3D-Central Force Problems I

Read: C-TDL, pages 643-660 for next lecture.

All 2-Body, 3-D problems can be reduced to

- * a 2-D angular part that is exactly and <u>universally</u> soluble
- * a 1-D radial part that is <u>system-specific</u> and soluble by 1-D techniques in which you are now expert

<u>Next 3 lectures:</u> $\begin{bmatrix}\text{Correspondence Principle}\\ \text{Commutation Rules}\end{bmatrix} \longrightarrow$ all matrix elements

<u>Roadmap</u>

1. define radial momentum $\mathbf{p_r} = \mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p} - i\hbar)$

2. define orbital angular momentum $\vec{\mathbf{L}} = \vec{\mathbf{q}} \times \vec{\mathbf{p}}$

general definition of angular momentum and of "vector operators"

$$\left(\text{also } \mathbf{L}\times\mathbf{L} = i\hbar\mathbf{L} \text{ and } \left[\mathbf{L}_i, \mathbf{L}_j\right] = i\hbar\sum_k \varepsilon_{ijk}\mathbf{L}_k\right)$$

3. separate $\mathbf{p}^2$ into radial and angular terms: $\mathbf{p}^2 = \mathbf{p}_r^2 + \mathbf{r}^{-2}\mathbf{L}^2$

4. find Complete Set of Commuting Observables (CSCO) useful for block-diagonalizing **H**

$$\left[\mathbf{H},\mathbf{L}^2\right] = \left[\mathbf{H},\mathbf{L}_i\right] = \left[\mathbf{L}^2,\mathbf{L}_i\right] = 0 \qquad \mathbf{H},\mathbf{L}^2,\mathbf{L}_i \qquad \text{CSCO}$$

$|L, M_L\rangle$ universal basis set

5. separate radial part of **H**:

$$\mathbf{H}_\ell(r) = \underbrace{\frac{\mathbf{p}_r^2}{2\mu} + V(\mathbf{r}) + \frac{\hbar^2\ell(\ell+1)}{2\mu\mathbf{r}^2}}_{V_\ell(\mathbf{r})}$$

effective radial potential

1-D Schröd Eq.

6. ALL Matrix Elements of Angular Momentum Components Derived from Commutation Rules.

7. Spherical Tensor Classification of **all** operators.

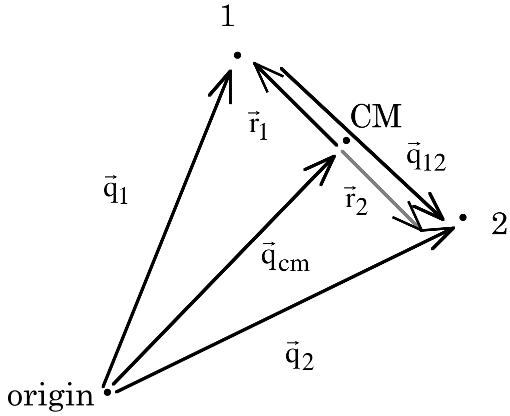
$\Downarrow$

8. Wigner-Eckart Theorem $\rightarrow$ all angular matrix elements of all operators.

I hate differential operators. Replace them using exclusively simple Commutation Rule based Operator Algebra.

Lots of derivations based on classical VECTOR ANALYSIS — much will be set aside as NONLECTURE

Central Force Problems: 2 bodies where interaction force is along the vector $\vec{q}_1 - \vec{q}_2$



$$\vec{q}_2 = \vec{q}_1 + \vec{q}_{12}$$
$$\vec{q}_{12} = \vec{q}_2 - \vec{q}_1 = \hat{i}(x_2 - x_1) + \hat{j}(y_2 - y_1) + \hat{k}(z_2 - z_1)$$

$$r \equiv |\vec{q}_{12}| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

also C.M. Coordinate system

$$\vec{r}_1 = \vec{q}_1 - \vec{q}_{cm} \qquad \left[|r_1|/r = m_2/M\right]$$
$$\vec{r}_2 = \vec{q}_2 - \vec{q}_{cm} \qquad \left[|r_2|/r = m_1/M\right]$$

$$\mathbf{H} = \mathbf{H}_{\text{translation}} + \mathbf{H}_{\text{center of mass}}$$

free translation of C of M of system of mass $M = m_1 + m_2$

motion of fictitious particle of mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$ in coordinate system with origin at C of M (CTDL page 713)

