16.901: Homework # 10
Solution

In this homework, you will complete the Matlab script, **fem_dif1d.m**, to solve the one-dimensional heat transfer problem,

$$T_{xx} + q(x) = 0, \qquad q(x) = 50e^x,$$

with boundary conditions $T(\pm 1) = 100$. This is the same problem we have been looking at in the lecture notes. The script is available on the webpage under the Homework #10 link. You need to complete the two lines in the script that calculate the forcing term integrals,

$$\int_{-L/2}^{L/2} \phi_j q \, dx.$$

The nodal basis for linear finite elements has been used in the script. The lines you need to add are in a loop over the elements. For each element, you calculate two forcing term contributions: the forcing term contribution to the weighted residual associated with node 1 of the element, and the forcing term contribution to the weighted residual associated with node 2 of the element. To do these contributions, the following integral will be required,

$$\int_{x_1}^{x_2} xe^x \, dx = \left[xe^x - e^x\right]_{x_1}^{x_2}.$$

1. Complete the Matlab script and run the simulation for 5 elements and 10 elements. For your homework, include the plots of the solutions as well as a hard copy of your completed script.

   Solution: Since $\phi_j(x)$ is zero in most of the domain, the weighted integrals reduce to,

   $$\int_{-L/2}^{L/2} \phi_j q \, dx = \int_{x_{j-1}}^{x_j} \phi_j q \, dx + \int_{x_j}^{x_{j+1}} \phi_j q \, dx,$$

   where the integral has been split into the portions from elements $j-1$ and j. As implemented in the Matlab script, a loop is performed over the elements and the contributions to any weighted residual are calculated and added to the appropriate array location. Thus, for element i, this would involve calculating one contribution to the weighted residual for ϕ_i,

   $$\int_{x_i}^{x_{i+1}} \phi_i q \, dx,$$

   and another contribution to the weighted residual for ϕ_{i+1},

   $$\int_{x_i}^{x_{i+1}} \phi_{i+1} q \, dx.$$

   Recalling that in element i,

   $$\phi_i(x) = \frac{x_{i+1} - x}{\Delta x}, \qquad \phi_{i+1}(x) = \frac{x - x_i}{\Delta x},$$

   then the weighted residual contributions for this problem are,

   $$\begin{aligned}
   \int_{x_i}^{x_{i+1}} \phi_i q \, dx &= 50 \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{\Delta x} e^x \, dx, \\
   &= \frac{50}{\Delta x} \left[x_{i+1}e^x - xe^x + e^x\right]_{x_i}^{x_{i+1}} \, dx, \\
   &= \frac{50}{\Delta x} \left(e^{x_{i+1}} - x_{i+1}e^{x_i} + x_i e^{x_i} - e^{x_i}\right)
   \end{aligned}$$

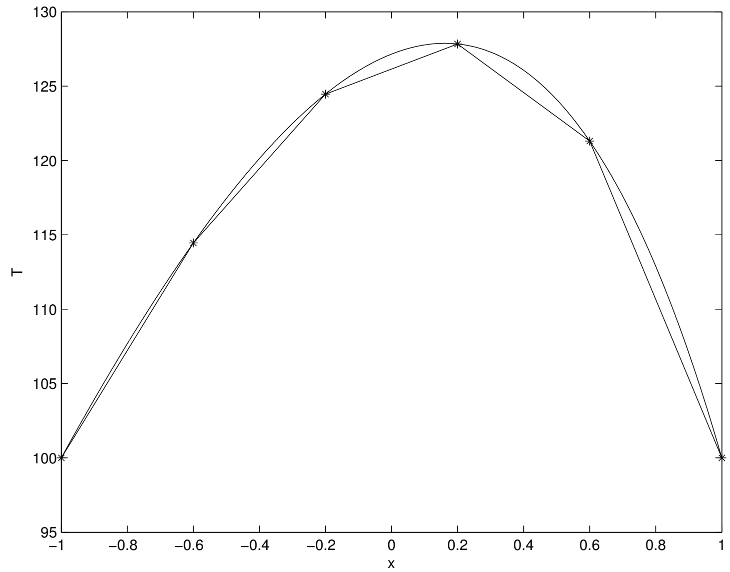
$$
\begin{aligned}
\int_{x_i}^{x_{i+1}} \phi_{i+1} q \, dx &= 50 \int_{x_i}^{x_{i+1}} \frac{x - x_i}{\Delta x} e^x \, dx, \\
&= \frac{50}{\Delta x} \left[x e^x - e^x - x_i e^x \right]_{x_i}^{x_{i+1}} \, dx, \\
&= \frac{50}{\Delta x} \left(x_{i+1} e^{x_{i+1}} - e^{x_{i+1}} - x_i e^{x_{i+1}} + e^{x_i} \right)
\end{aligned}
$$

