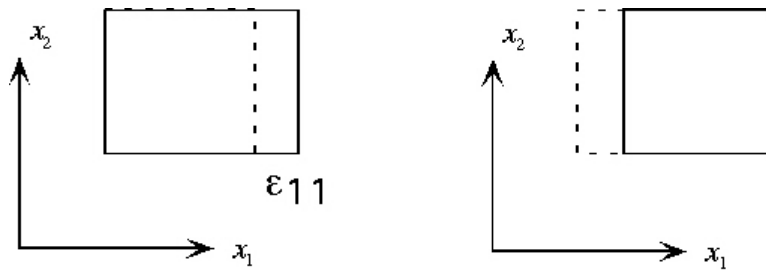
change in length $<$ 10% (0.1), change in angles $<$ 5% (0.05)

Good for range of use of most engineering materials, most structural applications.

Allows us to neglect higher order terms and leads to:

<u>Strain - Displacement Relations (for small strains)</u>

Conceptually, want to separate out rigid body translations from deformations

ε_{11} = relative elongation in x_1 direction

ε_{22} = relative elongation in x_2 direction.

ε_{33} = relative elongation to x_3 direction.

Consider infinitesimal element, side length dx_1 undergoing displacements and deformations in the x_1 direction defined by

u_1 is a <u>field variable</u> = $u_1(x_1, x_2, x_3)$, i.e. displacements vary with position

let u_1 be the displacement of the left-hand side of the infinitesimal element.

And $\left(u_1 + \dfrac{\partial u_1}{\partial x_1} dx_1 \right)$ is the displacement of right-hand side
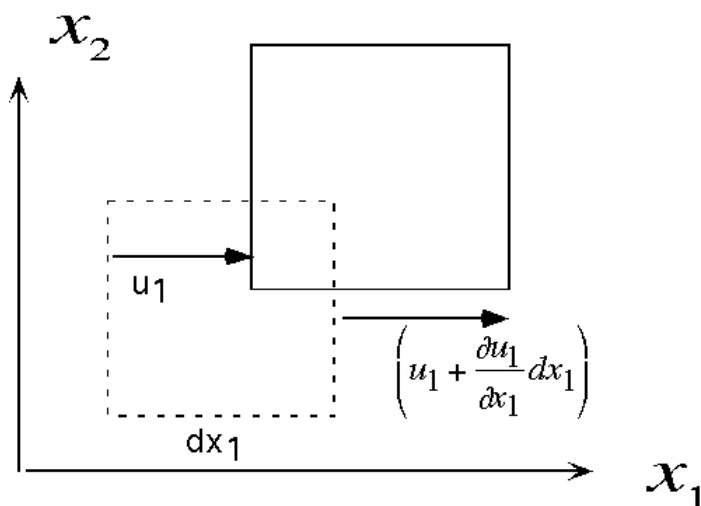
Recall:

$$\varepsilon_{11} \approx \varepsilon_1 = \frac{\ell \ deformed - \ell \ undeformed}{\ell \ undeformed}$$

$$\varepsilon_{11} = \frac{\left[dx_1 + u_1 + \frac{\partial u}{\partial x_1} dx_1 - u_1 \right] - dx_1}{dx_1}$$

$\ell \ undeformed$

Rigid body translation

$$= \frac{\partial u_1}{\partial x_1}$$

Similarly, ε_{22} = elongation in x_2

$$\varepsilon_{22} = \frac{\ell \ deformed - \ell \ undeformed}{\ell \ undeformed}$$

$$= \frac{\left[dx_2 + u_2 + \frac{\partial u_2}{\partial x_2} dx_2 - u_2 \right] - dx_2}{dx_2}$$

