20.430/6.561/10.539/2.795
Fields, Forces, and Flows in Biological Systems
Fall 2015

Problem Set # 7 **Issued: Friday 11/13/15**
Due: Friday, 5pm – 11/20/15

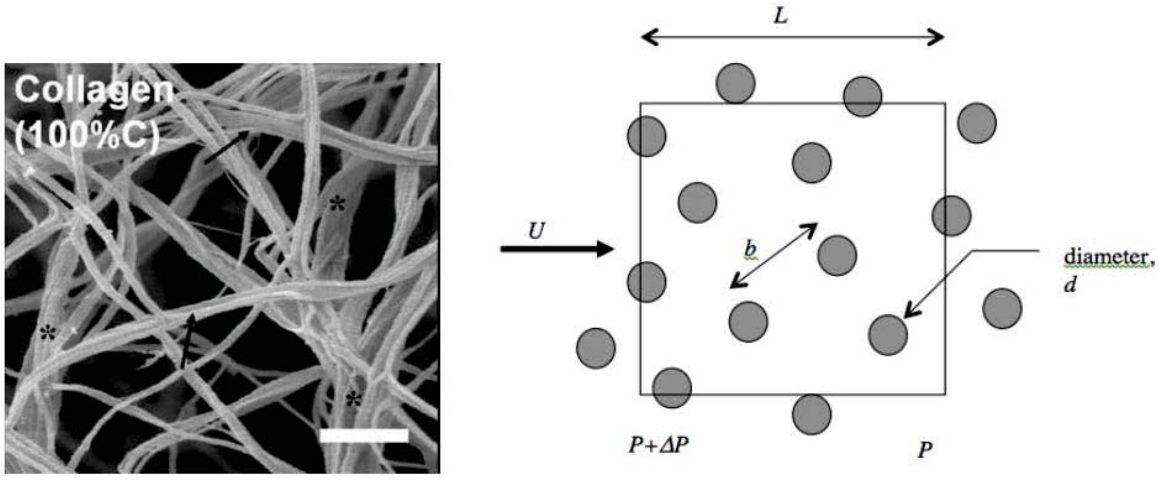
Reading Assignment: FFF Textbook, Chapter 5 Sections 5.1-5.4 and 5.6 and 5.11, and Chapter 7 Sections 7.5.1 and 7.5.2 (Darcy's Law)

Turn in Problem set in dropbox to the right of elevators on the 2nd floor of Building 16. Please turn in Problem 1 into Box 1 and Problem 2 into Box 2.

Problem 1: Fiber matrix permeability

Consider the flow of liquid through a fibrous matrix such as the cytoskeleton or extracellular matrix. To a first approximation, this situation can be modeled as flow through a collection of rigid cylinders of circular cross-section (representing the macromolecular constituents of the extracellular matrix or the fibers of the cytoskeleton) each aligned parallel to one another, and surrounded by liquid with a viscosity equal to that of water (see figure). The objective of the problem is to obtain an estimates for the "hydraulic permeability" of this matrix.

a. First, estimate using a scaling analysis how the drag force per unit length acting on one isolated fiber scales with the velocity of flow perpendicular to its axis (U), the fiber diameter (d) and the viscosity of the fluid (μ), on the assumption that inertial effects can be neglected[1].
b. Now consider a matrix comprised of identical, parallel fibers. Given that the average spacing between fibers is b, obtain a scaling relation for the pressure drop due to flow past a collection of fibers over a distance L where $L >> b$. (Hint: Consider a force balance on the fluid contained in a cube of volume L^3.)



Left: Electron micrograph of a collagen gel (scale bar = 1 μm) (Lai and Barocas, *J Biomech Engr*, 2012). Right: Schematic of matrix cross-section. All fibers are assumed to be aligned perpendicular to the page.

Note that the H*ydraulic permeability*, k, is defined by the following expression for the pressure gradient in a porous medium:

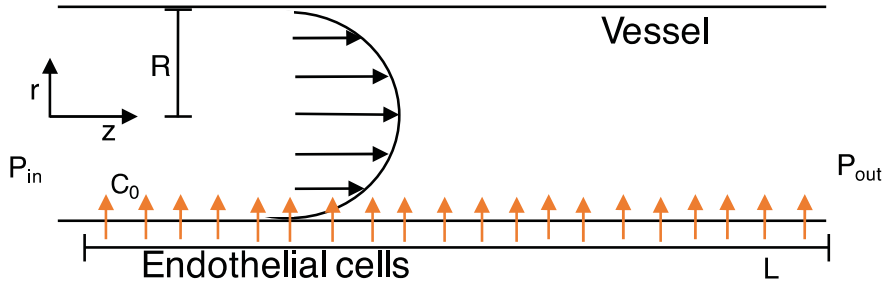
$$\nabla p = -\frac{U}{k}$$

[1]Strictly speaking, inertia cannot be neglected in this situation since the inertia of the fluid at large distances from the cylinder eventually become important. See e.g., Lamb, *Hydrodynamics*, p. 614 for a detailed account of this point. But neglecting it still provides a useful approximate result.

