MULTIPLE-INPUT DESCRIBING FUNCTIONS AND NONLINEAR SYSTEM DESIGN

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PREFACE

The theory of automatic control has been advanced in important ways during recent years, particularly with respect to stability and optimal control. These are significant contributions which appeal to many workers, including the writers, because they answer important questions and are both theoretically elegant and practically useful. These theories do not, however, lay to rest all questions of importance to the control engineer. The designer of the attitude control system for a space vehicle booster which, for simplicity, utilizes a rate-switched engine gimbal drive, must know the characteristics of the limit cycle oscillation that the system will sustain and must have some idea of how the system will respond to attitude commands while continuing to limit-cycle. The designer of a chemical process control system must be able to predict the transient oscillations the process may experience during start-up due to the limited magnitudes of important variables in the system. The designer of a radar antenna pointing system with limited torque capability must be able to predict the rms pointing error due to random wind disturbances on the antenna, and must understand how these random disturbances will influence the behavior of the system in its response to command inputs. But more important than just being able to evaluate how a given system will behave in a postulated situation is the fact that these control engineers must *design* their systems to meet specifications on important characteristics. Thus a complicated exact analytical tool, if one existed, would be of less value to the designer than an approximate tool which is simple enough in application to give insight into the trends in system behavior as a function of system parameter values or possible compensations, hence providing the basis for system design. As an analytical tool to answer questions such as these in a way which is useful to the system designer, the multiple-input describing function remains unexcelled.

This book is intended to provide a comprehensive documentation of describing function theory and application. It begins with a unified theory of quasi-linear approximation to nonlinear operators within which are embraced all the useful describing functions. It continues with the application of these describing functions to the study of a wide variety of characteristics of nonlinear-system operation under different input conditions. Emphasis is given to the *design* of these nonlinear systems to meet specified operating characteristics. The book concludes with a complete tabular and graphical presentation of the different describing functions discussed in the text, corresponding to a broad family of nonlinear functions. Dealing as it does with the single subject of describing functions, the book would seem to be very specialized in scope. And so it is. Yet the range of practical and important questions regarding the operation of nonlinear systems which this family of describing functions is capable of answering is so broad that the writers have had to set deliberate limits on the lengths of chapters to keep the book within reasonable size. Thus the subject is specialized to a single analytical tool which has exceedingly broad applicability.

This presentation is intended both for graduate students in control theory and for practicing control engineers. Describing function theory is applicable to problems other than the analysis and design of feedback control systems, and this is illustrated by some of the examples and problems in the book. But the principal application has been to control systems, and this has been the major focus of the book. The presentation is too comprehensive, and the subject too specialized, for the book to serve as the textbook in most graduate control courses, but it can serve very well as one of several reference books for such courses. In a graduate control course in the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology, the subject of this book is covered in a period of four or five weeks—twelve to fifteen lecture hours. The presentation of this book is not abbreviated primarily by omitting whole sections; rather, the principal ideas of almost every section are summarized briefly in class. A selection of these concepts is further developed through the problems. Such a presentation does not bring the student to the point of mastery of the subject, but it can give him a good understanding of the principal ideas underlying describing function theory and application. With this, the student can recognize the areas of useful applicability and can readily use the book as a reference to help him address the problems that arise in his professional experience.

The practicing control engineer should find the book valuable as a complete reference work in the subject area. If his background in mathematics is not sufficient to enable him to follow the theoretical development of Chapter 1 comfortably, he can omit that chapter and will still find a complete presentation in every chapter except Chapters 7 and 8, based on the physically motivated concept of harmonic analysis of the nonlinearity output. Chapter 7, which includes random processes at the nonlinearity input, requires a statistical approach. But this too reduces to a rather simple matter in the very important class of problems involving static single-valued nonlinearities. Chapter 8 treats transient responses by related forms of quasi-linearization which are developed completely within that chapter. Thus it is hoped that every control engineer will find the principal ideas presented in a manner which is meaningful and appealing to him.

It is a pleasure for the writers to acknowledge the contributions of people who helped in different ways to see this project to completion. We express sincere appreciation to Hazel Leiblein for typing large portions of the manuscript; to Allan Dushman and Laurie J. Henrikson for a careful reading of several chapters; and to Martin V. Dixon, who volunteered to prepare the graphed data on the relay with dead zone which are presented in Appendix F. Special appreciation is due our understanding wives, Linda and Winni, who accepted long evenings over a period of several years without the company of their husbands.

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1 NONLINEAR SYSTEMS AND DESCRIBING FUNCTIONS

1.0 INTRODUCTION

A system whose performance obeys the principle of superposition is defined as a *linear* system. This principle states that if the input $r_1(t)$ produces the response $c_1(t)$, and the input $r_2(t)$ yields the response $c_2(t)$, then for all a and b the response to the input $ar_1(t) + br_2(t)$ will be $ac_1(t) + bc_2(t)$; and this must be true for all inputs. A system is defined as *time-invariant* if the input $r(t + \tau)$ produces the response $c(t + \tau)$ for all input functions $r(t)$ and all choices for τ.

The simplest class of systems to deal with analytically is of course the class of *linear invariant* systems. For such systems the choice of time origin is of no consequence since any translation in time of the input simply translates the output through the same interval of time, and the responses to simple input forms can be superimposed to determine the responses to more complex input forms. This permits one in principle to *generalize from the response for any one input to the responses for all other inputs.* The elementary input function most commonly used as the basis for this generalization is the unit-impulse function; the response to this input is often called the system *weighting function.* All possible modes of behavior of a linear invariant system are represented in its weighting function. Having once determined this function, which is just the response to one particular input, the performance of such a system can hold no surprises.

A *linear variable* system, although it still obeys the principle of superposition, is appreciably more difficult to deal with analytically. The response to a single input function in this case does not suffice to define the responses to all inputs; rather, a one-parameter infinity of such responses is needed. A single elementary form of input such as the unit-impulse function is adequate, but this input must be introduced, and the response determined, with all translations in time. Furthermore, the calculation of such responses very often cannot be done analytically. For invariant systems, the calculation of the weighting function requires the solution of a linear invariant differential equation, and there is a well-established procedure for finding the homogeneous solution to all such equations. There is no comparable general solution procedure for linear variable differential equations, and so the determination of the time-variable weighting function for linear variable systems usually eludes analytic attack.

Any system for which the superposition principle does not hold is defined to be *nonlinear.* In this case there is *no possibility of generalizing from the responses for any class of inputs to the response for any other input.* This constitutes a fundamental and important difficulty which necessarily requires any study of nonlinear systems to be quite specific. One can attempt to calculate the response for a specific case of initial conditions and input, but make very little inference based on this result regarding response characteristics in other cases.

In spite of the analytic difficulties, one has no choice but to attempt to deal in some way with nonlinear systems, because they occupy very important places in anyone's list of practical systems. In fact, linear systems can be thought of only as approximations to real systems. In some cases, the approximation is very good, but most physical variables, if allowed to take large values, will eventually run out of their range of reasonable linearity. *Limiting* is almost universally present in control systems since most instrumented signals can take values only in a bounded range. Many error detectors, such as a *resolver* or *synchro differential,* have a restricted range of linearity. Most drive systems, such as *electrical* and *hydraulic actuators,* can be thought of as linear over only small ranges, and others, such as *gas jets,* have no linear range at all. The use of *digital data processing* in control systems inevitably involves signal quantization. These are examples of nonlinear effects which the system designer would prefer to avoid, but cannot. There are good reasons why he might also choose to design some nonlinear effects into his system. The use of a *two-* or *three-level switch* as a controller, switching the power supply directly into the actuator, often results in a considerable saving of space and weight, compared with a high-gain chain of linear amplification, ending with a power amplifier to drive the actuator. Also, controllers of sophisticated design, such as *optimal* or *final-value controllers*, often require nonlinear behavior.

As these few examples illustrate, the trend toward smaller and lighter-weight systems, the demand for higher-performance systems, and the increasing utility of digital operations in control systems, all conspire to broaden the place that nonlinear systems occupy. Thus the need for analytic tools which can deal with nonlinear systems in ways that are useful to the system designer continues to grow. This book treats a practical means of studying some of the performance characteristics of a broad class of nonlinear invariant systems. The techniques presented here can be, and have been, extended to some special cases of nonlinear variable systems, and the possibilities for doing so are relatively clear, once the basic analytic tool is well understood.

1.1 NONLINEAR-SYSTEM REPRESENTATION

Most systems can be considered an interconnection of components, or subsystems. In most cases, some of these subsystems are well characterized as linear, whereas others are more distinctly nonlinear. This results in a system configuration which is an interconnection of separable linear and nonlinear parts. The systems which are most commonly considered in this book are a further specialization of these: the class of systems which can be reduced to a *single-loop configuration with separable linear and nonlinear parts.* Some special cases of multiple-nonlinearity systems arranged in multiple loops which cannot be reduced are considered, but the configuration most commonly referred to is that of Fig. 1.1-1. This diagram could equally well represent a temperature control system, an inertial navigator platform gimbal servo, an aircraft stabilizer servo, a spacecraft telescope position control system, or a machine-tool positioning system. In each instance we might expect to

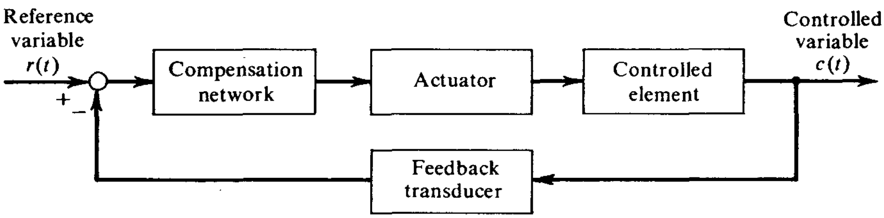

Figure 1.1-1 *General control system block diagram.*

find nonlinear effects in the actuator or feedback transducer, or even both, whereas the controlled element and loop compensation might well be linear.

To permit reference to certain classes of separable nonlinear elements, it is appropriate to classify them according to type in some sense. The broadest distinction to be made is between *explicit* and *implicit* nonlinearities. In the former case, the nonlinearity output is explicitly determined in terms of the required input variables, whereas in the latter case the output is defined only in an implicit manner, as through the solution of an algebraic or differential equation. Among the explicit nonlinearities, the next division is between *static* and *dynamic* nonlinearities. In the former case the nonlinearity output depends only on the input function, whereas in the latter, the output also depends on some derivatives of the input function. Among the static nonlinearities, a further distinction is drawn between *single-valued* and *multiple-valued* nonlinearities. In the case of a static, single-valued nonlinearity, the output is uniquely given in terms of the current value of the input, whereas more than one output may be possible for any given value of the input in the case of a static multiple-valued nonlinearity. The choice among the multiple values is made on the basis of the previous history of the input; thus such a nonlinearity is said to possess *memory*. One can imagine dynamic multiple-valued nonlinearities as well, but we shall not have occasion to refer to any such in this book. These are the major distinctions among nonlinearities from the point of view of the theory to be developed here. Other characteristics, such as *continuous* vs. *discontinuous*, are of little consequence here, but can be of supreme importance in other contexts.

An example of a static, single-valued, continuous, piecewise-linear nonlinearity is the *deadband gain*, or *threshold* characteristic (Fig. 1.1-2*a*). It could represent the acceleration input–voltage output relationship of a pendulous accelerometer, or the input-output characteristic of an analog angular position transducer. It is described by

$$y = \begin{cases} k(x-\delta) & \text{for } x \geq \delta \\ 0 & \text{for } -\delta \leq x < \delta \\ k(x+\delta) & \text{for } x < -\delta \end{cases} \tag{1.1-1}$$

where x and y denote the nonlinearity input and output, respectively. A static, multiple-valued, discontinuous, piecewise-linear nonlinearity is the *relay with deadband and hysteresis* (Fig. 1.1-2*b*). Arrows denote the direction in which this characteristic must be traversed in the determination of the output for a given input. The *history* of the input determines the value of the output in the multiple-valued regions. This characteristic is representative of the actuator switch in a temperature control system (in which case only the first quadrant portion applies) or the on-off gas jets in a spacecraft angular orientation system.

