Solutions to Home Assignment #2

Warm-Up Exercises

1.

3-D stress-strain equations: $\sigma_{mn} = E_{mnpq}\varepsilon_{pq}$

2-D stress-strain equations: $\sigma_{\alpha\beta} = E^{*}_{\alpha\beta\sigma\gamma}\varepsilon_{\sigma\gamma}$

The material is orthotropic, so there are 9 independent E_{mnpq}'s. Writing out the 3-D stress-strain equations in full results in the following equations.

$$\sigma_{11} = E_{1111}\varepsilon_{11} + E_{1122}\varepsilon_{22} + E_{1133}\varepsilon_{33} \quad \text{——} \; ①$$

$$\sigma_{22} = E_{1122}\varepsilon_{11} + E_{2222}\varepsilon_{22} + E_{2233}\varepsilon_{33} \quad \text{——} \; ②$$

$$\sigma_{33} = E_{1133}\varepsilon_{11} + E_{2233}\varepsilon_{22} + E_{3333}\varepsilon_{33} \quad \text{——} \; ③$$

$$\sigma_{23} = 2E_{2323}\varepsilon_{23} \quad \text{——} \; ④$$

$$\sigma_{13} = 2E_{1313}\varepsilon_{13} \quad \text{——} \; ⑤$$

$$\sigma_{12} = 2E_{1212}\varepsilon_{12} \quad \text{——} \; ⑥$$

Similarly, the 2-D equations become:

$$\sigma_{11} = E^*_{1111}\varepsilon_{11} + E^*_{1122}\varepsilon_{22} \qquad \text{———} \quad ⑦$$

$$\sigma_{22} = E^*_{1122}\varepsilon_{11} + E^*_{2222}\varepsilon_{22} \qquad \text{———} \quad ⑧$$

$$\sigma_{12} = 2E^*_{1212}\varepsilon_{12} \qquad \text{————} \quad ⑨$$

Under plane stress, $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. Therefore, equations ③, ④ and ⑤ are :

$$③ \rightarrow \quad \sigma_{33} = E_{1133}\varepsilon_{11} + E_{2233}\varepsilon_{22} + E_{3333}\varepsilon_{33} = 0$$

$$\Rightarrow \quad \varepsilon_{33} = -\frac{E_{1133}}{E_{3333}}\varepsilon_{11} - \frac{E_{2233}}{E_{3333}}\varepsilon_{22} \qquad \text{———} \quad ⑩$$

$$④ \rightarrow \quad \sigma_{23} = 2E_{2323}\varepsilon_{23} = 0$$

$$\Rightarrow \quad \varepsilon_{23} = 0 \qquad \text{————} \quad ⑪$$

$$⑤ \rightarrow \quad \sigma_{13} = 2E_{1313}\varepsilon_{13} = 0$$

$$\Rightarrow \quad \varepsilon_{13} = 0 \qquad \text{————} \quad ⑫$$

Equations ⑪ and ⑫ gives no information about the relationship between E_{mnpq} and $E^*_{\alpha\beta\sigma\gamma}$. Equation ⑩ can be inserted into equations ① and ② to find the relationship between E_{mnpq} and $E^*_{\alpha\beta\sigma\gamma}$.

$$① \Rightarrow \quad \sigma_{11} = E_{1111}\varepsilon_{11} + E_{1122}\varepsilon_{22} + E_{1133}\left(\frac{E_{1133}}{E_{3333}}\varepsilon_{11} + \frac{E_{2233}}{E_{3333}}\varepsilon_{22}\right)$$

$$= \left(E_{1111} - \frac{E_{1133}^2}{E_{3333}}\right)\varepsilon_{11} + \left(E_{1122} - \frac{E_{1133}E_{2233}}{E_{3333}}\right)\varepsilon_{22} \quad — ⑬$$

$$② \Rightarrow \quad \sigma_{22} = E_{1122}\varepsilon_{11} + E_{2222}\varepsilon_{22} + E_{2233}\left(\frac{E_{1133}}{E_{3333}}\varepsilon_{11} + \frac{E_{2233}}{E_{3333}}\varepsilon_{22}\right)$$

$$= \left(E_{1122} - \frac{E_{2233}E_{1133}}{E_{3333}}\right)\varepsilon_{11} + \left(E_{2222} - \frac{E_{2233}^2}{E_{3333}}\right)\varepsilon_{22} \quad — ⑭$$

Thus, comparing equations ⑬ and ⑦, and equations ⑭ and ⑧ one can obtain the relationship between E_{mnpq} and $E^*_{\alpha\beta\sigma\tau}$. In addition, comparing equations ⑥ and ⑨ allows one to obtain the relationship between E^*_{1212} and E_{1212}. These relationships are:

$$\boxed{\begin{aligned} E^*_{1111} &= E_{1111} - \frac{E_{1133}^2}{E_{3333}} \\ E^*_{2222} &= E_{2222} - \frac{E_{2233}^2}{E_{3333}} \\ E^*_{1212} &= E_{1212} \\ E^*_{1122} &= E_{1122} - \frac{E_{2233}E_{1133}}{E_{3333}} \end{aligned}}$$

* Note: $E^*_{1122} = E^*_{2211}$ ← stress-strain relations are symmetric!

2. The same relationships between E_{mnpq} and $E^*_{\alpha\beta\sigma\tau}$ hold for isotropic materials. However, instead of 9 independent E_{mnpq}'s, there are only 2 independent E_{mnpq}'s.

Since isotropic materials have the same property in all directions;

$$E_{1111} = E_{2222} = E_{3333}$$

$$E_{1122} = E_{2233} = E_{1133}$$

$$E_{1212} = E_{2323} = E_{1313}$$

This gives us 3 independent E_{mnpq}'s. We need one more relation to eliminate another E_{mnpq}. As noted in BMP, p.180, E_{1111}, E_{1122} and E_{1212} can be related using the Lamé constants.

$$E_{1111} = \lambda + 2\mu \qquad \text{——— ①}$$

$$E_{1122} = \lambda \qquad \text{——— ②}$$

$$E_{1212} = \mu \qquad \text{——— ③}$$

Substituting ② and ③ into ①, we find

$$E_{1111} = E_{1122} + 2E_{1212}$$

$$\Rightarrow \quad E_{1212} = \frac{E_{1111} - E_{1122}}{2}$$

Thus, there are two independent E_{mnpq}'s. Plugging these relations into the relationship between E_{mnpq} and $E^*_{\alpha\beta\sigma\tau}$ obtained in Prob. #1, we find

$$\boxed{\begin{aligned} E^*_{1111} &= E_{1111} - \frac{E_{1122}^2}{E_{1111}} \\ E^*_{2222} &= E_{1111} - \frac{E_{1122}^2}{E_{1111}} \\ E^*_{1212} &= \frac{E_{1111} - E_{1122}}{2} \\ E^*_{1122} &= E_{1122} - \frac{E_{1122}^2}{E_{1111}} \end{aligned}}$$