LAB $\quad \hat{\mathbf{H}}_{\text{translation}} = \dfrac{\mathbf{P}^2_{\text{trans}}}{2(m_1 + m_2)} + V_{\text{constant}}$ — free translation of system with respect to lab (not interesting)

BODY $\quad \hat{\mathbf{H}}_{\text{C.M.}} = \dfrac{1}{2\mu}\mathbf{P}^2_{\text{cm}} + \underbrace{V(r)}_{\text{free rotation (no } \theta,\phi \text{ dependence)}}$ — motion of particle of mass μ with respect to origin at c. of m.

GOAL IS TO SIMPLIFY $\mathbf{P}^2_{\text{cm}}$
because that is only place where θ,ϕ degrees of freedom appear.

1. Define Radial Component of $\vec{P}_{cm}$

Correspondence Principle

* classical mechanics
* Cartesian Coordinates
* symmetrize to avoid failure to satisfy Commutation Rules

** verify that all three derived operators, **p**, $\mathbf{p}_r$ and **L**
- are Hermitian
- satisfy $[\mathbf{q},\mathbf{p}]=i\hbar$

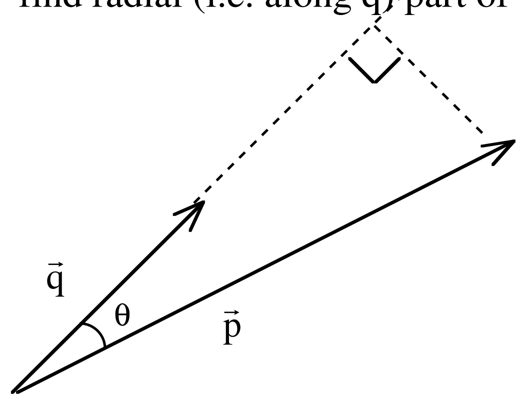
Purpose of this lecture is to walk you through the standard vector analysis and Quantum Mechanics Correspondence Principle procedures

$$\vec{q} \equiv \hat{i}x + \hat{j}y + \hat{k}z$$

$$\vec{p} \equiv \hat{i}p_x + \hat{j}p_y + \hat{k}p_z$$

$$r \equiv \left[x^2 + y^2 + z^2\right]^{1/2} = [q \cdot q]^{1/2} = |q|$$

find radial (i.e. along $\vec{q}$) part of $\vec{p}$



project $\vec{p}$ onto $\vec{q}$

$$q \cdot p = |q||p|\cos\theta$$

$$\cos\underbrace{(q,p)}_{\theta} = \frac{q \cdot p}{|q||p|}$$

radial component of p is obtained by projecting $\vec{p}$ onto $\vec{q}$

$$p_r = |p|\cos\theta = |p|\frac{q \cdot p}{|q||p|} = \frac{q \cdot p}{r}$$

so from standard vector analysis we get $p_r = r^{-1}\vec{q} \cdot \vec{p}$

This is a trial form for p_r, but it is necessary, according to Correspondence Principle, to symmetrize it.

$$\mathbf{p}_r = \frac{1}{4}\left[\mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{q})+(\mathbf{q}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{q})\mathbf{r}^{-1}\right]$$

arrange terms in all possible orders!

NONLECTURE (except for Eq. (4))

SIMPLIFY ABOVE Definition to $\mathbf{p_r} = \mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p} - i\hbar)$ ($\mathbf{r}$ is not a vector)

$[\vec{\mathbf{q}},\vec{\mathbf{p}}]$ is a vector commutator — be careful

$$[\vec{\mathbf{q}},\vec{\mathbf{p}}] = [\mathbf{x},\mathbf{p}_x]+[\mathbf{y},\mathbf{p}_y]+[\mathbf{z},\mathbf{p}_z] = 3i\hbar$$

$$\therefore \mathbf{p}\cdot\mathbf{q} = \mathbf{q}\cdot\mathbf{p} - [\vec{\mathbf{q}},\vec{\mathbf{p}}]$$

$$\mathbf{p}_r = \frac{1}{4}\left[\mathbf{r}^{-1}\left(2\mathbf{q}\cdot\mathbf{p}-[\vec{\mathbf{q}},\vec{\mathbf{p}}]\right)+\left(2\mathbf{q}\cdot\mathbf{p}-[\vec{\mathbf{q}},\vec{\mathbf{p}}]\right)\mathbf{r}^{-1}\right] \quad (1)$$

