16.901: Homework # 4
Solution

One family of implicit multi-step methods is the Adams-Moulton family. The first-order Adams-Moulton method is backward Euler and the second-order method is trapezoidal integration.

1. The third-order Adams-Moulton method is,

$$v^{n+1} = v^n + \Delta t\left(\frac{5}{12}f^{n+1} + \frac{8}{12}f^n - \frac{1}{12}f^{n-1}\right).$$

   Plot the eigenvalue stability region for this algorithm.

   Solution: To determine the eigenvalue stability region, f is assumed to be a linear forcing, $f = \lambda u$,

$$v^{n+1} = v^n + \lambda\Delta t\left(\frac{5}{12}v^{n+1} + \frac{8}{12}v^n - \frac{1}{12}v^{n-1}\right).$$

   Then, substituting the amplification factor, $v^n = g^n v^0$,

$$g^{n+1} = g^n + \lambda\Delta t\left(\frac{5}{12}g^{n+1} + \frac{8}{12}g^n - \frac{1}{12}g^{n-1}\right).$$

   Re-arranging, the non-zero roots satisfy,

$$g^2 = g + \lambda\Delta t\left(\frac{5}{12}g^2 + \frac{8}{12}g - \frac{1}{12}\right).$$

   To find the stability boundary, which is where $|g| = 1$, substitute $g = e^{i\theta}$ and solve for $\lambda\Delta t$,

$$\lambda\Delta t = \frac{e^{2i\theta} - e^{i\theta}}{\frac{5}{12}e^{2i\theta} + \frac{8}{12}e^{i\theta} - \frac{1}{12}}$$

   A plot of this stability region is shown in Figure 1.

2. Compare stability region of the third-order Adams-Moulton method to the stability region of the third-order backwards differentiation method discussed in lecture. Which method would likely be more efficient if the problem of interest had both small and large negative real eigenvalues? Why?

   Solution: The stability regions of the backward differentiation methods are shown in Figure 2. Clearly, while the third-order Adams-Moulton has a bounded region of stability, the third-order backwards differentiation method is stable for essentially any $\lambda\Delta t$ with a negative real part. Thus, while the Adams-Moulton stability region is larger than a correspondingly accurate explicit scheme (for example third-order Adams-Bashforth), it will still be strongly constrained when large and small negative real eigenvalues exists. The backwards differentiation method will be better.

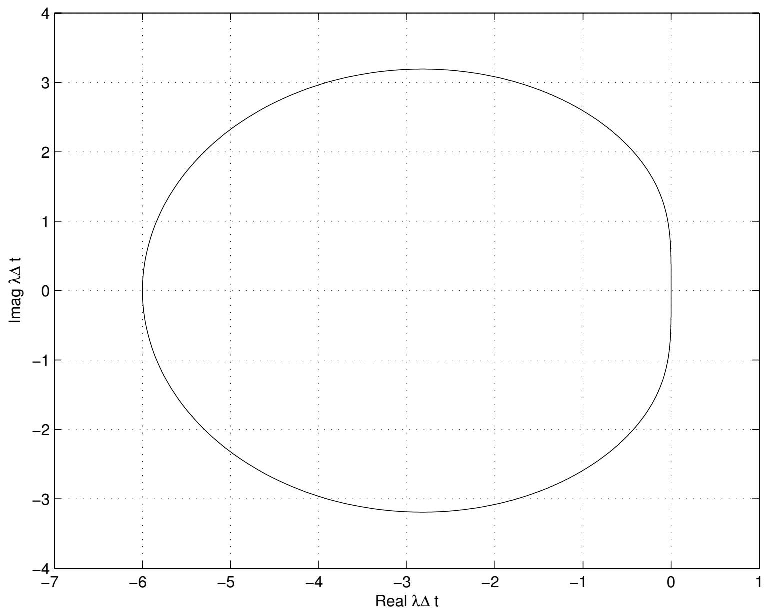

Figure 1: Stability of third-order Adams-Moulton method. Note: the interior of the curve is stable.

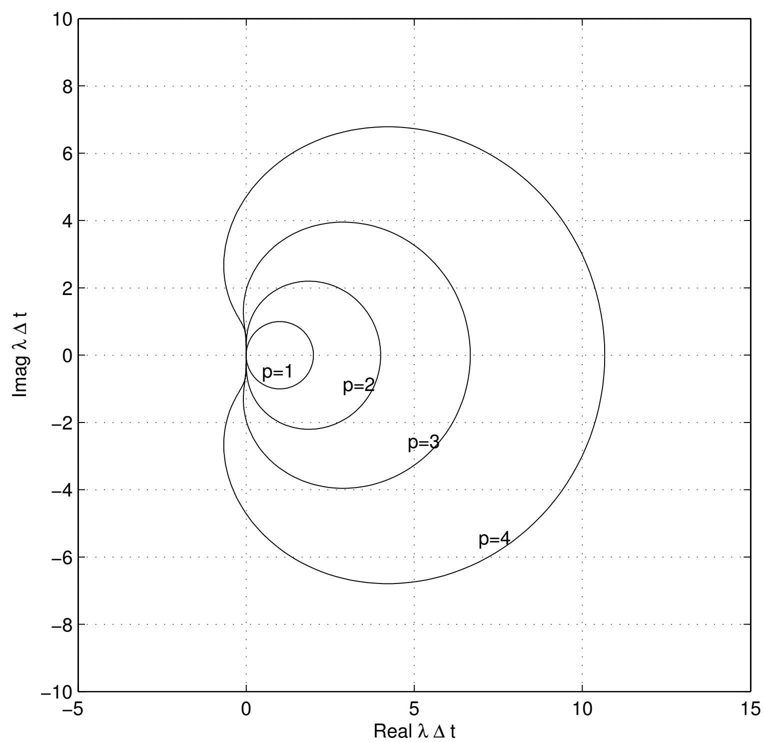

Figure 2: Backwards differentiation stability regions for $p = 1$ through $p = 4$ method. Note: interior of curves is unstable region.