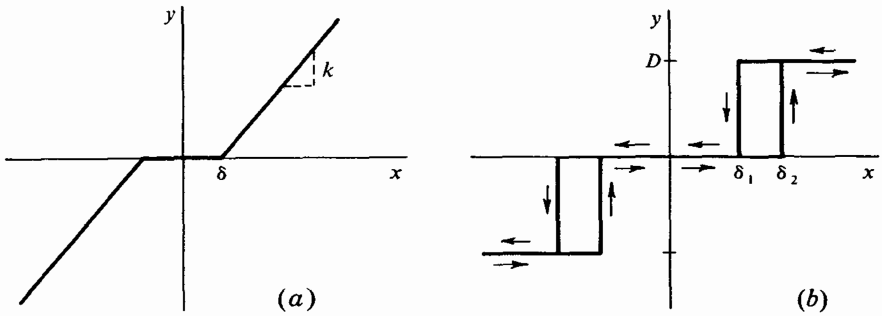

Figure 1.1-2 Examples of two static nonlinear characteristics. (a) Threshold; (b) relay with deadband and hysteresis.

Nonlinear differential equations illustrate *implicit dynamic* nonlinearities. For example, the equation

$$\dot{y}^{\frac{1}{3}} + 2y = x \tag{1.1-2}$$

represents such a nonlinearity, whereas

$$y = e^{\dot{x}+x} \tag{1.1-3}$$

portrays an *explicit dynamic* nonlinearity. It is to be noted that the process of converting implicit nonlinear differential relationships to explicit relationships is precisely the process of solving nonlinear differential equations—a process which is impossible for most equations of interest. For this reason, when they occur, we are usually forced to work directly with the implicit relationships themselves.

It is sometimes possible to recast a nonlinearity into a simpler form than that in which it is originally presented. An example is the implicit nonlinearity of Eq. (1.1-2). This differential relation can be represented by the feedback configuration of Fig. 1.1-3, just as if the equation were to be solved using

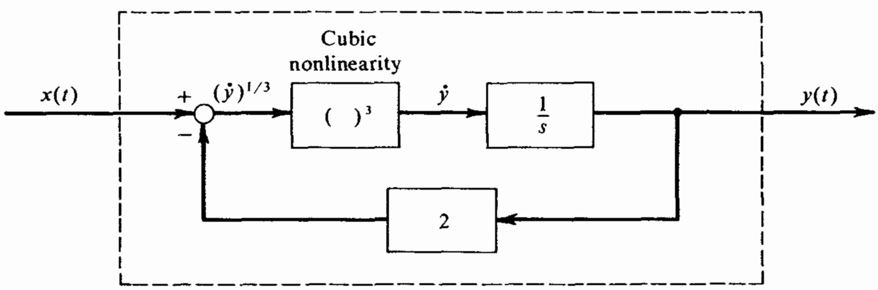

Figure 1.1-3 Closed-loop formulation of the implicit dynamic nonlinearity $\dot{y}^{\frac{1}{3}} + 2y = x$.

an analog computer. Only the explicit cubic nonlinearity appears in this diagram. Thus, if this nonlinear relation were part of a larger system, and the feedback path of Fig. 1.1-3 were absorbed into that system, we should have succeeded in trading an implicit dynamic nonlinearity for an explicit static single-valued nonlinearity. The static multiple-valued nonlinearity of Fig. 1.1-2*b* can be reduced to a static single-valued nonlinearity with a feedback path as shown in Fig. 1.1-4*a*. This, again, is an exact representation. An approximate representation of the hysteresis nonlinearity (multiple-valued) by the deadband gain nonlinearity (single-valued) in a feedback loop, together with an integrator and a high gain, is shown in Fig. 1.1-4*b*. The forward gain in this approximation must be chosen so that the bandwidth of the loop, when the deadband gain is operated in its linear range, is wide compared with the bandwidth of the system of which the hysteresis element is a part. In each case the feedback path of the transformed nonlinearity is then associated with the transfer of the rest of the system to separate the linear and nonlinear parts. The primary limitation on this technique of transforming a nonlinearity to simpler form is the fact that the feedback path of the transformed nonlinearity contains no filtering. This characteristic will be found undesirable for application of the theory developed in this book. The importance of this unfiltered feedback in any particular case depends on

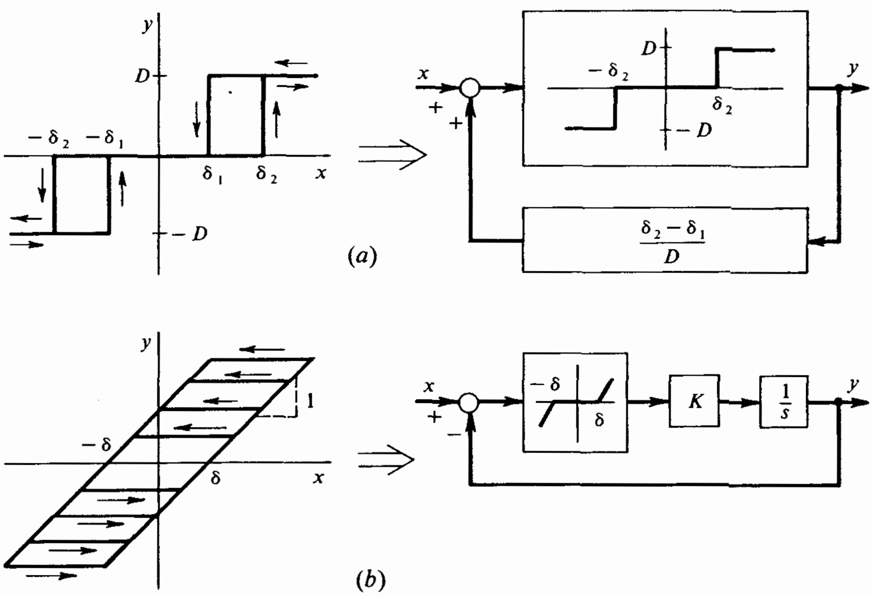

Figure 1.1-4 *Transformation of multiple-valued nonlinearities into single-valued nonlinearities with feedback loops.*

the relative amplitudes of the signals fed back to the nonlinearity through this path and through the linear part of the system.

If one wishes to study nonlinear differential equations which may arise out of any context whatsoever, it is often possible to collect the linear and nonlinear terms in the equation into separate linear and nonlinear parts, and then, using techniques similar to that of the preceding paragraph, arrange the resulting equation into a feedback configuration. The result is a closed loop having separated linear and nonlinear parts, which falls into the pattern of Fig. 1.1-1.

1.2 BEHAVIOR OF NONLINEAR SYSTEMS

The response of a linear invariant system to any forcing input can be expressed as a convolution of that input with the system weighting function. The response for any initial conditions of whatever magnitude can always be decomposed into the same set of normal modes which are properties of the system. The normal modes of all such systems are expressible as complex exponential functions of time, or as real exponentials and exponentially increasing or decreasing sinusoids. A special case of the latter is the undamped sinusoid, a singular situation. If a system displays an undamped sinusoidal normal mode, the amplitude of that mode in the response for a given set of initial conditions is, as for all other normal modes, dependent on the initial conditions.

The response characteristics of nonlinear systems, on the other hand, cannot be summarized in a way such as this. These systems display a most interesting variety of behavioral patterns which must be described and studied quite specifically. The most obvious departure from linear behavior is the *dependence of the response on the amplitude of the excitation*—either forcing input or initial conditions. Even in the absence of input, nonlinear systems have an important variety of response characteristics. A fairly common practical situation is that in which the system responds to small (in some sense) initial conditions by returning in a well-behaved manner to rest, whereas it responds to large initial conditions by diverging in an unstable manner. In other cases, the response to certain initial conditions may lead to a continuing oscillation, the characteristics of which are a property of the system, and not dependent on initial conditions. Such an oscillation may be viewed as a trajectory in the state space of the system which closes on itself and thus repeats; it is called a *limit cycle*. Nonlinear systems may have more than one limit cycle, and the one which is established will depend on the initial conditions, but the characteristics of each member of this discrete set of possible limit cycles are not dependent on initial conditions—they are properties of the system. This phenomenon is not possible in linear systems.

Since a limit cycle, once excited, will continue in the absence of further excitation, it constitutes a point of very practical concern for the system designer, and an analytic tool to study limit cycles is of evident importance to him.

A simple example of a limit cycling system is the Van der Pol oscillator, which obeys the equation

$$\ddot{x} - \eta(1 - x^2)\dot{x} + \omega_0^2 x = 0 \qquad \eta > 0 \tag{1.2-1}$$

If $x(t)$ is always small compared with 1, this oscillator appears unstable since the effective damping is negative. Thus the small signals will tend to grow. If $x(t)$ were an oscillation with amplitude much greater than 1, it seems qualitatively clear that the average effective damping would appear positive, and the large signals would tend to decrease. At some amplitude of oscillation, where the average $|x(t)|$ is of the order of 1, the average effective damping would appear to be zero, and the oscillation would continue with that amplitude. This heuristic argument is given quantitative significance in the following chapters.

The response of nonlinear systems to forcing inputs presents an infinite variety of possibilities, just a few examples of which will be cited here. An input which plays a central role in linear invariant system theory is the sinusoid. Its importance is due primarily to the fact that the stability of a closed-loop linear system can be determined from the steady-state response of the open-loop system to the set of sinusoids of all frequencies. The amplitude of the sinusoids is of no consequence since only the ratio of the input to output is needed, and for linear systems this is independent of amplitude. The response of a nonlinear system to sinusoidal inputs is an important characteristic of the system too, because of the near-sinusoidal character of the actual inputs that many systems may see. But the nature of this response characteristic is much more complex in this case. The steady-state response of a nonlinear system to a sinusoidal input is dependent upon amplitude as well as frequency, in general. And even more interesting characteristics may appear. The response may be multiple-valued—a large-amplitude mode and a small-amplitude mode both possible for the same sinusoidal input. Or the predominant output response to an input sinusoid may even be at a different frequency, either a subharmonic or superharmonic of the input frequency. A system which limit-cycles in the absence of input may continue to do so in the presence of a sinusoidal input, both frequency components being apparent in the response, or the limit cycle may be quenched by the presence of the input. Or again, a system which does not limit-cycle in the absence of input may break out into a limit cycle in the presence of a sinusoidal input.

Most of these characteristics—the multiple-mode response, the possible quenching or exciting of limit cycles—may also be true of nonlinear systems responding to other periodic inputs or to random inputs. And so the story could be continued; but the only point to be made here is that nonlinear systems display a most interesting variety of behavioral characteristics. The brief listing of some of them given here has favored those which can best be studied by the methods of this book.

1.3 METHODS OF NONLINEAR-SYSTEM STUDY

A number of possible means of studying nonlinear-system performance may be cited. All are of importance since different systems may be most amenable to analysis by different methods. Also, it was noted earlier that nonlinear-system study must be quite specific since generalization on performance characteristics seems impossible. It may be expected, then, that different analytic techniques will be best suited to the study of different performance characteristics. In effect, different techniques can be used to ask different questions about system performance.

The most common methods of nonlinear-system study are listed here, with brief comment for the purpose of viewing the describing function against a background of alternative methods. Of greatest interest is the *usefulness* of each of these methods in the *design* of practical nonlinear systems.

PROTOTYPE TEST

The most certain means of studying a system is to build one and test it. The greatest advantage of this is that it *avoids* the necessity of choosing *a mathematical model* to represent the important characteristics of the system. The disadvantages are clear: the technique is of limited help in design because of the *time* required to construct a series of trial systems, the *cost* involved, and in many cases the *danger* involved in trying a system about which little is known.