$$= \frac{1}{4}\left[\underbrace{\mathbf{r}^{-1}4\mathbf{q}\cdot\mathbf{p}-\mathbf{r}^{-1}2\mathbf{q}\cdot\mathbf{p}}_{\text{add and subtract } 2\mathbf{r}^{-1}\mathbf{q}\cdot\mathbf{p}}+2\mathbf{q}\cdot\mathbf{p}\mathbf{r}^{-1}-6i\hbar\mathbf{r}^{-1}\right] \quad (2)$$

$$= \mathbf{r}^{-1}\mathbf{q}\cdot\mathbf{p}-\frac{3}{2}i\hbar\mathbf{r}^{-1}+\frac{1}{2}\left[\mathbf{q}\cdot\mathbf{p},\mathbf{r}^{-1}\right] \quad (3)$$

LEMMA: need more general Commutation Rule for which $[\mathbf{q}\cdot\mathbf{p},\mathbf{r}^{-1}]$ is a special case

1st simplify: $[f(\mathbf{r}),\mathbf{q}\cdot\mathbf{p}] = \mathbf{q}\cdot[f(\mathbf{r}),\vec{\mathbf{p}}]+\cancelto{0}{[f(\mathbf{r}),\vec{\mathbf{q}}]}\cdot\vec{\mathbf{p}}$

but, from 1-D, we could have shown

$$[f(\mathbf{x}),\mathbf{p}]\phi = f(\mathbf{x})\frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}}\phi - \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{x}}(f(\mathbf{x})\phi)$$

$$= \frac{\hbar}{i}[f(\mathbf{x})\phi' - f'\phi - f\phi'] = i\hbar f'(\mathbf{x})\phi$$

$$[f(\mathbf{x}),\mathbf{p}] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \qquad \text{for 1-D}$$

Thus, in 3-D, the chain rule gives

$$[f(\mathbf{r}),\vec{\mathbf{p}}] = i\hbar\left[\hat{i}\frac{\partial f}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{x}} + \hat{j}\frac{\partial f}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{y}} + \hat{k}\frac{\partial f}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{z}}\right]$$

evaluate these first

$$\frac{\partial \mathbf{r}}{\partial \mathbf{x}} = \frac{\partial}{\partial x}[x^2+y^2+z^2]^{1/2} = x[x^2+y^2+z^2]^{-1/2} = \mathbf{x}/\mathbf{r}$$

etc. for $\frac{\partial \mathbf{r}}{\partial \mathbf{y}}$ & $\frac{\partial \mathbf{r}}{\partial \mathbf{z}}$

Thus $[f(\mathbf{r}),\vec{\mathbf{p}}] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\left[\hat{i}\frac{\mathbf{x}}{\mathbf{r}} + \hat{j}\frac{\mathbf{y}}{\mathbf{r}} + \hat{k}\frac{\mathbf{z}}{\mathbf{r}}\right] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\frac{\vec{\mathbf{q}}}{\mathbf{r}}$

$$[f(\mathbf{r}),\vec{\mathbf{q}}\cdot\vec{\mathbf{p}}] = \mathbf{q}\cdot[f(\mathbf{r}),\mathbf{p}] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\left(\frac{\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2}{\mathbf{r}}\right) = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\mathbf{r}$$

$$[f(\mathbf{r}),\mathbf{q}\cdot\mathbf{p}] = i\hbar\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\mathbf{r}$$ this is a scalar, not a vector, equation (4)

But we wanted to evaluate the commutation rule for f(**r**) = $\mathbf{r}^{-1}$

$$[\mathbf{r}^{-1},\mathbf{q}\cdot\mathbf{p}] = i\hbar\frac{\partial}{\partial \mathbf{r}}\left(\frac{1}{\mathbf{r}}\right)\mathbf{r} = -i\hbar\mathbf{r}^{-1} \qquad (5)$$

plug this result into (3)

$$\mathbf{p_r} = \mathbf{r}^{-1}\mathbf{q}\cdot\mathbf{p} - \frac{3}{2}i\hbar\mathbf{r}^{-1} + \frac{1}{2}(i\hbar\mathbf{r}^{-1})$$

RESUME HERE

$$\mathbf{p_r} = \mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p} - i\hbar) \qquad (6)$$

This is the compact but non-symmetric result we got starting with a carefully symmetrized starting point – as required by Correspondence Principle.