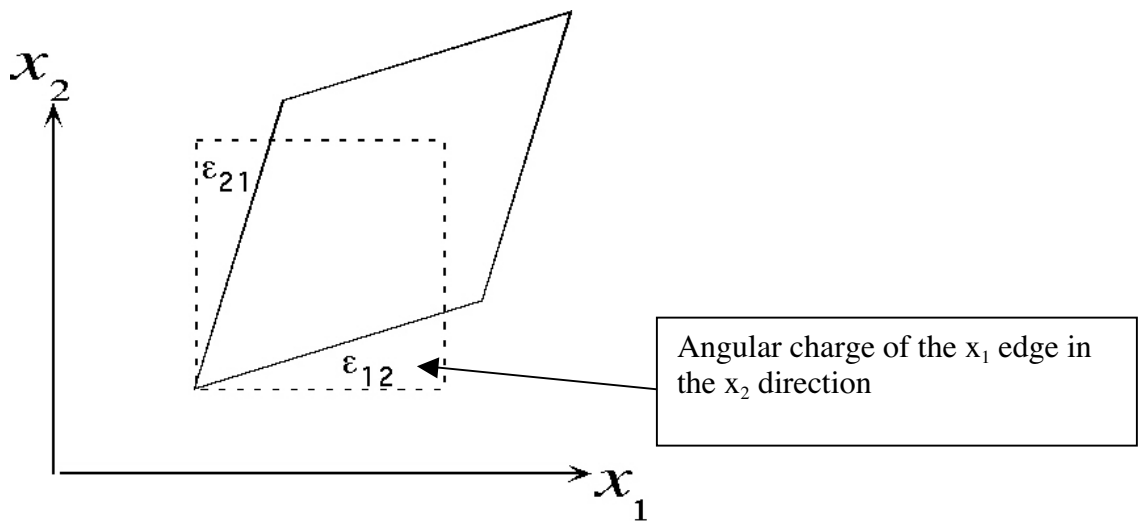$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

and $\varepsilon_{33} = \frac{\partial u_3}{\partial u_3}$

Now for shear strain

Need to be careful which angles we choose to define shear strain

$$\varepsilon_{12} = \frac{1}{2} angle\ change\left(=\frac{1}{2}\emptyset_{12}\right)\ \text{etc.}$$

$$=\frac{1}{2}\left(\ \bigtimes undeformed\ -\ \bigtimes deformed\ \right)$$

$$=\frac{1}{2}\left(\frac{\Pi}{2}-\left(\frac{\Pi}{2}-\phi_{12}\right)\right)$$

e.g:

Angular charge of the x_1 edge in the x_2 direction

$\varepsilon_{12} + \varepsilon_{21} = total\ angle\ change\ in\ x_1 - x_2\ plane$

$\varepsilon_{13} + \varepsilon_{31} = total\ angle\ change\ in\ x_1 - x_3\ plane$

$\varepsilon_{23} + \varepsilon_{32} = total\ angle\ change\ in\ x_2 - x_3\ plane$

NOTE: The strain tensor is defined such that there seems to be two parts in each angle change. But the <u>stress tensor</u> is symmetric and we would like the strain tensor also to be symmetric. Thus:

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(angle\ change\ in\ x_1 - x_2\ plane\right)$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(angle\ change\ in\ x_1 - x_3\ plane\right)$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(angle\ change\ in\ x_2 - x_3\ plane\right)$$

Consider diagonal of parallelogram undergoing pure shear:

Angles change by same amount.