COMPUTER SIMULATION

The capability of modern computers—analog, digital, and hybrid—is such that very complete simulations of complex systems can be made and studied in a practical way. Dependence upon computer simulation, however, is not a very good first step in nonlinear-system design. The attempt to design through a sequence of blind trials on a computer is costly and unsatisfying. Each computer run answers the question, "What is the response to one particular set of initial conditions and one particular input?" One must always wonder if he has tried enough initial-condition and input combinations to have turned up all interesting response characteristics of the system. The computer, used simply to make a direct simulation of the system, is thus a limited tool in the early phases of system design. However, since any other tool which is more useful for that purpose will almost surely involve approximations, computer simulation to check the design and verify system performance is an appropriate, if not essential, *final step* before building the real system.

CLOSED-FORM SOLUTIONS

There are a number of nonlinear differential equations, mostly of second order, for which exact solutions have been found or for which certain properties of the solutions have been tabulated. These constitute a very small number of *special cases*, and it is rare indeed when a control-system problem or other problem arising out of a significant physical situation can be made to fit one of these cases.

PHASE-PLANE SOLUTION

The dynamic properties of a system can be described in terms of the differential equations of state, and an attempt made to solve for the trajectories of the system in the state space. But this is just another way to solve nonlinear differential equations, and it is rarely possible to effect the solution. For the special cases of *first-* and *second-order systems*, however, this approach is useful because the two dimensions available on a flat piece of paper are adequate to display completely the state of these systems. Thus graphical techniques can be used to solve for the state, or phase, trajectories. This allows the response to be calculated for any set of initial conditions and for certain simple input functions. More important, however, is the fact that certain properties of the trajectories, such as their slopes, can be displayed over the whole phase plane. This information helps to alleviate concern over whether enough specific trajectories have been calculated to exhibit all interesting response characteristics. Thus, when such phase trajectories can be determined and their characteristics portrayed over the whole phase space, for certain inputs, one has a most valuable attack on the problem. But this can rarely be achieved for systems of greater than second order.

LYAPUNOV'S DIRECT METHOD

One of the most important properties of a system, *stability*, can in principle be evaluated without calculating the detailed responses of the system from given initial conditions with given inputs. All that is necessary is an indication of whether the state trajectories in the vicinity of an equilibrium point tend to move generally toward or away from that point. This concept has most evident application to systems operating without command inputs; certain simple forms of input can also be considered in some cases. An analytic procedure for ignoring the detailed characteristics of the trajectories themselves, and just observing whether or not they tend in a generalized sense toward an equilibrium point, is given by the direct method of Lyapunov. A positive definite scalar function of the state variables which has certain required properties is defined. It is referred to as a Lyapunov function; we shall denote it as $V(x)$, where x is the vector of state variables. The time rate of change of this function, $\dot{V}(x)$, is calculated for motion of x along the system state trajectories. The sign of this derivative function in each region of the state space determines whether the state trajectories in that region tend generally toward or away from the origin of the space which is taken at an equilibrium point. Stability or instability of the system can be demonstrated by showing connected regions, including and surrounding the equilibrium point in which $\dot{V}(x)$ has consistently a negative or positive sign.

Failure of any number of choices for the form of the Lyapunov function to demonstrate stability or instability conclusively indicates nothing regarding system properties; it just means that the functions tried did not fit properly the characteristics of the system. Only for linear systems do we have well-defined procedures for choosing functions which give useful indications of stability. For nonlinear systems one can try different functional forms, but the search for a good one often goes unrewarded. To quote Popov, who has worked extensively with the method, "The study of stability by means of Lyapunov theorems is in principle universal, but in practice limited" (Ref. 11).

It is even possible to construct $V(x)$ functions whose time derivatives would indicate *bounds on a system limit cycle.* The concept is very appealing, but its implementation has so far failed to produce usefully tight quantitative bounds (Refs. 6, 12).

SERIES-EXPANSION SOLUTION

A whole family of techniques exists which develop the solutions of nonlinear differential equations or express the dynamic properties of nonlinear systems in expansions of various forms. These expansions may be a series of nonlinear-system operators, a power series in some small system parameter, a power series in the running variable—time in the case of dynamic systems—or of some other form. The central question related to these expansions is the speed with which the series *converge.* One can often solve nonlinear differential equations by simply assuming a series form for the solution, such as a power series in the running variable, and solving for the coefficients in the series which cause the solution to obey the differential equation. But the solution form chosen in this way is completely arbitrary, and one has no reason to expect that it will fit the actual solution efficiently. For example, if a system actually has a solution of the form

$$y(t) = A \sin \omega t \tag{1.3-1}$$

the assumed solution form

$$y_a(t) = a + bt + ct^2 + \cdots \tag{1.3-2}$$

cannot generate the solution for an interval of time comparable even with one period of the oscillation with a reasonable number of terms in the series.

More rapidly convergent expansions can be made if one can solve for the approximate response of the system and develop the solution in a series of functions which fit this response efficiently. If such a solution to a nonlinear-system problem is to be achieved, the leading term in the expansion must be the solution to a simpler problem which we are able to solve, and each succeeding term must be derivable from this in some tractable manner. If we confess that the only problems we are really able to solve are linear problems (this statement is intended to be a bit overgeneral), we must expect that the leading term in most useful series solutions will be the solution of a linear problem, and subsequent terms in the expansion will attempt to account for the nonlinear characteristics of the system. Such expansions can then be expected to converge rapidly only if the system is "slightly nonlinear," that is, if the system properties are describable to a good approximation as properties of a linear system. But this is not true of some of the simplest and most commonplace of nonlinear systems, such as a relay-controlled servo. Thus series methods, although they will continue to hold an important place in nonlinear-system theory, are almost certain to be restricted in applicability.

LINEARIZATION

The problem of studying a nonlinear system can be avoided altogether by simply replacing each nonlinear operation by an approximating linear operation and studying the resulting linear system. This allows one to say a great deal about the performance of the approximating system, but the relation of this to the performance of the actual system depends on the validity of the linearizing approximations. Linearization of nonlinear operations ordinarily can be justified only for *small departures* of the variables from nominal operating values. This is pictured in Fig. 1.3-1. Any response which carries variables through a range which exceeds the limits of reasonable linear approximation cannot be described using this technique unless the system is repeatedly relinearized about new operating points, and the resulting solutions patched together. In addition, some commonplace nonlinearities, among them the two-level switch, have a discontinuity at the point which should be

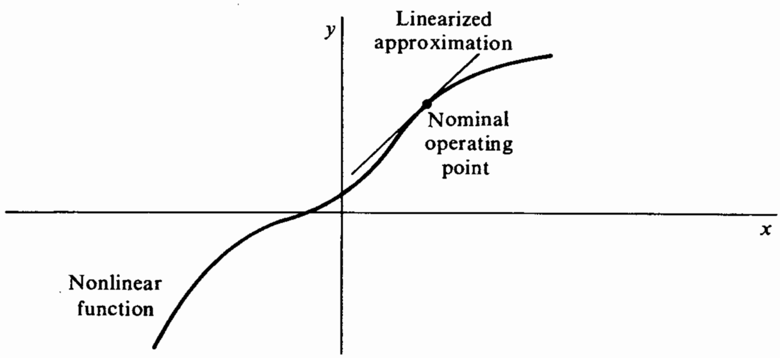

Figure 1.3-1 Illustration of true linearization.

chosen as the operating point. Linearization in the ordinary sense is not possible in these cases.

QUASI-LINEARIZATION

If the small-signal constraint of true linearization is to be relieved, but the advantages of a linear approximation retained, one must determine the operation performed by the nonlinear element on an *input signal of finite size* and approximate this in some way by a linear operation. This procedure results in different linear approximations for the same nonlinearity when driven by inputs of different forms, or even when driven by inputs of the same form but of different magnitude. The approximation of a nonlinear operation by a linear one which depends on some properties of the input is called *quasi-linearization.* It is a kind of linearization since it results in a linear description of the system, but it is not true linearization since the characteristics of the linear approximation change with certain properties of the signals circulating through the system.

This notion is illustrated in Fig. 1.3-2 for a general saturation-type nonlinearity. True or small-signal linearization about the origin would approximate the nonlinear function by a fixed gain which is the slope of the nonlinear function at the origin. However, if the signal at the input to this nonlinearity ranges into the saturated regions, it seems intuitively proper to say that the effective gain of the nonlinearity is lower than that for small signals around the origin. Such a gain, which depends on the magnitude of the nonlinearity input, is illustrated in the figure, and results from a quasi-linearization of the nonlinear function.

Quasi-linearization enjoys a very substantial advantage over true linearization in that there is *no limit to the range of signal magnitudes* which can be accommodated. Moreover, a completely linearized model can exhibit only

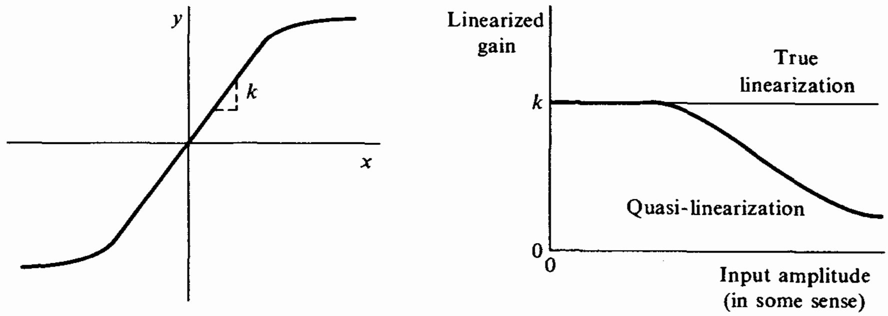

Figure 1.3-2 Illustration of quasi-linearization.

linear behavior, whereas a quasi-linearized model exhibits the basic characteristic of nonlinear behavior: *dependence of performance on signal amplitude*. On the other hand, a quasi-linearized model is more difficult to employ. A linearized model depends only on the system and the choice of nominal operating point. A quasi-linearized model depends on the system and certain properties of the signals circulating in that system. This gives rise to the inevitable requirement for simultaneous solution of two problems: (1) the quasi-linearized model is used to solve for the signals in the system, and (2) certain properties of these signals are used to define the quasi-linearized model. In spite of these difficulties, and more yet to be discussed, quasi-linearization stands as a most valuable tool in nonlinear-system study. A substantial number of interesting and important characteristics of nonlinear-system behavior can be studied better with this technique than with any other.

1.4 THE DESCRIBING FUNCTION VIEWPOINT

Quasi-linearization must be done for *specified input signal forms*. Any form of input to the nonlinear operator can be considered, the output calculated, and the result approximated by the result of a linear operation. However, for feedback-system configurations which are of primary interest to control engineers, the signal at the input to the nonlinearity depends both on the input to the system and the signal fed back within the system. The presence of the fed-back signal complicates considerably the determination of the form of signal which appears at the input to the nonlinearity. Practical solution of this problem for feedback-system configurations depends on avoiding the calculation of the signal form by *assuming* it to have a form which is guessed in advance. The forms which may reasonably be expected to appear at the nonlinearity input are those resulting from the filtering effect of the linear part of the loop. This leads us to consider three basic signal forms with which to derive quasi-linear approximators for nonlinear operators:

1. *Bias.*[1] A constant component might be found in the signal at the nonlinearity input due to a bias in nonlinearity output which is propagated through the linear part and around the loop. Or, if the linear part contains one or more integrations, it can support a biased output even in the absence of a biased input.

2. *Sinusoid.* The limit of all periodic functions as the result of low-pass linear filtering is a sinusoid. Thus any periodic signal at the nonlinearity output would tend to look more like a sinusoid after propagation through the linear part and back to the nonlinearity input.

3. *Gaussian process.* Random processes with finite power density spectra tend toward gaussian processes as the result of low-pass linear filtering.[2] The restriction to finite power density spectra rules out bias and sinusoidal signals. But these have already been singled out for separate attention. Thus any random signal at the nonlinearity output may be expected to look more nearly gaussian after propagation through the linear part back to the nonlinearity input.