* This result is identical to result obtained from standard vector analysis IN THE LIMIT OF $\hbar \to 0$.

Still must do 2 things: show $[\mathbf{r},\mathbf{p_r}] = i\hbar$
show $\mathbf{p_r}$ is Hermitian

$$\left[\mathbf{r},\mathbf{p_r}\right] = \left[\mathbf{r},\mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p} - i\hbar)\right]$$

$$= \mathbf{r}^{-1}[\mathbf{r},\mathbf{q}\cdot\mathbf{p}] - \mathbf{r}^{-1}\underbrace{[\mathbf{r},i\hbar]}_{0} + \underbrace{\left[\mathbf{r},\mathbf{r}^{-1}\right]}_{0}(\mathbf{q}\cdot\mathbf{p} - i\hbar)$$

$$= \mathbf{r}^{-1}[\mathbf{r},\mathbf{q}\cdot\mathbf{p}] \qquad \text{Use Eq. (4)}$$

$$[\mathbf{r},\mathbf{q}\cdot\mathbf{p}] = i\hbar\mathbf{r}$$

$$\therefore [\mathbf{r},\mathbf{p_r}] = i\hbar$$

*

* we do not need to confirm that $\mathbf{p_r}$ is Hermitian because it was constructed from a symmetrized form which is guaranteed to be Hermitian.

Correspondence Principle!

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2. Verify that Classical Definition of Angular Momentum is Appropriate for QM.

$$\vec{L} = \vec{q} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} \qquad (7)$$

We will see that this definition of an angular momentum is acceptable as far as the correspondence principle is concerned, but it is not sufficiently general.

NONLECTURE

What about symmetrizing $\vec{L}$?

$$L_x = yp_z - zp_y = p_z y - p_y z$$

$$= -(\vec{p} \times \vec{q})_x$$

PRODUCTS OF NON-CONJUGATE QUANTITIES

$$\therefore p \times q = -L$$

$\mathbf{q} \times \mathbf{p} + \mathbf{p} \times \mathbf{q} = 0$ symmetrization is impossible!

$\mathbf{q} \times \mathbf{p} - \mathbf{p} \times \mathbf{q} = 2\vec{\mathbf{L}}$ antisymmetrization is unnecessary!

But is $\vec{\mathbf{L}}$ Hermitian as defined?

BE CAREFUL: $(\mathbf{q} \times \mathbf{p})^\dagger \neq \mathbf{p}^\dagger \times \mathbf{q}^\dagger$!

go back to definition of vector cross product

$$\mathbf{L}_x = \mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y$$

$$\mathbf{L}_x^\dagger = \mathbf{p}_z^\dagger \mathbf{y}^\dagger - \mathbf{p}_y^\dagger \mathbf{z}^\dagger = \mathbf{p}_z\mathbf{y} - \mathbf{p}_y\mathbf{z} = \mathbf{y}\mathbf{p}_z - \mathbf{z}\mathbf{p}_y = \mathbf{L}_x$$

($\mathbf{p}$,$\mathbf{q}$ are Hermitian)

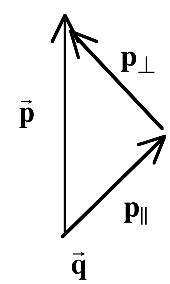
$\therefore \vec{\mathbf{L}}$ is Hermitian as defined.

RESUME

3A. Separate $\mathbf{p}^2$ into radial and angular terms.

GOAL: $\mathbf{p}^2 = \mathbf{p}_\mathbf{r}^2 + \mathbf{r}^{-2}\mathbf{L}^2$ (8)

vector analysis $\vec{\mathbf{p}} = \vec{\mathbf{p}}_{||} + \vec{\mathbf{p}}_{\perp}$ (|| and $\perp$ with respect to $\vec{\mathbf{q}}$)



Classically

$$\vec{\mathbf{p}} = \mathbf{r}^{-2}\left[\underbrace{\vec{\mathbf{q}}(\mathbf{q}\cdot\mathbf{p})}_{\text{component } || \text{ to } \vec{\mathbf{q}}} - \underbrace{\vec{\mathbf{q}} \times \overbrace{(\mathbf{q}\times\mathbf{p})}^{\vec{\mathbf{L}}}}_{\text{component in } \vec{\mathbf{q}},\vec{\mathbf{p}} \text{ plane which is } \perp \text{ to } \vec{\mathbf{q}}}\right] \quad (9)$$

(is the sign correct?)