Problem 2: Endothelial signaling into the bloodstream

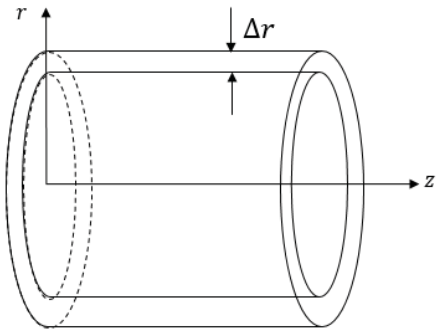
Note: For additional info on similarity transformations, see Section 5.11 of FFF

When tissue damage or infection occurs, adjacent endothelial cells lining blood vessels secrete signaling factors into the bloodstream to attract immune cells such as leukocytes, which can roll and extravasate into the tissue as discussed in lecture. Consider a patch of endothelial cells lining a section of the vessel that is secreting signaling TNFα proteins into flowing blood. Assume that there is a fixed concentration, C_0, of TNFα at the endothelial cell wall that is going into the vessel volume by diffusion.

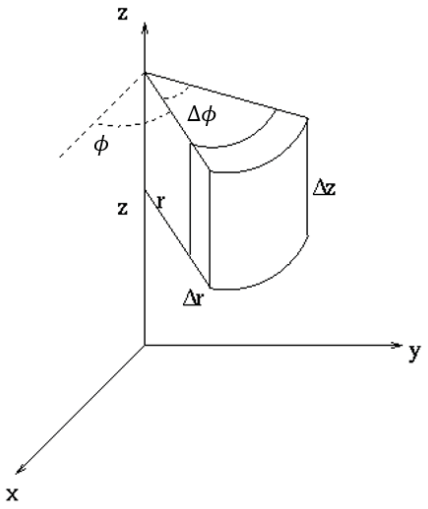


Further assume that the vessel flow is fully developed throughout the entire length and is driven by a pressure drop ΔP across a characteristic length L. At z = 0 there is initially no TNFα diffused into the vessel.

a) First, without considering the secreted mdecule, conduct a momentum balance on a thin cylindrical shell, then apply Newton's law of viscosity to derive the velocity distribution inside the cylindrical vessel (as in, do not start with Navier-Stokes!) Hint: the starting momentum balance requires both convective terms from bulk fluid, and viscous terms from molecular motion. Once you find an expression for the general momentum balance you can use simplifying assumptions to get rid of insignificant terms.



b) Perform a mass balance on a differential volume element to derive a partial differential equation for TNFα in the vessel. You can assume that TNFα is transported by convection only in the axial direction and diffusion only in the radial direction. State a non-dimensional parameter that allows you to justify neglecting axial diffusion. Then insert your velocity flow profile and state the boundary conditions and assumptions that would allow you to theoretically solve for C(r,z), but do not try to solve. Hint: your equation should be first order in z and second order in r.



c) An important approximation we can make to simplify the system is to assume that the molecule diffuses rather slowly into the bloodstream, so we can approximate the diffusion in the radial direction as into a "semi-infinite" volume.

i. Linearize the velocity profile by first transforming the r coordinate into y with (r = R-y), then using the stated assumption.

ii. Now make the same coordinate transformation with the governing equation, and simplify the cylindrical coordinates equation to a cartesian approximation. (Recall in PS2 we simply rewrote the coordinate system, but now you can see under what circumstances we can do that). Rewrite the boundary conditions accordingly. Hint: The centerline of the cylinder is now effectively "infinitely" far from y=0.

We note that the equation is still a PDE in two variables, but there may be hope. We can make a transformation of y and z into a single "similarity" variable. The similarity variable is chosen as a functional ratio between the two variables y and z, such that the behavior of the system is the same provided that the ratio is satisfied, regardless of the specific values of y and z. In other words, we want to find $y = \delta(z)$ over which the concentration of TNFα is constant. Note that this

is essentially a diffusive boundary layer, analogous to the fluid boundary layer discussed in lecture. The similarity variable is then $\eta = y/\delta(z)$.

d) Use scaling to derive $\delta(z)$, the characteristic diffusive length, then use that expression to obtain the similarity variable η.

e) Now that you have your similarity variable, transform the PDE into an ODE in terms of the similarity variable, with appropriate boundary conditions. Hint: the solution will have the form $\frac{d^2c}{d\eta^2} + a\eta^2 \frac{dc}{d\eta} = 0$. We have escaped the PDE!

f) Solve the resulting ODE (you can use an analytical solver).

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