These three forms of signal, which we have some reason to expect to find at the input to the nonlinearity, are the principal bases for the calculation of approximators for nonlinear operators in this book. The quasi-linear approximating functions, which describe approximately the transfer characteristics of the nonlinearity, are termed *describing functions.* The major limitation on the use of these describing functions to describe system behavior is the requirement that the actual signal at the nonlinearity input approximate the form of signal used to derive the describing functions.

Within this requirement that the linear part of the system filter the output of the nonlinearity sufficiently, describing function theory provides answers to quite general questions about nonlinear-system operation. The response of systems to the whole class of inputs consisting of linear combinations of these limiting signal forms can be calculated. Even more general system inputs can be handled; the only requirement is that the input to the nonlinearity be of appropriate form. This includes, of course, the special case of zero input. The important problem of limit cycle determination is most expeditiously solved by describing function analysis. Situations involving certain combinations of limit cycle and forced response can also be treated: an example of a limit cycling system responding to a ramp command input and a random disturbance is discussed in the text. This must be considered

[1] Throughout the text, the term "bias" implies a constant, or dc, signal.

[2] This limiting behavior of many random processes is discussed briefly in Appendix H.

by any reasonable standard to be a very complicated problem, but it is susceptible to practical solution by the describing function technique. A significant further advantage is the fact that the solution procedure *does not break down*, in the sense of becoming very much more complicated, *for systems of higher order* than second or third, as is true of some analytic techniques. In the case of sinusoidal and constant signals, the order of the linear part of the system is of very little consequence. In the case of random signals, the analytic expressions for mean-squared signal levels become more complicated with increasing order of the system, but systems of order five, six, or seven are perfectly practical to deal with. And if graphical, rather than analytic, integration of the spectrum is used, systems of arbitrary order can be handled with nearly equal facility.

But the principal advantage of describing function theory is not that it permits the approximate calculation of the response of a given system to a given input or class of inputs; this can always be done by computer simulation. The real advantage, which justifies the development of an approximate theory such as this, is that it serves as a valuable aid to the *design* of nonlinear systems. There are certain situations in which describing functions permit a *direct synthesis* of nonlinear systems to optimize some performance index. In other cases, the *compensation* required to meet some performance specification becomes apparent upon application of describing function theory. In any case, the *trends in system performance characteristics* as functions of system parameters are probably more clearly displayed using describing function theory than with any other attack on nonlinear-system design. An analytic tool yielding this kind of information, even approximately, is of greater value to the system designer than an exact analytic tool which yields only specific information regarding the behavior of the system under specific circumstances.

The describing function technique has its limitations as well. The fundamental limitation is that the *form of the signal* at the input to the nonlinearity *must be guessed in advance*. For feedback configurations, this guess is usually taken to be one of the limiting signal forms discussed above for the reasons cited there. A less obvious limitation, which is probably true of every method of nonlinear-system study, is the fact that *the analysis answers only the specific questions asked of it*. If the designer does not ask about all important aspects of the behavior of a nonlinear system, describing function analysis will not disclose this behavior to him. For example, if one uses the two-sinusoid-input describing function to study subharmonic resonance, he would conclude—as many writers have—that a system with an odd static single-valued nonlinearity cannot support a subharmonic resonance of even order. Actually, the describing function is telling him that such a resonance cannot exist with just the two assumed sinusoids at the input to the nonlinearity. An even subharmonic resonance can indeed exist in such a system, but it will be a biased asymmetric mode. Or again, use of the single-sinusoid-input describing function may indicate that a system has two stable limit cycles, and one might expect to see either one, depending on initial conditions. In some cases, however, use of the bias-plus-sinusoid-input describing function would show that in the presence of one of the limit cycles the system has an unstable small-signal mode. Thus the system is unable to sustain that limit cycle. We conclude that the analysis is tailored to the evaluation of particular response characteristics. The burden of deciding what characteristics should be inquired into rests with the system designer.

Another difficulty which the user of describing function theory must be alert to is the possibility of *multiple solutions*. Formulation of a problem using describing functions results in a simultaneous set of nonlinear algebraic relations to be solved. More than one solution may exist. These solutions represent different possible modes of response, some of which in some cases may be shown to be unstable. But the characteristics of these different solutions may be quite different, and the designer could be badly misled if he did not inquire into the possibility of other solutions. As an illustration of this, the gaussian-plus-sinusoid-input describing function can be used to determine how much random noise must be injected into a system to quench a limit cycle. The equations defining the behavior of the system may have a solution for a zero limit cycle amplitude and a certain rms value of the noise. However, one cannot conclude from this that the calculated rms value of noise will quench the limit cycle until he has assured himself that there is not also another solution for the same rms noise and a nonzero limit cycle amplitude.

A final limitation on the use of describing function theory is the fact that there is no satisfactory evaluation of the *accuracy* of the method. Research into this problem on the part of quite a few workers has resulted in some intuitively based criteria which are rather crude and some analytically based procedures which are impractical to use. All we have, then, is the fact that a great deal of experience with describing function techniques has shown that they work very well in a wide variety of problems. Furthermore, in those cases in which the technique does not work well, it is almost always obvious that the linear part of the system is providing very little low-pass filtering of the nonlinearity output. Finally, since the design of a nonlinear system must be based on the use of approximate analytic techniques, and these techniques will be inadequate to answer all questions regarding system behavior, the design must be checked—preferably by computer simulation—before it is approved. At that point in the design process one need not concern himself with checking the accuracy of the approximate analytic tools he has used in arriving at the design. Rather, his object is to check the design itself, to assure himself of its satisfactory performance in a variety of simulated situations. The recommended procedure is, then, to use describing functions as an aid to system design, watching only for the obvious situations in which the theory might not have good application, and then to check the design by simulation.

1.5 A UNIFIED THEORY OF DESCRIBING FUNCTIONS

A quasi-linear approximator for a nonlinear operator is formed by noting the operation performed by the nonlinearity on an input of specified form, and approximating this operation in some way by a linear operation. We shall take the input to the nonlinearity to have a rather general form: consider the input $x(t)$ to be the sum of any number of signals, $x_i(t)$, each of an identifiable type. In later application, these input components $x_i(t)$ are taken to be constant signals, sinusoids, and gaussian processes, for the reasons discussed in the preceding section. For the purpose of the present theoretical development, however, no specialization is required. Corresponding to this form of input, the most general form of linear approximator for the nonlinearity is a parallel set of linear operators, one to pass each component of the input. Each input component to be considered is stationary, and if we restrict our attention to invariant nonlinearities, the linear operators which comprise the quasi-linear approximator can be taken as invariant at the outset. The resulting approximator for the nonlinearity is shown in Fig. 1.5-1. The $w_i(t)$ in this figure are the weighting functions for the filters which pass the different input components.

Having chosen this form for the quasi-linear approximator, it remains to decide on what basis to make the approximation, that is, what criterion to use in choosing the $w_i(t)$. The criterion used in the present development is minimum mean-squared error; the filters in the linear approximator are designed to minimize the mean-squared difference between the output of that approximator and the output of the nonlinearity. There are a number of reasons for this choice. As is always true in optimum linear theory—whether optimum filtering, optimum control, optimum estimation, or other special case—the quadratic error criterion, of all reasonable criteria, leads to the most tractable formulation of the optimum design problem. This suggests that the minimum mean-squared error criterion is advantageous, not only because it is analytically tractable, but also because the development based on this criterion runs a very close parallel to other optimum linear theory based on the same criterion. Thus, those who are familiar with, for example, Wiener's optimum filter theory for the separation of a signal from noise will have no difficulty following this theory. In addition, a criterion is desired which is universally applicable to all forms of input signal. No other such criterion has been demonstrated to give superior results over a broad range

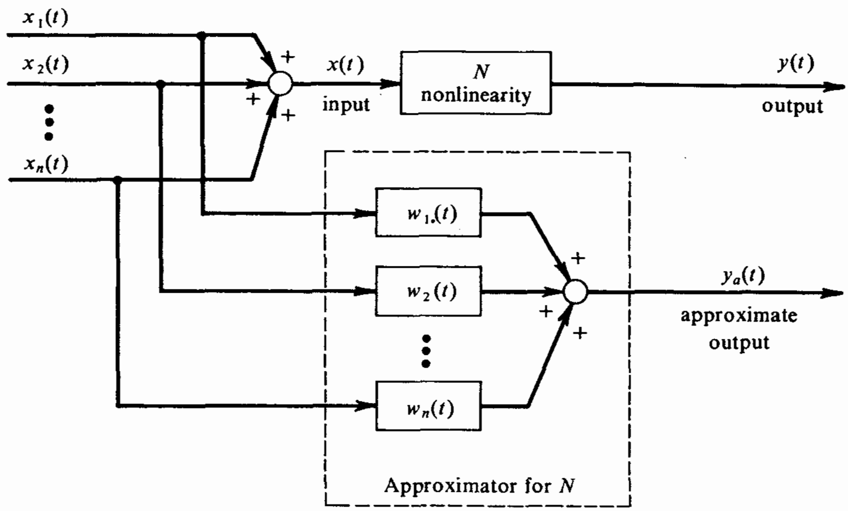

Figure 1.5-1 General linear approximator for a nonlinear operator.

of problems. This matter is discussed more fully in Chap. 2 with respect to sinusoidal inputs, and in Chap. 7 as it relates to random inputs.

This concept of forming a quasi-linear approximator to a nonlinear operator so as to minimize the mean-squared error was initiated by Booton (Ref. 3). He considered only a single input component—a random process with finite power density spectrum—and took the approximator to be a static gain. He showed separately that under the conditions he was considering, the optimum linear approximator, static or dynamic, was a static gain (Ref. 2). The concept was extended by Somerville and Atherton (Ref. 13) to treat an input of the type considered here: the sum of a number of components of identifiable form. They took the approximator to be a parallel set of static gains. In the present development we take the approximator to have the most general linear form: a parallel set of dynamic linear operators. Those cases in which the optimum linear operator is a static gain will appear as consequences of this theory.

The problem of determining the optimum linear approximator to a nonlinearity being driven by an input of specified form is treated here as a statistical problem. This is necessary to permit a unified attack. The repertory of input components to be considered must include random processes for which statistical techniques are essential. The deterministic signals to be considered as well can be formulated as simple forms of random processes; so a statistical approach embraces all forms of input. From this viewpoint, the mean-squared error which is to be minimized is seen as the expectation of the squared approximation error at any one time over all members of the ensemble of possible inputs. The superscript bar used in this development indicates in every case an ensemble average. The reader who needs additional background in the statistics of random processes is referred to the brief presentation in Appendix H or the more complete discussions in Refs. 4 and 7 to 9.