Finally, these contributions need to be included correctly in the right-hand side vector, which in the Matlab script is the vector, b. Moving the source term over to the right-hand side means that we need to subtract each of the contributions to the appropriate b entry. Specifically,

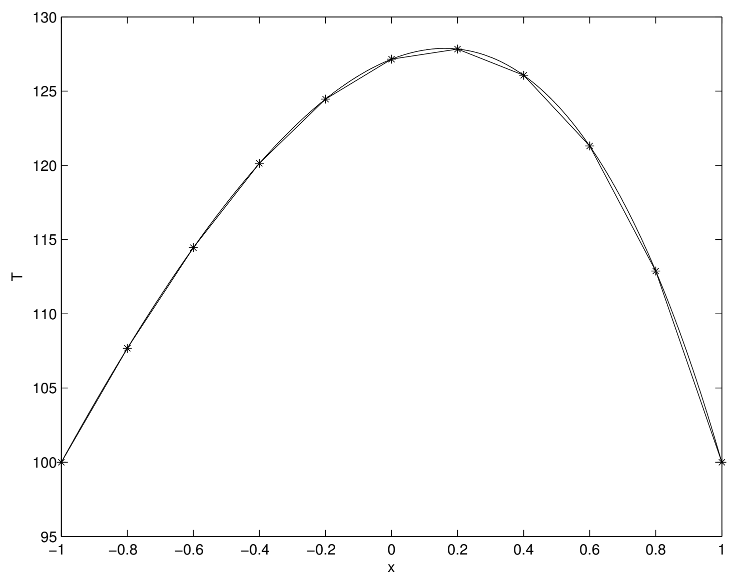
$$
\begin{aligned}
b(i) &= b(i) - \int_{x_i}^{x_{i+1}} \phi_i q \, dx, \\
b(i+1) &= b(i+1) - \int_{x_i}^{x_{i+1}} \phi_{i+1} q \, dx.
\end{aligned}
$$

2. What do you think the order of accuracy is for this finite element method? Justify your answer using the plots from the 5 and 10 element solutions. Note: the exact solution has been included in these plots as the solid line without symbols.

   Solution: The results are shown in Figure 1. Interestingly, the FEM results for linear elements are exact at the nodes. However, in between the nodes (i.e. within the elements), there is error since a linear function is being used to represent a higher-order (curved) solution. The error, i.e. $tildeT(x) - T(x)$, is shown in Figure 2 for both $N = 5$ and $N = 10$ solutions. A clear factor of four reduction is observed with the increased grid resolution leading to the conclusion that the method is second order accurate. Note: to construct this plot, each element was subdivided into 20 points and the FEM and exact solution were calculated at these points and compared.



(a) $N = 5$ elements



(b) $N = 10$ elements

Figure 1: Comparison of finite element solution to exact solution.

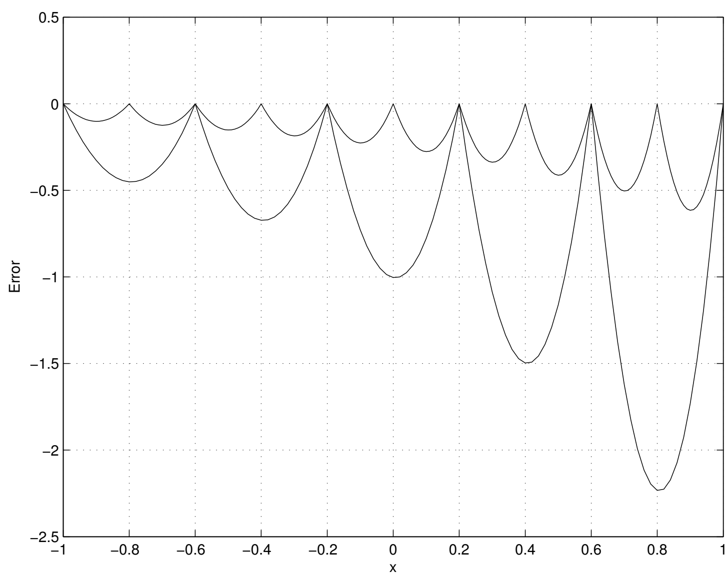


Figure 2: Error of $N = 5$ and $N = 10$ finite element soltuions.