$\pi/4 - \varepsilon_{12}$

$\pi/4 - \varepsilon_{12}$
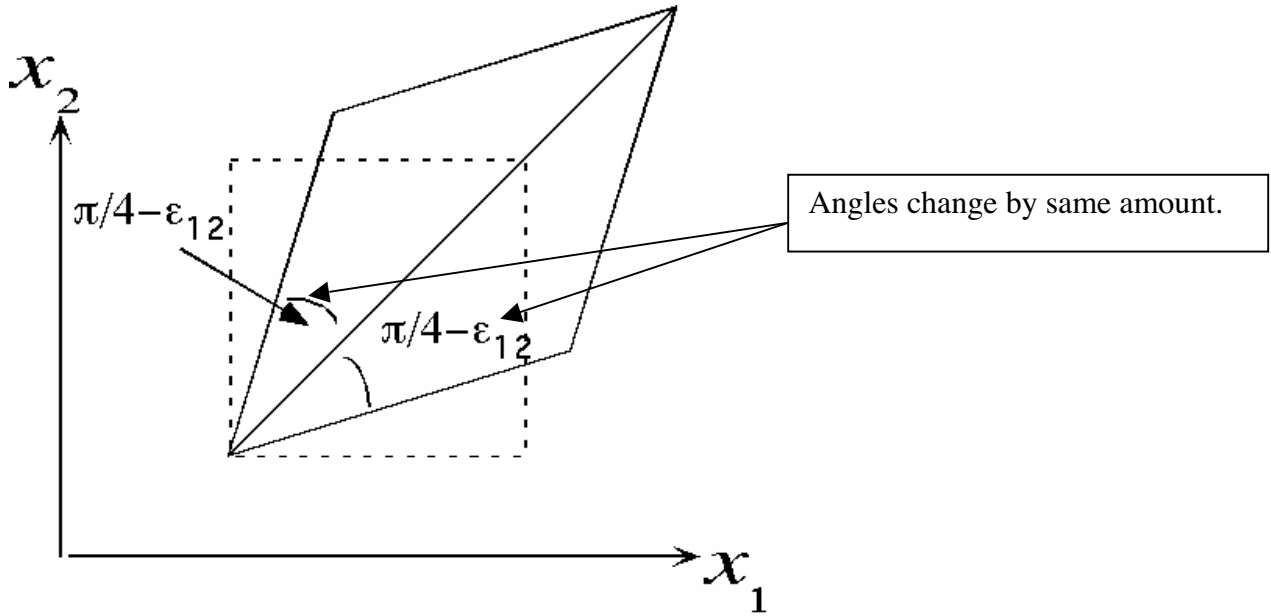
Formal Definitions

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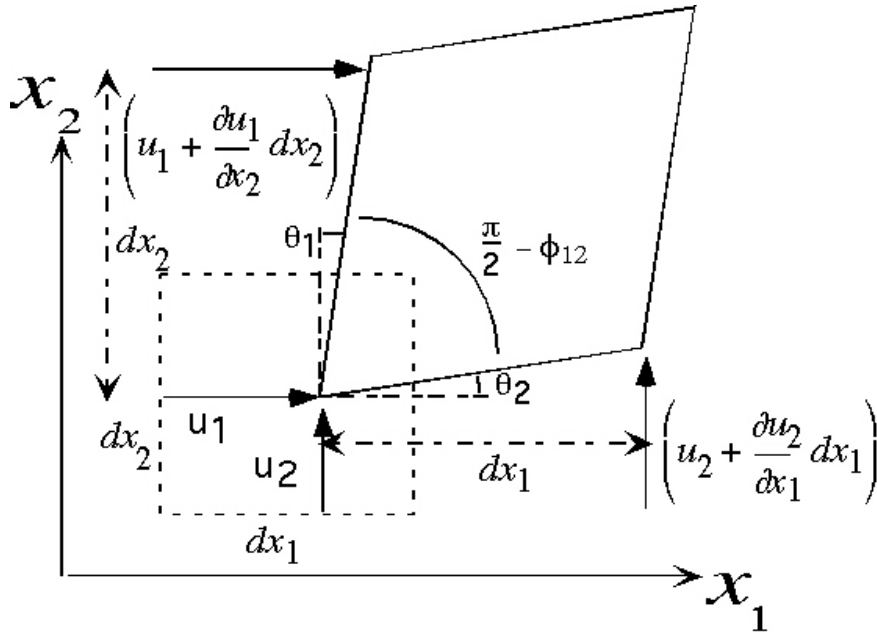
$$\phi_{12} = \theta_1 + \theta_2$$