THE OPTIMUM QUASI-LINEAR APPROXIMATOR

The linear approximator of the form shown in Fig. 1.5-1 which minimizes the mean-squared approximation error is now derived. The error in the approximation is

$$e(t) = y_a(t) - y(t) \tag{1.5-1}$$

and its mean-squared value

$$\overline{e(t)^2} = \overline{y_a(t)^2} - \overline{2y_a(t)y(t)} + \overline{y(t)^2} \tag{1.5-2}$$

Now

$$y_a(t) = \sum_{i=1}^{n} \int_0^\infty w_i(\tau)x_i(t-\tau)\,d\tau \tag{1.5-3}$$

so

$$\begin{aligned}\overline{y_a(t)^2} &= \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, w_i(\tau_1)w_j(\tau_2)\overline{x_i(t-\tau_1)x_j(t-\tau_2)} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, w_i(\tau_1)w_j(\tau_2)\varphi_{ij}(\tau_1-\tau_2)\end{aligned} \tag{1.5-4}$$

under the definition

$$\varphi_{ij}(\tau) = \overline{x_i(t)x_j(t+\tau)} \tag{1.5-5}$$

Also

$$\overline{y_a(t)y(t)} = \sum_{i=1}^{n} \int_0^\infty w_i(\tau)\overline{y(t)x_i(t-\tau)}\,d\tau \tag{1.5-6}$$

A necessary condition on the optimum set of weighting functions is derived from the observation that $\overline{e(t)^2}$ must be stationary with respect to variations in the $w_i(t)$ from the optimum set. To formulate an analytic statement of this requirement, we express each of the weighting functions as the optimum function plus a variation.

$$w_i(t) = w_{oi}(t) + \delta w_i(t) \tag{1.5-7}$$

The variations $\delta w_i(t)$ are arbitrary, except that they must represent physically realizable weighting functions. These expressions are used in Eqs. (1.5-4) and (1.5-6) to give

$$\overline{y_a(t)^2} = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^\infty d\tau_1\int_0^\infty d\tau_2[w_{oi}(\tau_1)+\delta w_i(\tau_1)][w_{oj}(\tau_2)+\delta w_j(\tau_2)]\varphi_{ij}(\tau_1-\tau_2) \tag{1.5-8}$$

$$\overline{y_a(t)y(t)} = \sum_{i=1}^{n}\int_0^\infty d\tau_1[w_{oi}(\tau_1)+\delta w_i(\tau_1)]\overline{y(t)x_i(t-\tau_1)} \tag{1.5-9}$$

With these expressions for the terms appearing in Eq. (1.5-2), the mean-squared error can be written in expanded form. First, the terms which do not involve any of the variational functions constitute the stationary value of $\overline{e(t)^2}$; this value will be shown to be a minimum, and thus an optimum, value.

$$\begin{aligned}\overline{e(t)_o{}^2} &= \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^\infty d\tau_1\int_0^\infty d\tau_2\, w_{oi}(\tau_1)w_{oj}(\tau_2)\varphi_{ij}(\tau_1-\tau_2)\\ &\quad - 2\sum_{i=1}^{n}\int_0^\infty d\tau_1\, w_{oi}(\tau_1)\overline{y(t)x_i(t-\tau_1)} + \overline{y(t)^2}\end{aligned} \tag{1.5-10}$$

Next, the first-degree terms in variational functions constitute the first variation in $\overline{e(t)^2}$. It is this first variation which must vanish to define the stationary point.

$$\begin{aligned}\delta\overline{e(t)^2} &= \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^\infty d\tau_1\int_0^\infty d\tau_2\, w_{oi}(\tau_1)\,\delta w_j(\tau_2)\varphi_{ij}(\tau_1-\tau_2)\\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^\infty d\tau_1\int_0^\infty d\tau_2\,\delta w_i(\tau_1)w_{oj}(\tau_2)\varphi_{ij}(\tau_1-\tau_2)\\ &\quad - 2\sum_{i=1}^{n}\int_0^\infty d\tau_1\,\delta w_i(\tau_1)\overline{y(t)x_i(t-\tau_1)}\\ &= 2\sum_{i=1}^{n}\int_0^\infty d\tau_1\,\delta w_i(\tau_1)\left[\sum_{j=1}^{n}\int_0^\infty d\tau_2\, w_{oj}(\tau_2)\varphi_{ij}(\tau_1-\tau_2) - \overline{y(t)x_i(t-\tau_1)}\right]\end{aligned} \tag{1.5-11}$$

In this reduction we have used the fact that $\varphi_{ji}(\tau_2-\tau_1)=\varphi_{ij}(\tau_1-\tau_2)$ for all i and j.

Finally, the second-degree term in variational functions is the second variation in $\overline{e(t)^2}$. The sign of this term determines the nature of the stationary point defined by the vanishing of the first variation.

$$\delta^2\overline{e(t)^2} = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^\infty d\tau_1\int_0^\infty d\tau_2\,\delta w_i(\tau_1)\,\delta w_j(\tau_2)\varphi_{ij}(\tau_1-\tau_2) \tag{1.5-12}$$

This expression is identical in form with Eq. (1.5-4), the mean-squared output of the linear approximator. Thus the second variation may be interpreted as the mean-squared value of the sum of the outputs of a parallel set of filters with weighting functions $\delta w_i(t)$ subjected to the inputs $x_i(t)$. This must be positive; so the stationary point is shown to be a minimum.

The minimum mean-squared error is thus achieved by that set $w_{oj}(\tau_2)$ which causes Eq. (1.5-11) to be zero. But this must be true for arbitrary choices of the variations, $\delta w_i(\tau_1)$. We can be assured of the vanishing of the first variation, then, only if the bracketed term in Eq. (1.5-11) is zero for each value of i over the range of the τ_1 integration. Thus

$$\sum_{j=1}^{n}\int_0^\infty w_{oj}(\tau_2)\varphi_{ij}(\tau_1-\tau_2)\,d\tau_2 = \overline{y(t)x_i(t-\tau_1)} \qquad \tau_1 \geq 0,\ i = 1, 2, \ldots, n \tag{1.5-13}$$

These are the conditions which define the optimum set of filter weighting functions. This result has a simple interpretation which is common to all optimum linear filter theory based on the mean-squared-error criterion. The left-hand member is $\overline{x_i(t)y_a(t+\tau_1)}$, the cross correlation between the ith input component and the approximate output. Because of the stationary character of the input components and nonlinearity output, the right-hand member can be written $\overline{x_i(t)y(t+\tau_1)}$, the cross correlation between the ith input component and the actual output of the nonlinearity. *The set of filters which minimizes the mean-squared approximation error is that set which equates, over the nonnegative range of their arguments, the input-output cross-correlation functions for the nonlinearity and its approximation. This cross-correlation equivalence is demanded for each component of the input* according to Eq. (1.5-13). This property of cross-correlation equivalence is also true, for example, of the Wiener filter.

Since the correlation between input and output is preserved in the approximate output, the error in the approximation must be uncorrelated with the input. This can readily be shown to be so; viz.,

$$\begin{aligned}
\overline{x(t)e(t+\tau)} &= \sum_{i=1}^{n}\overline{x_i(t)[y_a(t+\tau)-y(t+\tau)]}\\
&= \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\int_0^\infty w_{oj}(\tau_2)\overline{x_i(t)x_j(t+\tau-\tau_2)}\,d\tau_2 - \overline{y(t+\tau)x_i(t)}\right]\\
&= \sum_{i=1}^{n}\left[\sum_{j=1}^{n}\int_0^\infty w_{oj}(\tau_2)\varphi_{ij}(\tau-\tau_2)\,d\tau_2 - \overline{y(t)x_i(t-\tau)}\right]\\
&= 0 \qquad \text{for } \tau \geq 0
\end{aligned} \tag{1.5-14}$$

according to Eq. (1.5-13). The correlation between actual and approximate outputs is

$$\begin{aligned}
\overline{y(t)y_a(t+\tau)} &= \sum_{i=1}^{n} \int_0^\infty w_{oi}(\tau_1)\overline{y(t)x_i(t+\tau-\tau_1)}\,d\tau_1 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^\infty d\tau_1 \int_0^\infty d\tau_2\, w_{oi}(\tau_1)w_{oj}(\tau_2)\varphi_{ij}(\tau_1-\tau-\tau_2) \\
&= \overline{y_a(t)y_a(t-\tau)} \qquad \tau \le 0 \\
&= \overline{y_a(t)y_a(t+\tau)} \qquad \tau \ge 0
\end{aligned} \tag{1.5-15}$$

So the cross correlation between actual and approximate outputs is equal to the autocorrelation of the approximate output over the indicated range of τ. The restrictions on the range of τ for which Eqs. (1.5-14) and (1.5-15) hold are due to the restricted range of τ_1 for which Eq. (1.5-13) is a required condition.

The statistics of the approximation error can now be written.

$$\begin{aligned}
\overline{e(t)} &= \overline{y_a(t) - y(t)} \\
&= \sum_{i=1}^{n} \int_0^\infty w_{oi}(\tau_1)\overline{x_i(t-\tau_1)}\,d\tau_1 - \overline{y(t)}
\end{aligned} \tag{1.5-16}$$

If the input to the nonlinearity, $x(t) = \sum_{i=1}^{n} x_i(t)$, has a nonzero mean value, there is no prescribed way of associating this constant with the various components $x_i(t)$. The assignment of the mean to the $x_i(t)$ can be made arbitrarily with no loss of generality. The most convenient convention to employ is to assign all of the mean of $x(t)$ to one of the $x_i(t)$ which is just a constant, or bias, function. With this convention, all but one of the $x_i(t)$ are unbiased functions; the remaining $x_i(t)$ is a constant function equal to the mean of $x(t)$. Equation (1.5-16) then becomes

$$\overline{e(t)} = \overline{x(t)} \int_0^\infty w_{oi}(\tau_1)\,d\tau_1 - \overline{y(t)} \tag{1.5-17}$$

where $w_{oi}(\tau_1)$ is the weighting function of the filter which passes the bias input component. This weighting function has yet to be determined on the basis of minimizing the mean-squared approximation error; we shall find that the result also has the desirable property of reducing the mean error as expressed in Eq. (1.5-17) to zero.

The mean-squared approximation error, which is minimized by the set of filters that satisfy the conditions of Eq. (1.5-13), is

$$\begin{aligned}
\overline{e(t)_o^2} &= \overline{y_a(t)^2} - 2\overline{y_a(t)y(t)} + \overline{y(t)^2} \\
&= \overline{y_a(t)^2} - 2\overline{y_a(t)^2} + \overline{y(t)^2} \\
&= \overline{y(t)^2} - \overline{y_a(t)^2}
\end{aligned} \tag{1.5-18}$$

using Eq. (1.5-15) with $\tau = 0$. But this also gives

$$\overline{y_a(t)^2} = \overline{y(t)^2} - \overline{e(t)_o^2} \tag{1.5-19}$$

which demonstrates the fact that *linear filters which minimize the mean-squared approximation error always underestimate the mean-squared output of the nonlinearity.* This should not be interpreted necessarily as a failure of the approximator. The power in the output of the nonlinearity is spread over a wider band of frequencies than that of the approximate output, and thus suffers greater attenuation in passing through the linear part of the system. It is most important for the approximated system to yield nearly correct statistics for the input to the nonlinearity since the approximation itself depends on these quantities, and to achieve this we should expect that the mean-squared output of the approximator would have to be less than the mean-squared output of the nonlinearity.

The most general expression for the filter weighting functions is the solution of Eq. (1.5-13), a coupled set of simultaneous integral equations. However, the case of almost universal interest is that in which the various components of the input are *statistically independent.* Conditions under which this property may be expected to hold are discussed later, in connection with specific input forms. Independence of the input components, together with the convention of assigning any bias in $x(t)$ to one input component, thus leaving all but one component with zero mean, assures that the different input components are uncorrelated.

$$\begin{aligned}\varphi_{ij}(\tau) &= \overline{x_i(t)x_j(t+\tau)} \\ &= \overline{x_i(t)}\;\overline{x_j(t+\tau)} \\ &= 0 \qquad i \neq j\end{aligned} \tag{1.5-20}$$

This serves to uncouple the components of Eq. (1.5-13) so that they become simply

$$\int_0^\infty w_{oi}(\tau_2)\varphi_{ii}(\tau_1 - \tau_2)\,d\tau_2 = \overline{y(t)x_i(t-\tau_1)} \qquad \tau_1 \geq 0, i = 1, 2, \ldots, n \tag{1.5-21}$$

This expression, which is hereafter used to define the filter weighting functions, defines a more specific form of cross-correlation equivalence than Eq. (1.5-13). For independent input components, the cross correlation between input and output of each filter is independently equated, over the nonnegative range of their arguments, to the cross correlation between that component of input and the output of the nonlinearity. It is still true, as was noted earlier, that the cross correlation between each component of input and the approximate output is being equated to the cross correlation between that component of input and the actual output of the nonlinearity; but in the case of independent input components the entire cross correlation between $x_i(t)$ and $y_a(t)$ is due to the cross correlation between $x_i(t)$ and the output of the filter which passes that particular input component. So there is a kind of isolation among the parallel paths which make up the approximator to the nonlinearity; the signals passing through these paths are uncorrelated. However, each of the filter weighting functions depends not only on the form of the nonlinearity and the characteristics of the input component which that filter passes, but on the characteristics of all other input components as well, through the $y(t)$ term appearing in Eq. (1.5-21). Thus the basic nature of a nonlinear operator, the failure of the property of superposition, is evidenced in this approximator to the nonlinearity.

