Fall 2005 10.34 Exam I. Friday Oct. 7, 2005.

Read through the entire exam before beginning work, and budget your time.

Perform all calculations by hand, showing all steps. You may use a calculation for simple multiplication and addition of numbers, however.

Problem 1. Consider the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

(*a*) Compute the product $A\underline{v}$ for the vector $\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

<u>Answer</u>

Using the rule $(A\underline{v})_j = \sum_{k=1}^{N} a_{jk}v_k$, we have

$$A\underline{v} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} [(2)(1)+(1)(2)+(-1)(3)] \\ [(1)(1)+(3)(2)+(1)(3)] \\ [(-1)(1)+(1)(2)+(2)(3)] \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \\ 7 \end{bmatrix}$$

(*b*) Compute the LU decomposition $A = LU$ where L is a lower-triangular matrix and U is an upper-triangular matrix.

<u>Answer</u>

To generate $A = LU$ we perform Gaussian elimination without partial pivoting. First, we zero the (2,1) component, calculating

$$\lambda_{21} = \frac{a_{21}}{a_{11}} = \frac{1}{2} = 0.5$$

and then by performing the row operation $2 \leftarrow 2 - \lambda_{21} \times 1$, we have the new matrix

$$A^{(2,1)} = \begin{bmatrix} 2 & 1 & -1 \\ [1-(0.5)(2)] & [3-(0.5)(1)] & [1-(0.5)(-1)] \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2.5 & 1.5 \\ -1 & 1 & 2 \end{bmatrix}$$

Next, we zero the (3,1) component by calculating $\lambda_{31} = \frac{a_{31}^{(2,1)}}{a_{11}^{(2,1)}} = -\frac{1}{2} = -0.5$ and performing the row operation on $A^{(2,1)}$ $3 \leftarrow 3 - \lambda_{31} \times 1$, to yield

$$A^{(3,1)} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2.5 & 1.5 \\ [-1-(-0.5)(2)] & [1-(-0.5)(1)] & [2-(-0.5)(-1)] \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2.5 & 1.5 \\ 0 & 1.5 & 1.5 \end{bmatrix}$$

Next, we zero the (3,2) component by calculating $\lambda_{32} = \frac{a_{32}^{(3,1)}}{a_{22}^{(3,1)}} = \frac{1.5}{2.5} = \frac{3/2}{5/2} = \frac{3}{5} = 0.6$

and performing the row operation on $A^{(3,1)}$ $3 \leftarrow 3 - \lambda_{32} \times 2$, to yield the upper triangular matrix

$$U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2.5 & 1.5 \\ 0 & [1.5-(0.6)(2.5)] & [1.5-(0.6)(1.5)] \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2.5 & 1.5 \\ 0 & 0 & 0.6 \end{bmatrix}$$

To generate the lower triangular matrix L, we place ones along the principal diagonal and store the values of the λ_{mn}'s below the diagonal,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \lambda_{21} & 1 & 0 \\ \lambda_{31} & \lambda_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0.6 & 1 \end{bmatrix}$$

(c) Compute the determinant of A.

<u>Answer</u>

Here, we save some time by noting that $|A| = |LU| = |L||U|$, and remembering that the determinant of a triangular matrix is the product of its diagonal elements. Therefore, $|L| = 1$ and

$$|A| = |U| = U_{11}U_{22}U_{33} = (2)(2.5)(0.6) = 3$$

(d) Compute the solution $\underset{\sim}{x}$ to the linear system $A\underset{\sim}{x} = \underset{\sim}{b}$ for $\underset{\sim}{b} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

<u>Answer</u>

We can compute the solution quickly by using the LU factorization. Substituting for A,

$$LU\underline{x} = \underline{b} \qquad \Rightarrow \qquad \begin{aligned} L\underline{c} &= \underline{b} \\ U\underline{x} &= \underline{c} \end{aligned}$$

First, we compute $\underline{c}$ by solving $L\underline{c} = \underline{b}$ through forward substitution,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0.6 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} & & c_1 &= 2 \\ (0.5)c_1 + (1)c_2 &= 0 & c_2 &= -(0.5)(c_1) = -1 \\ (-0.5)c_1 + (0.6)c_2 + c_3 &= -1 & c_3 &= -1 - 0.6c_2 + 0.5c_1 = 0.6 \end{aligned}$$

Then, we solve $U\underline{x} = \underline{c}$ through backward substitution,

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2.5 & 1.5 \\ 0 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0.6 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} (2)x_1 + (1)x_2 + (-1)x_3 &= 2 & x_1 &= \frac{2+1+1}{2} = 2 \\ (2.5)x_2 + (1.5)x_3 &= -1 & x_2 &= \frac{-1-(1.5)(1)}{2.5} = -1 \\ (0.6)x_3 &= 0.6 & x_3 &= 1 \end{aligned}$$

Therefore, the solution is $\underline{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Problem 2. Consider the system of three nonlinear algebraic equations,

$$f_1(\underline{x}) = x_1^2 + x_2 - \frac{1}{2}x_3^2 = 0$$

$$f_2(\underline{x}) = x_1 + x_2^3 + \frac{1}{3}x_3^3 = 0$$

$$f_3(\underline{x}) = -\frac{1}{2}x_1^2 + x_2 + x_3^2 = 0$$

(a) Compute the Jacobian matrix, where the elements are functions of $\underline{x}$.

