LECTURE S19

Measuring the size of a Signal

Often, we would like to descibe the size of a signal:

1) How much power is being transmitted?

2) How big is my signal compared to the noise?

3) How well does a control system track commanded reference signals? That is, how large is the error signal.

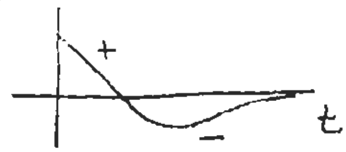
We would like to have a measure of the size of a signal that

1) Conforms to our notions of "big" and "small" signals.

2) Is easy to compute, and has useful properties.

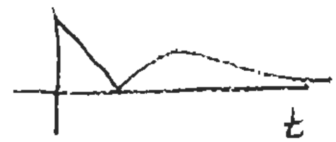
Possible measures of size

a) $\text{size} = \int_{-\infty}^{\infty} g(t)\,dt$



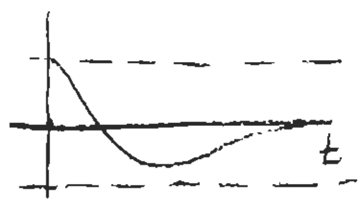
$\text{size} \gtrless 0$!

b) $\text{size} = \int_{-\infty}^{\infty} |g(t)|\,dt$



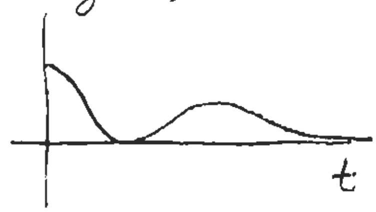
$\text{size} \geq 0$, but messy

c) $\text{size} = \max_t |g(t)|$



$\text{size} \geq 0$.

d) $\text{size} = \int_{-\infty}^{\infty} g^2(t)\,dt$



$\text{size} \geq 0$

Which is best? (b), (c), and (d) are all potentially useful, but (d) turns out to be the most useful:

- Works well analytically
- Corresponds to power or energy.

So define

$$\|g\|^2 = \int_{-\infty}^{\infty} |g(t)|^2 \, dt$$

$$\|g\| = \sqrt{\|g\|^2} = \text{"norm" (or "2-norm") of signal.}$$

This norm is a direct analog of the length of a vector:

$$\|\underline{x}\|^2 = \sum_i x_i^2$$

Parseval's Theorem

We have defined the norm (squared) of a signal as

$$\|g\|^2 = \int_{-\infty}^{\infty} g^2(t)\, dt$$

Would like to compute this in frequency domain, where control design is done.

Define

$$h(t) = g(t) * g(-t)$$
$$= \int_{-\infty}^{\infty} g(\tau)\, g(\tau - t)\, d\tau$$

So

$$h(0) = \int_{-\infty}^{\infty} g(\tau)\, g(\tau)\, d\tau = \|g\|^2$$

So just need to compute $h(0)$.

Note that

$$H(f) = G(f)G(-f) = G(f)G^*(f)$$

Inverse FT $H(f)$ to find $h(0)$:

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{j2\pi f t}\, df$$

$$\Rightarrow h(0) = \int_{-\infty}^{\infty} H(f)\, df$$

$$= \int_{-\infty}^{\infty} G(f)\, G^*(f)\, df$$

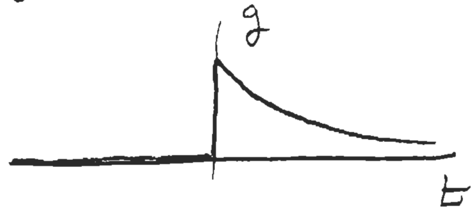
Therefore,

$$\boxed{\begin{aligned} \|g\|^2 &= \int_{-\infty}^{\infty} g^2(t)\, dt \\ &= \int_{-\infty}^{\infty} |G(f)|^2\, df \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2\, d\omega \end{aligned}}$$

This is <u>Parseval's Theorem</u> –

very important.

Example $\quad g(t) = e^{-at}\,\sigma(t), \quad a > 0$



$$\int_{-\infty}^{\infty} g^2(t)\,dt = \int_0^{\infty} e^{-2at}\,dt = \frac{1}{2a}$$

$$G(j\omega) = \frac{1}{j\omega + a}$$

$$\Rightarrow \quad G(j\omega)\,G(-j\omega) = \frac{1}{j\omega + a}\,\frac{1}{-j\omega + a} = \frac{1}{\omega^2 + a^2}$$

$$\|G\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2}\,d\omega$$

$$= \frac{1}{2\pi}\,\frac{1}{a}\,\tan^{-1}\frac{\omega}{a}\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi}\,\frac{1}{a}\left(\frac{\pi}{2} - \left(\frac{-\pi}{2}\right)\right) = \frac{1}{2a} \quad \checkmark$$

As expected, the two methods give the same result.

Note: This idea can be extended to Laplace transforms, where

$$\|g\|^2 = \frac{1}{2\pi j}\int_{-j\infty}^{+j\infty} G(s)\,G(-s)\,ds$$

Looks harder, but is easier! To evaluate:

1) Expand $G(s)G(-s)$ as partial fractions:

$$G(s)\,G(-s) = \frac{-1}{(s+a)(s-a)}$$

$$= \underbrace{\frac{1/2a}{s+a}}_{\text{pole in left half plane}} + \underbrace{\frac{-1/2a}{s-a}}_{\text{pole in right half plane}}$$

2) Integral $= \sum$ residues of left half plane poles

"residue" = numerator coefficient of first-order poles,