DESCRIBING FUNCTIONS CORRESPONDING TO SPECIFIC INPUT FORMS

We now specialize Eq. (1.5-21) to the cases of the limiting input signal forms under consideration, each in the presence of other independent input components.

Bias First determine the optimum filter to operate on the constant input component, the $x_i(t)$ which is the mean value of $x(t)$. To emphasize the nature of this input function we denote it by B, a bias, and write this component of the input explicitly.

$$x(t) = B + x_r(t) \tag{1.5-22}$$

Here the input to the nonlinearity is written as the sum of the bias component and the remainder, $x_r(t)$, which is the sum of all other components of $x(t)$. The remainder can be an arbitrary collection of functions provided that they are statistically independent of B. This is not to say that the *statistics* of the remainder are independent of B. When this nonlinearity is treated as part of a feedback system so that its input is related to its output, the statistics of the remainder, such as the standard deviation of a gaussian component, become functionally related to each other and to B. But the independence referred to here, which justifies the use of Eq. (1.5-21), simply requires that there be no correlation between the instantaneous value of the remainder at any one time and the value of the bias. This is always true since the bias is a deterministic quantity and the remainder is unbiased. Thus

$$\overline{Bx_r(t)} = B\overline{x_r(t)} = 0 \tag{1.5-23}$$

The autocorrelation function for the bias input component, $x_i(t) = B$, is

$$\begin{aligned}\varphi_{ii}(\tau) &= \overline{x_i(t)x_i(t+\tau)}\\ &= \overline{B^2}\\ &= B^2\end{aligned} \tag{1.5-24}$$

Equation (1.5-21) in this case becomes

$$\int_0^\infty w_B(\tau_2)B^2\,d\tau_2 = \overline{y(t)}B \tag{1.5-25}$$

where the weighting function for the optimum filter to pass the bias input component has been written $w_B(\tau_2)$. Since we are considering only stationary statistics, the mean output of the nonlinearity is a constant, and the integral equation is clearly satisfied by

$$w_B(\tau_2) = \frac{1}{B}\overline{y(t)}\,\delta(\tau_2) \tag{1.5-26}$$

where $\delta(\tau_2)$ *is the unit-impulse function of argument* τ_2. A filter whose weighting function, or response to a unit-impulse input, is a unit impulse scaled by K, is recognized to be just a static gain of magnitude K. This is not the only possible solution of Eq. (1.5-25), but any other solution would also represent a filter with the dc gain $(1/B)\overline{y(t)}$. The simplest solution is then a static gain of this magnitude. The generality involved in the original assumption of a dynamic filter to pass a bias component was not necessary since the only characteristic of that filter which is of consequence is its zero frequency gain. However, the analytic development was facilitated by the consistent assumption of a dynamic linear operator for each input component.

The linear filter operating on the bias input component which minimizes the mean-squared approximation error is thus found to be the static gain which equates the mean output of the approximator to that of the nonlinearity. This approximator may therefore be described as the unbiased minimum-variance quasi-linear approximator for the nonlinearity. This gain is called the *describing function* for the bias input component and is denoted by N_B.

$$\begin{aligned} N_B &= \frac{1}{B}\overline{y(t)} \\ &= \frac{1}{B}\overline{y(0)} \end{aligned} \tag{1.5-27}$$

This step utilizes the fact that only stationary inputs to the nonlinearity are considered. Since the nonlinearity is invariant, the output statistics are also stationary. Thus all expectations are independent of time, and can be calculated at any convenient time, such as $t = 0$. This is done repeatedly in what follows.

Sinusoid Now find the optimum filter to pass a sinusoidal component in the presence of other independent input components. The sinusoidal component of the input is thus written explicitly.

$$x(t) = A \sin(\omega t + \theta) + x_r(t) \tag{1.5-28}$$

Note that the remainder is quite arbitrary provided that it is uncorrelated with the sinusoidal component we are singling out for attention; in particular, it may possibly include other sinusoidal components. The sinusoid considered here has a deterministic amplitude and frequency. If the phase angle of the sinusoid were known, there would remain no need to find a filter to approximate the transfer characteristics of the nonlinearity in passing the known sinusoid. One could then just replace the nonlinearity with a function generator which would produce the known output of the nonlinearity corresponding to the known sinusoidal input. This cannot in fact be done, since we do not have any a priori knowledge of the phase angle, and there is assumed to be no deterministic mechanism operating in the system which will fix the phase angle. The amplitude of a sinusoidal component at the input to a nonlinear device enclosed in a feedback loop is also unknown in advance, but the amplitude is determined by the nature of the system and its inputs. The phase angle, on the other hand, remains undetermined. It is measured against some arbitrary time reference. The basic performance of the system is independent of the choice of time reference, and thus of the phase angle of the sinusoidal component. There is no reason to anticipate that some phase angles will occur with greater likelihood than others; so the phase angle θ in Eq. (1.5-28) is taken as a random variable with uniform distribution over 2π radians. An important consequence of this particular distribution for θ, as noted in Appendix H, is that the resulting ensemble of all possible sinusoids of the form $A \sin(\omega t + \theta)$, with A and ω determined, is statistically stationary.

Since the phase angle θ is the only random variable associated with a sinusoidal input component, independence among several sinusoidal components means independence of their phase angles. A great many problem situations involving multiple sinusoids at the nonlinearity input do have this property of independence of phase. Some important situations in which this is not true are also discussed in Chap. 5. For the present, the describing function is determined under the assumption of independence.

For $x_i(t) = A \sin(\omega t + \theta)$, with A and ω determined, and θ uniformly distributed over 2π radians, the autocorrelation function is

$$\begin{aligned}
\varphi_{ii}(\tau) &= \overline{x_i(t)x_i(t+\tau)} \\
&= \overline{x_i(0)x_i(\tau)} \\
&= \frac{1}{2\pi}\int_0^{2\pi} A^2 \sin\theta \sin(\omega\tau + \theta)\, d\theta \\
&= \tfrac{1}{2}A^2 \cos\omega\tau
\end{aligned} \tag{1.5-29}$$

The left-hand member of Eq. (1.5-21) is in this case

$$\int_0^\infty w_A(\tau_2)\tfrac{1}{2}A^2 \cos \omega(\tau_1 - \tau_2)\, d\tau_2$$
$$= \tfrac{1}{2}A^2 \cos \omega\tau_1 \int_0^\infty w_A(\tau_2) \cos \omega\tau_2\, d\tau_2 + \tfrac{1}{2}A^2 \sin \omega\tau_1 \int_0^\infty w_A(\tau_2) \sin \omega\tau_2\, d\tau_2 \tag{1.5-30}$$

The right-hand member is

$$\begin{aligned}\overline{y(t)x_i(t - \tau_1)} &= \overline{y(0)x_i(-\tau_1)} \\ &= \overline{y(0)A \sin (\theta - \omega\tau_1)} \\ &= A \cos \omega\tau_1 \overline{y(0) \sin \theta} - A \sin \omega\tau_1 \overline{y(0) \cos \theta}\end{aligned} \tag{1.5-31}$$

Satisfaction of the equation for all nonnegative values of τ_1 requires

$$\tfrac{1}{2}A \int_0^\infty w_A(\tau_2) \cos \omega\tau_2\, d\tau_2 = \overline{y(0) \sin \theta} \tag{1.5-32}$$

and

$$\tfrac{1}{2}A \int_0^\infty w_A(\tau_2) \sin \omega\tau_2\, d\tau_2 = -\overline{y(0) \cos \theta} \tag{1.5-33}$$

The right-hand members of Eqs. (1.5-32) and (1.5-33) are statistics associated with the random variables $y(0)$ and θ; they are constants. These equations are seen to be satisfied by

$$w_A(\tau_2) = \frac{2}{A}\overline{y(0) \sin \theta}\, \delta(\tau_2) + \frac{2}{A\omega}\overline{y(0) \cos \theta}\, \mathring{\delta}(\tau_2) \tag{1.5-34}$$

where $\mathring{\delta}(\tau_2)$ is the derivative of the unit-impulse function, the so-called "doublet." The integral of $f(t)\,\delta(t)$ across the point $t = 0$ is $f(0)$, if $f(t)$ is continuous at $t = 0$. Similarly, the integral of $f(t)\,\mathring{\delta}(t)$ across $t = 0$ is $-\mathring{f}(0)$, if $\mathring{f}(t)$ is continuous at $t = 0$, which can be verified by an integration by parts. The transfer function corresponding to this weighting function is the sum of a proportional plus derivative path.

$$W_A(s) = \frac{2}{A}\left[\overline{y(0) \sin \theta} + \left(\frac{1}{\omega}\overline{y(0) \cos \theta}\right)s\right] \tag{1.5-35}$$

This is not the only solution to Eqs. (1.5-32) and (1.5-33), but any other solution would represent a filter which has the same transfer at the frequency of the input sinusoid. As in the case of the bias input component, the assumption of an arbitrary dynamic linear operator to pass a sinusoidal input component is more general than necessary since only the transfer at the frequency of the input sinusoid is of consequence. At any one frequency, a linear filter is just a complex gain which modifies the amplitude and phase of the input. The real and imaginary parts of this complex gain may be thought of as in-phase and quadrature gains. Using notation appropriate to this interpretation, the describing function for a sinusoidal input component in the presence of any other independent components is written

$$\begin{aligned} N_A &= n_p + jn_q \\ n_p &= \frac{2}{A}\overline{y(0)\sin\theta} \\ n_q &= \frac{2}{A}\overline{y(0)\cos\theta} \end{aligned} \tag{1.5-36}$$

For nonlinearities which are static and single-valued, so that $y(0)$ is given unambiguously in terms of $x(0)$, this quadrature gain is always zero. This can be seen from the following calculation. With $x(t)$ given by Eq. (1.5-28), we write

$$\begin{aligned} &\overline{y(0)\cos\theta} \\ &\quad = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_n f_n(p_1, \ldots, p_n) y[A\sin\theta + x_r(0)]\cos\theta \\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_n f_n(p_1, \ldots, p_n) \int_0^{2\pi} y[A\sin\theta + x_r(0)]\cos\theta\, d\theta \end{aligned} \tag{1.5-37}$$

where $1/2\pi$ is the probability density function for θ; $p_1, \ldots, p_n$ are the random variables which define the remainder of the input, $x_r(0)$; and $f_n(p_1, \ldots, p_n)$ is their joint probability density function. The θ integration in Eq. (1.5-37) is an integral over one period of a periodic function. The interval of integration is arbitrary provided only that it encompasses exactly one period.

$$\begin{aligned} &\int_0^{2\pi} y[A\sin\theta + x_r(0)]\cos\theta\, d\theta \\ &\quad = \int_{-\pi/2}^{3\pi/2} y[A\sin\theta + x_r(0)]\cos\theta\, d\theta \\ &\quad = \int_{-\pi/2}^{\pi/2} y[A\sin\theta + x_r(0)]\cos\theta\, d\theta + \int_{\pi/2}^{3\pi/2} y[A\sin\theta + x_r(0)]\cos\theta\, d\theta \end{aligned} \tag{1.5-38}$$

Change the variable of integration in the second term to $\theta' = \theta - \pi$.

$$\begin{aligned} \text{Second term} &= -\int_{-\pi/2}^{\pi/2} y[-A\sin\theta' + x_r(0)]\cos\theta'\, d\theta' \\ &= -\int_{-\pi/2}^{\pi/2} y[A\sin\theta'' + x_r(0)]\cos\theta''\, d\theta'' \\ &= -\text{first term} \end{aligned} \tag{1.5-39}$$

The further change of variable $\theta'' = -\theta'$ was employed in this demonstration of the fact that Eq. (1.5-37), and thus the imaginary part of the describing function for a sinusoidal input to a static single-valued nonlinearity, is zero. For such a nonlinearity, the describing function for a sinusoidal input in the presence of any other uncorrelated inputs is a real static gain, the proportional gain of Eq. (1.5-36).