$\mathbf{r}^{-2}$ is needed in both terms to remain dimensionally correct.

talk through this vector identity

1st term ($\mathbf{p}_{\parallel}$): $q \cdot p = |q||p|\cos\theta$

$\vec{q}/|q|$ = unit vector along $\vec{q}$

2nd term ($\mathbf{p}_{\perp}$): $q \times p$ points $\perp$ up out of paper

q: thumb; p: finger; $\perp$: palm

$\underbrace{q}_{\text{thumb}} \times \underbrace{q \times p}_{\text{finger}}$ is in plane of paper in opposite direction of $p_{\perp}$, hence minus sign.

Is it necessary to symmetrize Eq. (9)?

NONLECTURE

Examine Eq. (9) for QM consistency

x component

$$p_x = r^{-2}\left[x\left(xp_x + yp_y + zp_z\right) - \left(yL_z - zL_y\right)\right]$$

but $$yL_z - zL_y = y\left(xp_y - yp_x\right) + z\left(xp_z - zp_x\right)$$

$$p_x = r^{-2}\left[\left(x^2 + y^2 + z^2\right)p_x + \cancelto{0}{(xy - yx)}p_y + \cancelto{0}{(xz - zx)}p_z\right] = p_x$$

similarly for p_y, p_z

Symmetrize? No, because 2 parts of $\vec{p}$ are already shown to be Hermitian.

RESUME

3B. Evaluate $\mathbf{p}\cdot\mathbf{p}$

$$\mathbf{p}^2 = \vec{\mathbf{p}}\mathbf{r}^{-2}\left[\mathbf{q}(\mathbf{q}\cdot\mathbf{p}) - \mathbf{q}\times(\mathbf{q}\times\mathbf{p})\right] \tag{10}$$

$$\left[\text{goal is } \mathbf{p}^2 = \mathbf{p}_\mathbf{r}^2 + \mathbf{r}^{-2}\mathbf{L}^2\right]$$

commute $\vec{\mathbf{p}}$ through $\mathbf{r}^{-2}$ to be able to take advantage of classical vector triple product

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NONLECTURE

$$\left[\vec{\mathbf{p}},\mathbf{r}^{-2}\right] = -i\hbar\left[\hat{i}\frac{\partial}{\partial\mathbf{x}}\mathbf{r}^{-2} + \hat{j}\frac{\partial}{\partial\mathbf{y}}\mathbf{r}^{-2} + \hat{k}\frac{\partial}{\partial\mathbf{z}}\mathbf{r}^{-2}\right]$$

$$= 2i\hbar\mathbf{r}^{-4}\vec{\mathbf{q}} \qquad \left[\text{Recall } \left[f(\mathbf{x}),\mathbf{p}_\mathbf{x}\right] = i\hbar\frac{\partial \mathbf{f}}{\partial\mathbf{x}}\right]$$

$$\text{because } \frac{\partial}{\partial\mathbf{x}}\mathbf{r}^{-2} = -2\mathbf{r}^{-3}\frac{\partial\mathbf{r}}{\partial\mathbf{x}} = -2\mathbf{r}^{-3}\left(\frac{1}{2}\right)\frac{2\mathbf{x}}{\mathbf{r}} = -2\mathbf{x}/\mathbf{r}^4$$

$$\text{thus } \vec{\mathbf{p}}\mathbf{r}^{-2} = \mathbf{r}^{-2}\left(\vec{\mathbf{p}} + 2i\hbar\mathbf{r}^{-2}\vec{\mathbf{q}}\right) \tag{11}$$

$$\mathbf{p}^2 = \mathbf{r}^{-2}\left(\vec{\mathbf{p}} + 2i\hbar\mathbf{r}^{-2}\vec{\mathbf{q}}\right)\left[\mathbf{q}(\mathbf{q}\cdot\mathbf{p}) - \mathbf{q}\times(\mathbf{q}\times\mathbf{p})\right] \tag{12}$$