<u>Answer</u>

The Jacobian matrix $J(\underline{x})$ has elements $J_{mn} = \left.\frac{\partial f_m}{\partial x_n}\right|_{\underline{x}}$. Thus, for this system we have

$$J(\underline{x}) = \begin{bmatrix} 2x_1 & 1 & -x_3 \\ 1 & 3x_2^2 & x_3^2 \\ -x_1 & 1 & 2x_3 \end{bmatrix}$$

(b) Using $\underline{x}^{[0]} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, compute the new estimate $\underline{x}^{[1]}$ generated by Newton's method. *__Hint__: Does $J(\underline{x}^{[0]})$ look familiar?*

Answer

The linear system that we wish to solve is $J(\underline{x}^{[0]})\underline{p}^{[0]} = -\underline{f}(\underline{x}^{[0]})$. For this particular $\underline{x}^{[0]}$, the Jacobian is equal to the matrix A from problem 1,

$$J(\underline{x}^{[0]}) = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

This means that we can use the LU decomposition from problem 1 to avoid performing Gaussian elimination again. The function vector is

$$\underline{f}(\underline{x}^{[0]}) = \begin{bmatrix} x_1^2 + x_2 - \frac{1}{2}x_3^2 \\ x_1 + x_2^3 + \frac{1}{3}x_3^3 \\ -\frac{1}{2}x_1^2 + x_2 + x_3^2 \end{bmatrix} = \begin{bmatrix} 1 + 1 - \frac{1}{2} \\ 1 + 1 + \frac{1}{3} \\ -\frac{1}{2} + 1 + 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2.333 \\ 1.5 \end{bmatrix}$$

The linear system that we wish to solve is of the form $A\underline{x} = \underline{b}$ with

$$\underline{b} = -\underline{f}(\underline{x}^{[0]}) = \begin{bmatrix} -1.5 \\ -2.333 \\ -1.5 \end{bmatrix}$$

Repeating the forward and backward substitution process of (1.d), we have a full Newton-update vector and new solution estimate,

$$\underline{p}^{[0]} = \begin{bmatrix} -2.167 \\ 0.667 \\ -2.167 \end{bmatrix} \qquad \underline{x}^{[1]} = \underline{x}^{[0]} + \underline{p}^{[0]} = \begin{bmatrix} -1.167 \\ 1.667 \\ -1.167 \end{bmatrix}$$

(c) Would this guess be accepted in a robust reduced-step Newton algorithm?

Answer

The square of the 2-norm (length) of the function vector at $\underline{x}^{[0]}$ is

$$\left|\underline{f}(\underline{x}^{[0]})\right|^2 = \underline{f}(\underline{x}^{[0]}) \cdot \underline{f}(\underline{x}^{[0]}) = 9.9429$$

For the new estimate, $\underline{f}(\underline{x}^{[1]}) = \begin{bmatrix} 2.379 \\ 2.9356 \\ 2.3479 \end{bmatrix}$ and thus

$$\left|\underline{f}(\underline{x}^{[1]})\right|^2 = \underline{f}(\underline{x}^{[1]}) \cdot \underline{f}(\underline{x}^{[1]}) = 19.6436$$

As $\left|\underline{f}(\underline{x}^{[1]})\right|^2 > \left|\underline{f}(\underline{x}^{[0]})\right|^2$, we would not accept this new estimate, but would rather iteratively halve the step length until we find that the 2-norm is reduced at the new estimate.

Problem 3. Consider again the matrix A of problem 1.

(*a*) What properties of the eigenvalues and eigenvectors of A can you infer simply by inspection of A, *i.e.* with no additional computations?

<u>Answer</u>

Since A is real-symmetric, we know that its eigenvalues are all real and its eigenvectors are mutually orthogonal.

(*b*) Compute an upper bound on the largest possible magnitude (modulus) of an eigenvalue of A. That is, find a value λ_{max} such that for all eigenvalues λ_j of A, we are guaranteed to have $|\lambda_j| \le \lambda_{max}$.

<u>Answer</u>

Here, we use Gershorgin's theorem and the fact that we know all eigenvalues of A to be real. Gershorgin's theorem states that for each λ, such that $A\underline{w} = \lambda\underline{w}$, one of the following inequalities must apply,

$$\begin{aligned} |\lambda - 2| \le |-1| + |1| = 2 \quad &\Rightarrow \quad 0 \le \lambda \le 4 \\ |\lambda - 3| \le |1| + |1| = 2 \quad &\Rightarrow \quad 1 \le \lambda \le 5 \\ |\lambda - 2| \le |-1| + |1| = 2 \quad &\Rightarrow \quad 0 \le \lambda \le 4 \end{aligned}$$

Thus, we have $0 \le \lambda \le 5$, so that all eigenvalues must be non-negative and must be less than or equal to 5. 5 is thus an upper bound on the magnitude of the eigenvalues of A. In fact, the eigenvalues are 0.2679, 3.0000, 3.7321.