This gain is subject to an interesting interpretation.

$$N_A = \frac{1}{\pi A}\int_0^{2\pi}\left\{\int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_n f_n(p_1, \ldots, p_n) y[A \sin\theta + x_r(0)]\right\} \sin\theta\, d\theta \tag{1.5-40}$$

The definitions of all variables and functions used here are identical with those of Eq. (1.5-37). The quantity in the braces—the integral with respect to $p_1, \ldots, p_n$—is the expectation of the output of the nonlinearity with θ fixed. This expectation is itself a nonlinear function of θ which appears only in the form of $A \sin\theta$. Call this new function $y'(A \sin\theta)$. In terms of this, the describing function for a sinusoidal input component in the presence of any other independent input components is written

$$N_A = \frac{1}{\pi A}\int_0^{2\pi} y'(A \sin\theta) \sin\theta\, d\theta \tag{1.5-41}$$

But this is just an ordinary harmonic analysis of the modified nonlinearity $y'(x)$. This interpretation is also true of N_A for more general nonlinearities than static single-valued. In that case both the sine and cosine components of the output of the modified nonlinearity must be calculated, and this calculation is considerably more difficult. If the remainder includes a random process, the determination of the expectation of the nonlinearity output with θ fixed is somewhat obscure when the present output depends not only on the present input, but also on the history of that input. However, if the input consists only of a bias and any number of sinusoids, the calculation of the modified nonlinearity is straightforward. In summary, then, *the describing function for a sinusoidal input component may be viewed as the amplitude and phase relationship between an input sinusoid and the fundamental harmonic component of the expectation of the output of the nonlinearity taken with respect to all statistical parameters except* θ. This definition of the gain of a nonlinearity to a sinusoid in the presence of other input components was employed by Vander Velde (Ref. 14), based on an intuitive argument. This property has since been utilized for computational purposes by a number of writers, among them Atherton (Ref. 1), Gusev (Ref. 5), Popov (Ref. 10), and Somerville and Atherton (Ref. 13).

Random signal The last of the signal forms which we are considering is the gaussian process.

$$x(t) = r(t) + x_r(t) \tag{1.5-42}$$

$r(t)$ is a member of a stationary ensemble, and the remainder, $x_r(t)$, is uncorrelated with $r(t)$. For $x_i(t) = r(t)$, the autocorrelation function is simply written as

$$\varphi_{ii}(\tau) = \varphi_{rr}(\tau) \tag{1.5-43}$$

and the corresponding form of Eq. (1.5-21) as

$$\begin{aligned}\int_0^\infty w_R(\tau_2)\varphi_{rr}(\tau_1 - \tau_2)\,d\tau_2 &= \overline{y(0)r(-\tau_1)} \\ &= \varphi_{ry}(\tau_1) \qquad \tau_1 \geq 0\end{aligned} \tag{1.5-44}$$

The right-hand member of this equation, defining the weighting function for the filter which passes a gaussian input component, is the cross correlation between the gaussian input component and the output of the nonlinearity. For a general nonlinearity, the output at time zero, $y(0)$, may depend not only on the current value of the input, but on certain properties of the past history of that input, or on certain derivatives of the input at time zero. The cross-correlation function in Eq. (1.5-44) is an average over all inputs to, and corresponding outputs from, the nonlinearity. The evaluation of this expectation requires the joint probability density function for all the random variables needed to define $y(0)$. Needless to say, this constitutes a formidable task even for dynamic nonlinearities of simple-appearing form.

Even if one is able to evaluate $\varphi_{ry}(\tau_1)$ in some cases, a substantial chore remains. The solution to the integral equation will not be obvious; the equation must be solved in the more general sense. This solution is not difficult if the transform of $\varphi_{ry}(\tau_1)$ can be taken and if the result is a rational function of the transform variable, or can be well approximated by a rational function. If so, the solution to the integral equation can be written down explicitly, since the equation is of the form of the Wiener-Hopf equation. The solution is derived in a number of books, including Refs. 4 and 7 to 9. If transform techniques cannot usefully be employed, the only practical alternative is likely to be numerical solution with computer help. The solution will be some general function for $w_R(\tau_2)$. Thus the optimum linear filter to approximate the effect of the general nonlinearity in passing a gaussian input component is not a static gain, but is indeed some dynamic linear filter, as one would surely expect.

Fortunately, this situation is simplified considerably in the very important case of a static single-valued nonlinearity. In this case the output of the nonlinearity depends only on the current value of the input, and the right-hand member of Eq. (1.5-44) can be reduced to a convenient form if it is written out in detail.

$$\overline{y(0)r(-\tau_1)} = \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_n \int_{-\infty}^{\infty} dr_1 \int_{-\infty}^{\infty} dr_2\, y[r_1 + x_r(0)]r_2 \times f_n(p_1, \ldots, p_n) f_2(r_1, 0; r_2, -\tau_1) \qquad (1.5\text{-}45)$$

In this integral, the p_i are the statistical parameters involved in the remainder, $x_r(0)$, and $f_n(p_1, \ldots, p_n)$ is the joint probability density function for these parameters. The remaining random variables are $r(0)$, whose general value is written as r_1, and $r(-\tau_1)$, whose general value is written as r_2. The joint probability density function for r_1 and r_2 is the bivariate normal distribution whose form is given in Eq. (H-70).

$$f_2(r_1, 0; r_2, -\tau_1) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2\sigma^2(1-\rho^2)}(r_1^2 - 2\rho r_1 r_2 + r_2^2)\right] \qquad (1.5\text{-}46)$$

where $$\rho = \frac{\varphi_{rr}(\tau_1)}{\sigma^2}$$

and $$\sigma^2 = \varphi_{rr}(0)$$

Defining a new variable,

$$r_3 = \frac{r_2 - \rho r_1}{\sigma\sqrt{1-\rho^2}} \qquad (1.5\text{-}47)$$

and writing Eq. (1.5-46) in terms of it,

$$f_2 = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2}\left(r_3^2 + \frac{r_1^2}{\sigma^2}\right)\right] \qquad (1.5\text{-}48)$$

Now Eq. (1.5-45), with r_2 eliminated in favor of r_3, is written

$$\begin{aligned}\overline{y(0)r(-\tau_1)} &= \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_n \int_{-\infty}^{\infty} dr_1 \int_{-\infty}^{\infty} dr_3\, y[r_1 + x_r(0)] f_n(p_1, \ldots, p_n) \\ &\quad \times (\sigma\sqrt{1-\rho^2}\, r_3 + \rho r_1) \frac{1}{2\pi\sigma} \exp\left[-\frac{1}{2}\left(r_3^2 + \frac{r_1^2}{\sigma^2}\right)\right] \\ &= \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_n \int_{-\infty}^{\infty} dr_1\, y[r_1 + x_r(0)] f_n(p_1, \ldots, p_n) \\ &\quad \times \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{r_1^2}{2\sigma^2}\right) \rho r_1 \\ &= \frac{\varphi_{rr}(\tau_1)}{\sigma^2} \overline{y(0)r(0)} \end{aligned} \qquad (1.5\text{-}49)$$

The cross correlation between a gaussian input component and the output of a static single-valued nonlinearity is found to be proportional to the autocorrelation of the gaussian signal. The preceding argument shows that this interesting property holds even if the input to the nonlinearity consists of the sum of the gaussian and other signals, provided that the other signals are not correlated with the gaussian. This is true only of an unbiased gaussian input, but that is the case of interest here under the convention of associating any bias in $x(t)$ with a single input component. For this case, the solution of Eq. (1.5-44) is obvious.

$$w_R(\tau_2) = \frac{1}{\sigma^2}\overline{y(0)r(0)}\,\delta(\tau_2) \tag{1.5-50}$$

So the optimum quasi-linear operator to pass a gaussian input component in the presence of any other uncorrelated components is just a static gain in the case of a static single-valued nonlinearity. This is a result which could not have been anticipated with certainty. This gain, written in describing function notation, is

$$N_R = \frac{1}{\sigma^2}\overline{y(0)r(0)} \tag{1.5-51}$$

These are the describing functions, based on minimum mean-squared approximation error, for the three signal forms which might be expected to appear at the input to the nonlinearity in a feedback system. These signals are chosen because they may be assumed on the basis of the filtering properties of the linear part of the system. It is possible to calculate the optimum quasi-linear approximator to operate on other signal forms as well, and this will be done in Chap. 8 following a different motivation. The point is that for a feedback system, if the forms of the signals at the nonlinearity input are not *assumed*, they must be *calculated*. And this is indeed a forbidding task in most cases.

MULTIPLE INPUTS OF THE SAME FORM

A property of these describing functions which will serve to define the most general form of nonlinearity input that need be considered should be noted here. The input has been taken to be any linear combination of independent signals of the following forms: bias, sinusoid, and gaussian random process. This includes the possibility of any number of signals of the same form. The potential complexity of this situation is considerably reduced if we note, first of all, that there is no reason to consider more than one bias component. Even if the constant signal at the input to the nonlinearity should arise from several different sources, there would be no way to distinguish them, and the sum of all the constant input components would be propagated by the nonlinearity as a single bias. Less obvious than this is the fact that gaussian signals enjoy the same property in the case of static single-valued nonlinearities. If the input to the nonlinearity includes several gaussian components, say, $r_1(t), r_2(t), \ldots, r_n(t)$, the describing function for the ith signal is given by Eq. (1.5-51) to be

$$N_{R_i} = \frac{1}{\sigma_i^2}\overline{y(0)r_i(0)} \tag{1.5-52}$$

which would appear to be different for each i. However, one can show by direct calculation that

$$\frac{1}{\sigma_i^2}\overline{y(0)r_i(0)} = \frac{1}{\sigma_R^2}\overline{y(0)R(0)} \tag{1.5-53}$$

where $$R(t) = r_1(t) + r_2(t) + \cdots + r_n(t)$$

and $$\sigma_R^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$$

So the nonlinearity propagates each of the gaussian components with the same gain—that gain which corresponds to a single gaussian input component consisting of the sum of the individual gaussian contributors. This result may be interpreted in terms of the fact that the sum of a number of independent gaussian signals is a gaussian signal with variance equal to the sum of the variances of the contributors. Once summed, these contributors cannot be distinguished on the basis of the distribution of amplitudes of the total signal. They may still be distinguishable on the basis of their harmonic content, and the linear part of the system can be designed to separate a gaussian signal from a gaussian noise if their power spectral densities are distinct, but the static nonlinearity is insensitive to this property.

The remaining signal form, the sinusoid, does not have this additive property. Thus the most general input for which the describing functions need be calculated is the sum of a single bias, a single gaussian random process, and an arbitrary number of sinusoids.