Make small angle approximation: $\tan\theta = \theta$

$$\theta_1 = \frac{u_1 + \frac{\partial u_1}{\partial x_2}dx_2 - u_1}{dx_2}$$

$$\theta_2 = \frac{u_2 + \frac{\partial u_2}{\partial x_1}dx_1 - u_2}{dx_1}$$

Hence:

$$\phi_{12}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\Pi}{2} - \left\{\frac{\Pi}{2} - \left[\frac{\left(u_2 + \frac{\partial u_2}{\partial x_1}dx_1 - u_2\right)}{dx_1} + \frac{\left(u_1 + \frac{\partial u_1}{\partial x_2}dx_2 - u_1\right)}{dx_2}\right]\right\}\right)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = \varepsilon_{21}$$

and similarly:

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \varepsilon_{32}$$

$$\varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) = \varepsilon_{13}$$

or $\quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\qquad$ The strain tensor for small strains
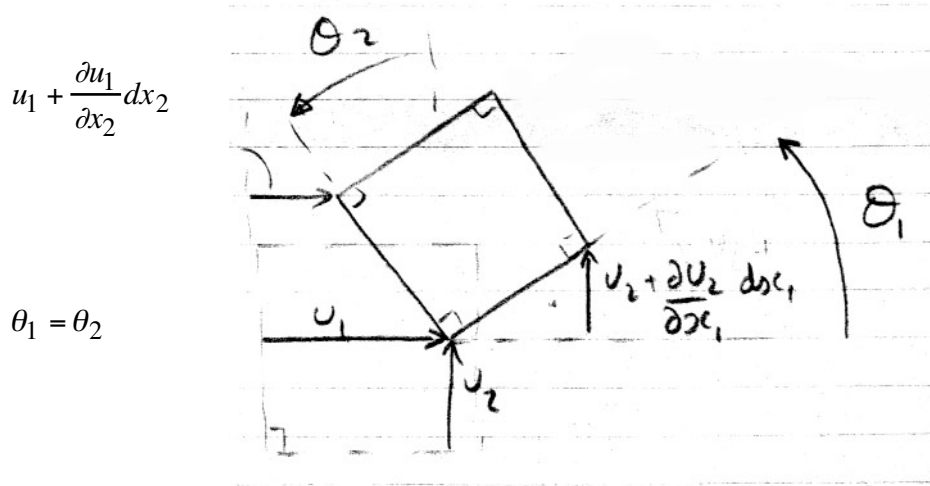
Strain displacement relation

6 independent strain components

(We have dealt with elongation and shear strains but remember, we were concerned to eliminate translation and rotational displacements)

(- translation accounted for by subtracting u_1, u_2 etc. from strain equations)

Rotation

$$u_1 + \frac{\partial u_1}{\partial x_2} dx_2$$

$$\theta_1 = \theta_2$$

$$\theta_1 = \frac{u_2 + \dfrac{\partial u_2}{\partial x_1} dx_1 - u_2}{dx_1} = \frac{\partial u_2}{\partial x_1} \qquad \text{as before}$$

> Negative because it acts to increase enclosed angle - consistent with shear strain

or $\qquad \theta_2 = \dfrac{-\left\{ u_1 + \dfrac{\partial u_1}{\partial x_2} dx_2 - u_1 \right\}}{dx_2} = \dfrac{-\partial u_1}{\partial x_2}$

But may also have shear deformation of the cube $\theta_1 \quad \theta_2$

Define average rotation (of diagonal)

$$\bar{\theta} = \frac{1}{2} \left\{ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right\} \qquad \text{note (-) sign}$$

<u>Compatibility</u>

One cannot independently define <u>6</u> strains from <u>3</u> displacements fields.

$$\begin{cases} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{cases}$$

The <u>strains must be related</u> (by equations) for them to be compatible.

for instance in $x_1 - x_2$ plane

take second partial of each:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} = \frac{\partial^3 u_1}{\partial x_1 dx_2^2} \qquad\qquad \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_2}{\partial x_1^2}$$

$$\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{1}{2}\left(\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \right)$$

Substitute first two into latter to get:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{2\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2}$$

