16.901: Homework # 2
Solution

1. Prove that any *consistent* multi-step method must have a root of $z = 1$ in its stability recurrence relationship. Note: this is a very short three to four line proof.

   Solution: The recurrence relationship is,

$$v^{n+1} + \sum_{i=1}^{s} \alpha_i v^{n+1-i} = 0.$$

   Assuming $v^n = z^n$ and letting $z = 1$, we find,

$$1 + \sum_{i=1}^{s} \alpha_i = 0.$$

   As discussed in Lecture 3, consistency requires that the scheme is first-order accurate, thus the local truncation error is $O(\Delta t^2)$. This requires that both the coefficient of the $O(1)$ (i.e. which multiplies u^n) and the $O(\Delta t)$ term (i.e. which multiplies u_t^n) are zero. As shown in the lecture, elimination of the $O(1)$ term requires that

$$1 + \sum_{i=1}^{s} \alpha_i = 0.$$

   Thus, since this matches the requirement for $z = 1$ to be a root of the recurrence relationship, $z = 1$ will always be a root for a consistent multi-step method.

2. The backwards differentiation algorithms are a class of implicit multi-step methods. The two-step backwards differentiation method has the following form,

$$v^{n+1} - \frac{4}{3}v^n + \frac{1}{3}v^{n-1} = \frac{2}{3}\Delta t f(v^{n+1}, t^{n+1}).$$

   Determine the leading truncation error term. What is the order of accuracy of this method?

   Solution: Using the notation from the notes, for this method,

$$N(v^{n+1}, v^n, v^{n-1}, \Delta t) = \frac{4}{3}v^n - \frac{1}{3}v^{n-1} + \frac{2}{3}\Delta t f(v^{n+1}, t^{n+1})$$

   The truncation error is,

$$\begin{aligned}
\tau &\equiv N(u^{n+1}, u^n, u^{n-1}, \Delta t) - u^{n+1} \\
&= \frac{4}{3}u^n - \frac{1}{3}u^{n-1} + \Delta t \frac{2}{3} f(u^{n+1}, t^{n+1}) - u^{n+1} \\
&= \frac{4}{3}u^n - \frac{1}{3}u^{n-1} + \Delta t \frac{2}{3} u_t^{n+1} - u^{n+1}
\end{aligned}$$

   Now, substituting the Taylor series,

$$\begin{aligned}
\tau &= \frac{4}{3}u^n - \frac{1}{3}\left[u^n - \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n - \frac{1}{6}\Delta t^3 u_{ttt}^n + O(\Delta t^4)\right] \\
&\quad + \Delta t \frac{2}{3}\left[u_t^n + \Delta t u_{tt}^n + \frac{1}{2}\Delta t^2 u_{ttt}^n + O(\Delta t^3)\right] \\
&\quad - \left[u^n + \Delta t u_t^n + \frac{1}{2}\Delta t^2 u_{tt}^n + \frac{1}{6}\Delta t^3 u_{ttt}^n + O(\Delta t^4)\right] \\
&= \frac{2}{9}\Delta t^3 u_{ttt}^n + O(\Delta t^4)
\end{aligned}$$

   Thus, the leading error term is $\frac{2}{9}\Delta t^3 u_{ttt}^n$ and therefore the local order of accuracy is $p = 2$.

3. Prove that the two-step backwards differentiation method is convergent.

Solution: Since the method is second-order accurate ($p = 2$), it is consistent. Then, according to the Dahlquist Equivalence Theorem, we must check if the method is stable to determine if it is convergent. To check if the method is stable, we look for unstable modes in the recurrence relationship. For the 2nd-order backwards differentiation method, the recurrence relationship is:

$$\begin{aligned} z^{n+1} - \frac{4}{3}z^n + \frac{1}{3}z^{n-1} &= 0, \\ \left(z^2 - \frac{4}{3}z + \frac{1}{3}\right) z^{n-1} v^0 &= 0. \end{aligned}$$

Clearly, there are $n - 1$ roots of $z = 0$. Then, the quadratic equation can be solved for the other two roots:

$$z^2 - \frac{4}{3}z + \frac{1}{3} = 0, \Rightarrow z = 1 \text{ and } z = \frac{1}{3}.$$

Since all of the roots have $|z| \leq 1$, the method is stable.