THE DESCRIBING FUNCTIONS FOR SMALL SIGNALS

The describing functions can now be written in more specific form for the most general input of interest: the sum of a bias, a gaussian signal, and an arbitrary number of sinusoids.

$$x(t) = B + r(t) + A_1 \sin(\omega_1 t + \theta_1) + \cdots + A_n \sin(\omega_n t + \theta_n) \tag{1.5-54}$$

These input components are considered statistically independent. Consider, for the purpose of this section, that the nonlinearity is static and single-valued, so that $y(0)$ depends only on $x(0)$. The random variables required to determine $x(0)$ are $r(0)$, which will be denoted simply as r, and the n phase angles θ_i. Being independent, these random variables have a joint probability density function which is the product of their individual density functions. The variable r has the normal distribution with zero mean and standard deviation σ, and each of the θ_i is uniformly distributed over

$(0,2\pi)$. Thus the describing functions for the bias component, one of the sinusoidal components—say, the first one—and the gaussian component are written, according to Eqs. (1.5-27), (1.5-36), and (1.5-51), as

$$N_B = \frac{1}{B}\int_{-\infty}^{\infty} dr \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n\, y(B + r + A_1 \sin\theta_1 + \cdots + A_n \sin\theta_n) \times \frac{1}{(2\pi)^{n+\frac{1}{2}}\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) \tag{1.5-55}$$

$$N_{A_1} = \frac{2}{A_1}\int_{-\infty}^{\infty} dr \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n\, y(B + r + A_1 \sin\theta_1 + \cdots + A_n \sin\theta_n) \times \sin\theta_1 \frac{1}{(2\pi)^{n+\frac{1}{2}}\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) \tag{1.5-56}$$

$$N_R = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} dr \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n\, y(B + r + A_1 \sin\theta_1 + \cdots + A_n \sin\theta_n) \times r \frac{1}{(2\pi)^{n+\frac{1}{2}}\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) \tag{1.5-57}$$

Given a specific form for the nonlinearity $y(x)$, these integrals can be evaluated to determine the general describing functions for a static single-valued nonlinearity.

For some purposes, an alternative form for these expressions is more useful. Making the change of variable,

$$x = B + r + A_1 \sin\theta_1 + \cdots + A_n \sin\theta_n \tag{1.5-58}$$

and eliminating r from the describing function expressions yields

$$N_B = \frac{1}{B}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n\, y(x) \frac{1}{(2\pi)^{n+\frac{1}{2}}\sigma} \times \exp\left[-\frac{1}{2\sigma^2}(x - B - A_1 \sin\theta_1 - \cdots - A_n \sin\theta_n)^2\right] \tag{1.5-59}$$

$$N_{A_1} = \frac{2}{A_1}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n\, y(x) \sin\theta_1 \frac{1}{(2\pi)^{n+\frac{1}{2}}\sigma} \times \exp\left[-\frac{1}{2\sigma^2}(x - B - A_1 \sin\theta_1 - \cdots - A_n \sin\theta_n)^2\right] \tag{1.5-60}$$

$$N_R = \frac{1}{\sigma^2}\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n\, y(x)(x - B - A_1 \sin\theta_1 - \cdots - A_n \sin\theta_n) \times \frac{1}{(2\pi)^{n+\frac{1}{2}}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x - B - A_1 \sin\theta_1 - \cdots - A_n \sin\theta_n)^2\right] \tag{1.5-61}$$

With this form for the describing functions it is easy to determine the limiting values as the corresponding input components go to zero. If the nonlinearity is asymmetric, the mean output may remain nonzero as the mean input B goes to zero. In that case, N_B goes to infinity as B goes to zero, and it is preferable to work directly with the average output rather than with the effective gain of the nonlinearity to the average input. That average output is BN_B, with N_B given either by Eq. (1.5-55) or Eq. (1.5-59). But if the nonlinearity is odd, so that

$$y(-x) = -y(x) \tag{1.5-62}$$

then the average output of the nonlinearity goes to zero as the average input B does. In that case N_B remains finite as B goes to zero, and Eqs. (1.5-55) and (1.5-59) are indeterminate forms at $B = 0$. This indeterminate form can be evaluated for Eq. (1.5-59). Treat the integral as the numerator and B, from the $1/B$ coefficient, as the denominator. Differentiate both numerator and denominator with respect to B as required by L'Hospital's rule, and form the ratio of these derivatives. The result is exactly the right-hand member of Eq. (1.5-61), the describing function for the gaussian input component.

$$\lim_{B\to 0} N_B = N_R(B = 0) \tag{1.5-63}$$

Similarly, Eq. (1.5-60) is indeterminate for $A_1 = 0$. Differentiation of numerator and denominator with respect to A_1 gives

$$\begin{aligned} \lim_{A_1\to 0} N_{A_1} = 2\int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n\, y(x) \sin^2 \theta_1 \\ \times \frac{1}{(2\pi)^{n+\frac{1}{2}}\sigma} \frac{1}{\sigma^2} (x - B - A_2 \sin \theta_2 - \cdots - A_n \sin \theta_n) \\ \times \exp\left[-\frac{1}{2\sigma^2}(x - B - A_2 \sin \theta_2 - \cdots - A_n \sin \theta_n)^2\right] \end{aligned} \tag{1.5-64}$$

The θ_1 integration can now be done, with the result

$$\begin{aligned} \lim_{A_1\to 0} N_{A_1} = \int_{-\infty}^{\infty} dx \int_0^{2\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_n\, y(x) \\ \times \frac{2\pi}{(2\pi)^{n+\frac{1}{2}}\sigma^3} (x - B - A_2 \sin \theta_2 - \cdots - A_n \sin \theta_n) \\ \times \exp\left[-\frac{1}{2\sigma^2}(x - B - A_2 \sin \theta_2 - \cdots - A_n \sin \theta_n)^2\right] \end{aligned} \tag{1.5-65}$$

But notice that if A_1 is set equal to zero in the expression for N_R [Eq. (1.5-61)], the θ_1 integration can be done, and the result is exactly the right-hand member of Eq. (1.5-65). So

$$\lim_{A_1 \to 0} N_{A_1} = N_R(A_1 = 0) \tag{1.5-66}$$

This demonstrates the very interesting fact that the effective gain of an odd memoryless nonlinearity to a small *bias* component in the presence of any other uncorrelated input components is the same as the effective gain of the nonlinearity to a small *sinusoid* of any frequency in the presence of the same additional input components. And this common gain is the same as the effective gain of that nonlinearity to a *gaussian* input component, again in the presence of the same additional input components. This suggests that the operation of such a nonlinearity on a small signal in the presence of other uncorrelated signals is *independent of the form of the small signal.* This property may seem even more remarkable when one notes that no assumption has been made as to the analyticity or differentiability of the nonlinear characteristic.

1.6 ABOUT THE BOOK

The argument developed in the preceding section provides a unified approach to all describing function theory. For the sake of the added insight it may provide, the more physically motivated historical development of describing functions, which is applicable to sinusoid and constant inputs, is also given in the following chapters. The original single-sinusoid-input describing function is derived in Chap. 2, and applied to the study of steady-state and transient oscillations in Chaps. 3 and 4. Multiple-input describing functions involving sinusoid and bias components at the input to the nonlinearity are derived and applied in Chaps. 5 and 6. The general input, including gaussian processes, sinusoids, and bias components, is treated in Chap. 7. Quasi-linearization based on other forms of nonlinearity input which are applicable to the study of transients in nonlinear systems is the subject of Chap. 8, and in Chap. 9, describing function theory is applied to the study of sampled-data nonlinear systems. Information which the reader may wish to refer to repeatedly has generally been presented in the Appendixes, where it can more conveniently be located.

Examples illustrating various applications of the theory are of simple engineering systems. Other applications, such as the solution of nonlinear differential equations arising in any other context, are treated in the same way, once the mathematical model is formulated. In each case, the system is presented as a mathematical model at the outset. This does not imply that the techniques can usefully be applied only to such simple "mathematical" systems and have little applicability to practical systems. *Every topic considered in this book is a practically useful topic.* It was noted earlier that the describing function method does not suffer as badly as most other analytic methods upon increasing the order of the system to which it is applied. Illustrations could be given by starting with a description of a physical system, deriving a mathematical model of the system, and applying describing function theory to the model. However, this would lengthen the presentation considerably and would serve no better to illustrate the subject of this book, describing function theory and usage. The modeling of a physical system for analytic study is fundamental to engineering analyses by any technique, and is not within the scope of this book.

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PROBLEMS

1-1. The system of Fig. 1-1 has a single nonlinear part, N, and several linear parts, L_1 to L_4. Rearrange this system into a single-loop configuration suitable for describing function analysis.

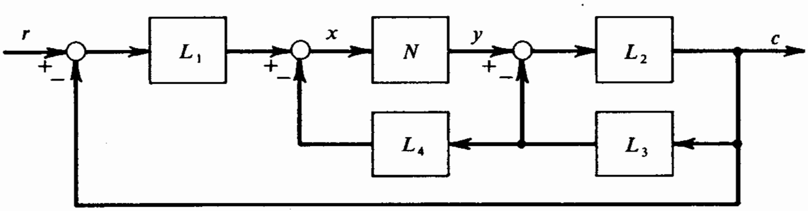

Figure 1-1

1-2. The differential equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + f(y)y = 0$$

has a constant coefficient, a, and a nonlinear coefficient, $f(y)$. Draw the closed loop corresponding to this differential equation, and identify the linear and nonlinear parts.

1-3. Characterize Van der Pol's equation

$$\ddot{y} - \alpha(1 - \beta y^2)\dot{y} + y = x$$

in two different unity-feedback-system configurations, one of which has a single static nonlinear element. [*Hint:* Use the relationship $(d/dt)(y^3) = 3y^2\dot{y}$ to recast the original equation.]

1-4. Transform the rectangular hysteresis nonlinearity (see picture in Appendix B, nonlinearity 45) into a single-valued nonlinearity with a feedback path.

1-5. What would you expect the quasi-linearized gain of a relay with dead zone (see picture in Appendix B, nonlinearity 3) to look like as a function of input amplitude?

1-6. The theory of Sec. 1.5 demonstrates that the describing functions for a static single-valued nonlinearity are static gains. Take an approximator for the nonlinearity in the form of a parallel set of static gains, rather than arbitrary linear operators as in Fig. 1.5-1, and find the expressions for these gains which minimize the mean-squared approximation error. This leads easily to Eq. (1.5-27), n_p in Eq. (1.5-36), and Eq. (1.5-51).

1-7. The most general linear operator that need be considered to approximate the effect of a nonlinearity in passing a sinusoidal input component is the sum of a proportional plus derivative gain. This follows from the fact that the steady-state transfer of any linear filter at a prescribed frequency is just a complex gain which is equivalent to a proportional plus derivative path.

Consider the input to a nonlinearity to be a sinusoid, $A \sin(\omega t + \theta)$, plus any uncorrelated remainder, $x_r(t)$. Use a proportional plus derivative operator to approximate the operation of the nonlinearity on the sinusoid, and derive expressions for the proportional and derivative gains which minimize the mean-squared approximation error. Thus derive Eq. (1.5-35).

1-8. Carry out the reduction of the expression for $\overline{\delta e(t)^2}$ indicated in Eq. (1.5-11).

1-9. What condition, used in deriving the describing function for a sinusoidal input [Eq. (1.5-36)], is violated in the case of two sinusoidal input components having rationally related frequencies?

1-10. The imaginary part of the describing function of a static single-valued nonlinearity for a sinusoidal input component was shown to be zero. Thus, in this case, Eq. (1.5-36) reads

$$N_A = \frac{2}{A}\overline{y(0)\sin\theta}$$

If only a sinusoid is present in the nonlinearity input, this is expressed

$$N_A = \frac{2}{A}\overline{y(A\sin\theta)\sin\theta}$$

where the expectation symbol implies an integration over the distribution of θ, which is uniform over the interval $(0,2\pi)$.

$$N_A = \frac{1}{\pi A}\int_0^{2\pi} y(A\sin\theta)\sin\theta\,d\theta$$

Alternatively, one could first compute the probability density function for $x = \sin\theta$ and then take the expectation by integration over the distribution of x. Carry out this alternative calculation, and derive the equivalent relation

$$N_A = \frac{2}{\pi A}\int_{-1}^{1}\frac{y(Ax)x}{\sqrt{1-x^2}}\,dx$$

1-11. The calculation which leads to Eq. (1.5-53) is quite tedious in the general case of an arbitrary number of gaussian input components. Consider the special case of a nonlinearity input consisting of the sum of a bias, a sinusoid, and two gaussian signals, and verify that relation in the case of the describing functions for the two gaussian components.

1-12. Verify the limiting forms of the describing functions for small inputs [Eqs. (1.5-63) and (1.5-66)] in the cases of the three-input describing functions for which analytic expressions are given in Sec. E-3.