get 4 terms

$$\mathbf{p}^2 = \underbrace{\mathbf{r}^{-2}(\mathbf{p}\cdot\mathbf{q})(\mathbf{q}\cdot\mathbf{p})}_{\mathbf{I}} - \underbrace{\mathbf{r}^{-2}\mathbf{p}\cdot\left[\mathbf{q}\times(\mathbf{q}\times\mathbf{p})\right]}_{\mathbf{II}} + \underbrace{\mathbf{r}^{-2}(2i\hbar)\mathbf{r}^{-2}(\mathbf{q}\cdot\mathbf{q})(\mathbf{q}\cdot\mathbf{p})}_{\mathbf{III}} - \underbrace{\mathbf{r}^{-2}(2i\hbar)\mathbf{r}^{-2}\mathbf{q}\cdot\left[\mathbf{q}\times(\mathbf{q}\times\mathbf{p})\right]}_{\mathbf{IV}}$$

$$\left.\begin{aligned}\mathbf{I} &= \mathbf{r}^{-2}(\mathbf{q}\cdot\mathbf{p} - 3i\hbar)(\mathbf{q}\cdot\mathbf{p})\\ \mathbf{III} &= \mathbf{r}^{-2}(2i\hbar)(\mathbf{q}\cdot\mathbf{p})\end{aligned}\right\}\mathbf{r}^{-2}(\mathbf{q}\cdot\mathbf{p} - i\hbar)(\mathbf{q}\cdot\mathbf{p}) = \mathbf{r}^{-1}\mathbf{p}_\mathbf{r}\underbrace{(\mathbf{q}\cdot\mathbf{p})}_{\mathbf{r}\mathbf{p}_\mathbf{r} + i\hbar}$$

$$\mathbf{II} = -\mathbf{r}^{-2}(\mathbf{p}\times\mathbf{q})\cdot(\mathbf{q}\times\mathbf{p}) = -\mathbf{r}^{-2}\left(\pm\mathbf{L}^2\right) = \mathbf{r}^{-2}\mathbf{L}^2$$

$$\mathbf{IV} = -\mathbf{r}^4(2i\hbar)\overbrace{(\mathbf{q}\times\mathbf{q})}^{0}\cdot(\mathbf{q}\times\mathbf{p})$$

$$\mathbf{p}^2 = \mathbf{r}^{-1}\underbrace{\mathbf{p}_\mathbf{r}(\mathbf{r}}_{\mathbf{r}\mathbf{p}_\mathbf{r}-[\mathbf{r},\mathbf{p}_\mathbf{r}]}\mathbf{p}_\mathbf{r} + i\hbar) + \mathbf{r}^{-2}\mathbf{L}^2 = \mathbf{r}^{-1}\left[\mathbf{r}\mathbf{p}_\mathbf{r} - i\hbar\right]\mathbf{p}_\mathbf{r} + \mathbf{r}^{-1}\mathbf{p}_\mathbf{r}i\hbar + \mathbf{r}^{-2}\mathbf{L}^2 = \mathbf{p}_\mathbf{r}^2 + \mathbf{r}^{-2}\mathbf{L}^2 \tag{13}$$

RESUME

This $\mathbf{p}^2 = \mathbf{p}_\mathbf{r}^2 + \mathbf{r}^{-2}\mathbf{L}^2$ equation

is a very useful and simple form for $\mathbf{p}^2$ – separated into additive radial and angular terms! If **H** can be separated into additive terms, then the eigenvectors can be factored.

SUMMARY

$\mathbf{p}_\mathbf{r} = \mathbf{r}^{-1}(\mathbf{q}\cdot\mathbf{p} - i\hbar)$ radial momentum

$\mathbf{p}^2 = \mathbf{p}_\mathbf{r}^2 + \mathbf{r}^{-2}\mathbf{L}^2$ separation of radial and angular terms

$$\mathbf{H} = \frac{\mathbf{p}_\mathbf{r}^2}{2\mu} + \left[\frac{\mathbf{L}^2}{2\mu\mathbf{r}^2} + V(\mathbf{r})\right]$$

eventually $V_\ell(\mathbf{r}) = \dfrac{\hbar^2 \ell(\ell+1)}{2\mu\mathbf{r}^2} + V(\mathbf{r})$

Next: properties of $\mathbf{L}_i$, $\mathbf{L}^2 \longrightarrow$ CSCO