In general, this can be written in tensor form:

$$\frac{\partial^2 \varepsilon_{nk}}{\partial x_m \partial x_\ell} + \frac{\partial^2 \varepsilon_{m\ell}}{\partial x_n \partial x_k} + \frac{\partial^2 \varepsilon_{n\ell}}{\partial x_m \partial x_k} - \frac{\partial^2 \varepsilon_{mk}}{\partial x_n \partial x_\ell} = 0$$

gives <u>6</u> equations (3 conditions)

<u>Transformation of Strain</u>

Identical to transformation of stress

$$\tilde{\varepsilon}_{mn} = \ell_{\tilde{m}p} \ell_{\tilde{n}q} \varepsilon_{pq}$$

Mohr's Circle in 2-D

Principal strains/directions

-no associated shear strains

Alternative notations for stress and strain

Engineering Notation   (see Crandall Dahl and Lardner)

$x_1 \rightarrow x$

$x_2 \rightarrow y$

$x_3 \rightarrow z$

Tensor   Engineering

σ_{11} σ_x

σ_{22} σ_y

σ_{33} σ_z

σ_{12} σ_{xy}

σ_{13} σ_{xz}

σ_{23} σ_{yz}

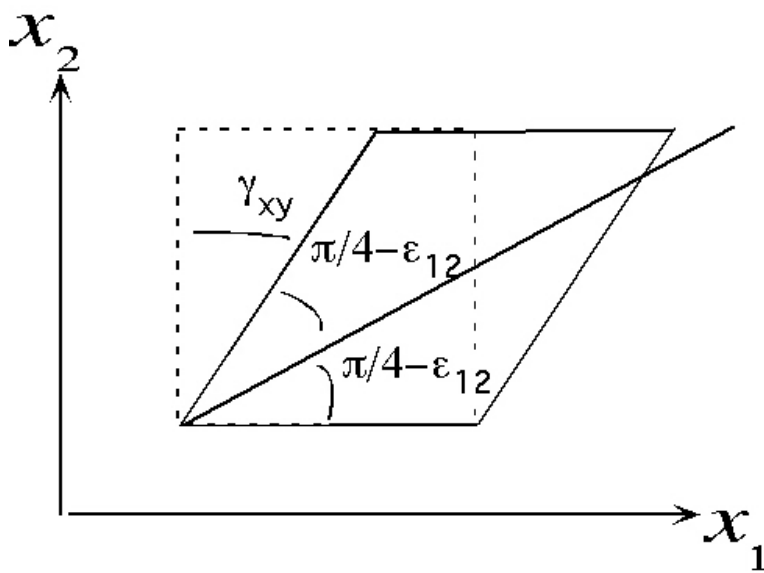In addition τ (tau) is often used for shear stress

$\tau_{xy} = \sigma_{xy}$

$\tau_{xz} = \sigma_{xz}$

$\tau_{yz} = \sigma_{yz}$

For stress this is just the substitution of different symbols, however for strain there is a fundamental difference from tensor notation.

Engineering shear strain = total angle change

Tensor shear strain = 1/2 (angle change)

Tensor   *Engineering*

ε_{11} $\rightarrow$ ε_x

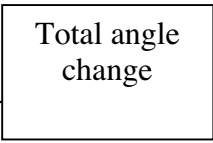ε_{22} $\rightarrow$ ε_y

ε_{33} $\rightarrow$ ε_z

ε_{12} $\rightarrow$ $\frac{1}{2}\varepsilon_{xy}$ *or* $\frac{1}{2}\gamma_{xy}$

ε_{13} $\rightarrow$ $\frac{1}{2}\varepsilon_{xz}$ *or* $\frac{1}{2}\gamma_{xz}$

ε_{23} $\rightarrow$ $\frac{1}{2}\varepsilon_{yz}$ *or* $\frac{1}{2}\gamma_{yz}$

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┌──────────────┐
│  Total angle │
│    change    │
└──────────────┘
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Strain matrix Notation

$$\varepsilon_{mn} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

- principal values are eigenvalues
- principal directions are eigenvectors

Stress matrix Notation

$$\sigma_{mn} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Next block we will link stress and strain by constitutive behavior and to do this we will take a closer look at the materials.