16.30/31, Fall 2010 — Recitation # 3

1 On Linearity and Time Invariance

Consider the systems described by the following input/output relationships:

$$y(t) = u(t) + 1, \tag{1}$$

and

$$y(t) = 2t^3 u(t). \tag{2}$$

Are these systems linear or nonlinear? Are they time-invariant or time-varying?

Linearity — A system is linear if any linear combination of two arbitrary inputs results in an output that is the linear combination of the outputs corresponding to the original inputs.

Let us consider first system (1). We have that:

$$u_1(t) = 1 \quad \Rightarrow y_1(t) = 2, \qquad \text{and} \qquad u_0(t) = 0 \quad \Rightarrow y_0(t) = 1.$$

Clearly, $u_0(t) = 0 \cdot u_1(t)$, but $y_0(t) \neq 0 \cdot y_1(t)$. This *counterexample* shows that the system is *not* linear.

Now consider system (2). Considering two generic inputs u_1 and u_2, we get

$$y_1(t) = 2t^3 u_1(t), \qquad y_2(t) = 2t^3 u_2(t).$$

Now consider the input $u_3(t) = \alpha u_1(t) + \beta u_2(t)$. The corresponding output can be computed as

$$y_3(t) = 2t^3(\alpha u_1(t) + \beta u_2(t)) = \alpha \cdot 2t^3 u_1(t) + \beta \cdot 2t^3 u_2(t),$$

that is, $y_3(t) = \alpha y_1(t) + \beta y_2(t)$. Hence, the system is linear.

Time invariance — A system is time-invariant if it commutes with a time delay, i.e., if the output of the system does not depend on the "origin" of the time axis. Recall that a time delay is a system such that its output is equal to the input, shifted in time by a certain delay (say T), i.e., $y(t) = u(t - T)$.

Let us check whether the systems above do in fact commute with a time delay. For system (1), we have:

$$u(t) \longrightarrow \text{System} \longrightarrow u(t) + 1 \longrightarrow \text{Time Delay} \longrightarrow u(t - T) + 1$$

and

$$u(t) \longrightarrow \text{Time Delay} \longrightarrow u(t - T) \longrightarrow \text{System} \longrightarrow u(t - T) + 1,$$

hence the system is time-invariant.

For system (2), we have:

$$u(t) \longrightarrow \text{System} \longrightarrow 2t^3 u(t) \longrightarrow \text{Time Delay} \longrightarrow 2(t - T)^3 u(t - T)$$

and

$$u(t) \longrightarrow \text{Time Delay} \longrightarrow u(t - T) \longrightarrow \text{System} \longrightarrow 2t^3 u(t - T).$$

Clearly, the two signals are different—the system is time-varying.

2 Linearization

Consider a simplified model of a car, moving on a road inclined by an angle γ with respect to the horizontal plane. Let $v(t)$ be the position of the car along the road at time t. The car is subject to the following forces, in the direction parallel to the road:

- Weight: $-mg\sin\gamma$;
- Aerodynamic drag: $-\frac{1}{2}\rho v(t)^2 S c_x$
- Wheel traction: u,

so that the equation of motion of the car (i.e., $F = ma$) can be written as

$$\dot{v}(t) = f(v(t), u(t)) = -\frac{1}{2m}\rho S c_x v(t)^2 - g\sin\gamma + \frac{1}{m}u(t).$$

(Note that we are considering γ as a fixed parameter.) This model is nonlinear. Let us derive a linearized model for the car's dynamics.

The first step is to define a reference trajectory v_e (and the corresponding input u_e),in such a way that

$$\dot{v}_e(t) = f(v_e(t), u_e(t)). \tag{3}$$

For example, let us consider a constant-speed reference trajectory, as one could do, e.g., for cruise control. In other words, let us choose $v_e(t) = \bar{v}$ where $\bar{v}$ is the reference speed, say 65 mph. The corresponding reference input can be computed as follows:

$$\dot{v}_e(t) = -\frac{1}{2m}\rho S c_x v_e(t)^2 - g\sin\gamma + \frac{1}{m}u_e(t),$$

i.e.,

$$0 = -\frac{1}{2m}\rho S c_x \bar{v}^2 - g\sin\gamma + \frac{1}{m}u_e(t),$$

and hence $u_e(t) = \frac{1}{2}\rho S c_x \bar{v}^2 + mg\sin\gamma$.

The second step is to rewrite the equations of motion as a Taylor series expansion about the reference. Formally,

$$\dot{v}(t) = f(v_e(t), u_e(t)) + \frac{\partial f}{\partial v}(v_e(t), u_e(t)) \cdot \delta v(t) + \frac{\partial f}{\partial u}(v_e(t), u_e(t)) \cdot \delta u(t) + o((\delta v, \delta u)^2),$$

where we defined $\delta v(t) := v(t) - v_e(t)$, and $\delta u(t) := u(t) - u_e(t)$. Assuming that δv and δu are "small" (in the sense that we can ignore the second-order and higher terms in the Taylor series), we can set

$$\dot{v}(t) \approx f(v_e(t), u_e(t)) + \frac{\partial f}{\partial v}(v_e(t), u_e(t)) \cdot \delta v(t) + \frac{\partial f}{\partial u}(v_e(t), u_e(t)) \cdot \delta u(t). \tag{4}$$

Subtracting (3) from (4), and replacing the "approximately equal" with an "equal" sign for simplicity, we get

$$\delta\dot{v}(t) = \underbrace{\frac{\partial f}{\partial v}(v_e(t), u_e(t))}_{A} \cdot \delta v(t) + \underbrace{\frac{\partial f}{\partial u}(v_e(t), u_e(t))}_{B} \cdot \delta u(t),$$

which is, in general, a time-varying linear system (we are tacitly assuming the output is equal to the state v, i.e., the output matrix C is the identity I, and $D = 0$).

Computing the partial derivatives yields:

$$A = \frac{\partial f}{\partial v}(v_e(t), u_e(t)) = -\frac{1}{2m}\rho S c_x 2 v_e(t)$$

$$B = \frac{\partial f}{\partial u}(v_e(t), u_e(t)) = \frac{1}{m}.$$

Notice that in general these matrices can be functions of time, even in the case in which the function f is not, due to the fact that the reference trajectory (and hence the linearization point) depends on time.

Ultimately, we get the linearized model as

$$\delta\dot{v}(t) = -\frac{1}{m}\rho S c_x \bar{v}\ \delta v + \frac{1}{m}\delta u.$$

16.30 / 16.31 Feedback Control Systems
Fall 2010