Structures - Class Exercise 1a.
1.101 Sophomore Design - Fall 2006

Given the differential equation: $\frac{d^2}{dx^2}w(x) = \frac{Q_o}{EI} \cdot (L - x)$ on $0 \le x \le L$ where capitalized letters are constants. The boundary conditions are $\left.\frac{dw}{dx}\right|_{x=0} = 0$ *and* $w|_{x=0} = 0$

Find $w(x) = ?$ $\left.\frac{dw}{dx}\right|_{x=L} = ?$ *and* $w|_{x=L} = ?$

From direct integration of the differential equation $w(x) = \frac{Q_o}{EI} \cdot \left(\frac{x^2}{2}L - \frac{x^3}{6}\right) + c_1 \cdot x + c_2$

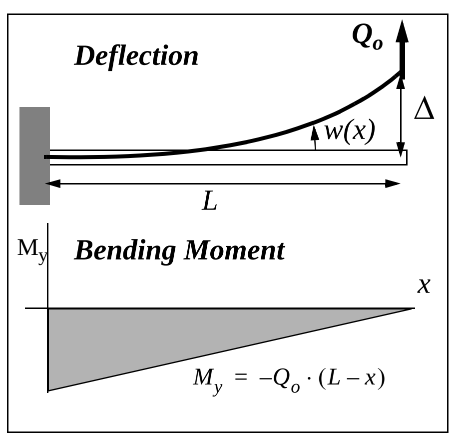
From the boundary conditions, the two constants of integration must be zero, so

$$w(x) = \frac{Q_o}{EI} \cdot \left(\frac{x^2}{2}L - \frac{x^3}{6}\right) \quad \text{and} \quad \frac{d}{dx}w(x) = \frac{Q_o}{EI} \cdot \left(xL - \frac{x^2}{2}\right)$$

Then, at $x = L$, the deflection, $w(L) = \Delta$, and the slope, $\left.\frac{dw}{dx}\right|_{x=L} = \phi(L)$ are

$$\Delta = w(L) = \frac{Q_o \cdot L^3}{3EI} \quad \text{and} \quad \phi(L) = \left.\frac{d}{dx}(w)\right|_{x=L} = \frac{Q_o \cdot L^2}{2EI}$$

What does all this have to do with the picture shown?



Q_o is the end load. $w(x)$ is the deflection. The differential equation we start with is the "moment-curvature" relation. For small deflections and rotations, $\frac{d^2}{dx^2}w(x)$ is the curvature (positive concave upwards). The right hand side is the bending moment which, by convention of 1.050, is taken as positive when the curvature is concave downwards. $M_y = -Q_o \cdot (L - x)$

Structures - Class Exercise 1b.
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Given the differential equation: $\frac{d^2}{dx^2}w(x) = \frac{-M_o}{EI}$ on $0 \le x \le L$ where all capitalized letters are constants. The boundary conditions are $\left.\frac{dw}{dx}\right|_{x=0} = 0$ $\quad w|_{x=0} = 0$

Find $\quad w(x) = ? \quad \left.\frac{dw}{dx}\right|_{x=L} = ? \quad and \quad w|_{x=L} = ?$

As in the previous exercise, we find, after applying the boundary conditions

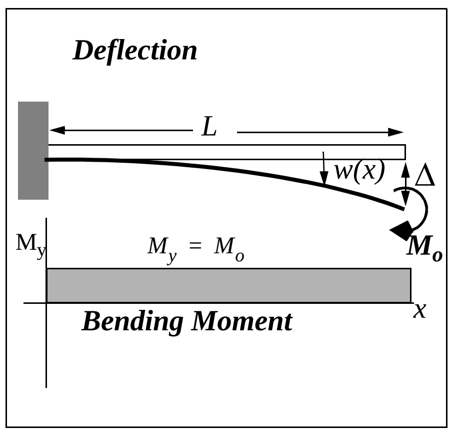
$$w(x) = \frac{-M_o x^2}{2EI} \quad \text{and} \quad \frac{d}{dx}w(x) = \frac{-M_o x}{EI}$$

Then, at $x = L$, the deflection, $w(L) = \Delta$, and the slope, $\left.\frac{dw}{dx}\right|_{x=L} = \phi(L)$ are

$$\Delta = w(L) = \frac{-M_o L^2}{2EI} \quad \text{and} \quad \phi(L) = \left.\frac{d}{dx}(w)\right|_{x=L} = \frac{-M_o L}{EI}$$

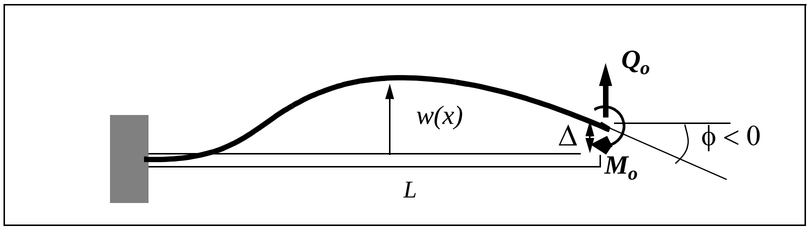
What does all this have to do with the picture shown?

Here, M_o is the *applied moment* at the end of the beam. Within the beam, $0<x<L$, the bending moment is constant and equal to the applied moment. It is positive according to our convention.



We now superimpose these two loading cases to solve a more general problem and one that is relevant to your design task.

For the cantilever under end load and end moment, the deflected shape might look as shown .



For this combined loading, we have:

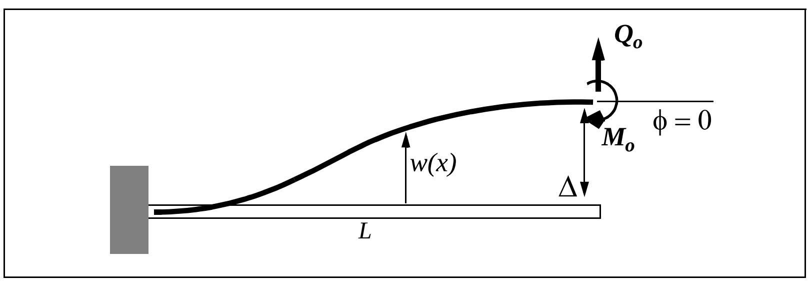
$$w(x) = \frac{Q_o}{EI} \cdot \left(\frac{x^2}{2}L - \frac{x^3}{6}\right) + \frac{-M_o x^2}{2EI}$$

$$\frac{d}{dx}w(x) = \frac{Q_o}{EI} \cdot \left(xL - \frac{x^2}{2}\right) + \frac{-M_o x}{EI}$$

At the end of the beam, we have:

$$\Delta = \frac{Q_o \cdot L^3}{3EI} - \frac{M_o L^2}{2EI} \qquad \text{and} \qquad \phi(L) = \frac{Q_o \cdot L^2}{2EI} - \frac{M_o L}{EI}$$

We want the solution for the special case when the slope at $x = L$ is zero.

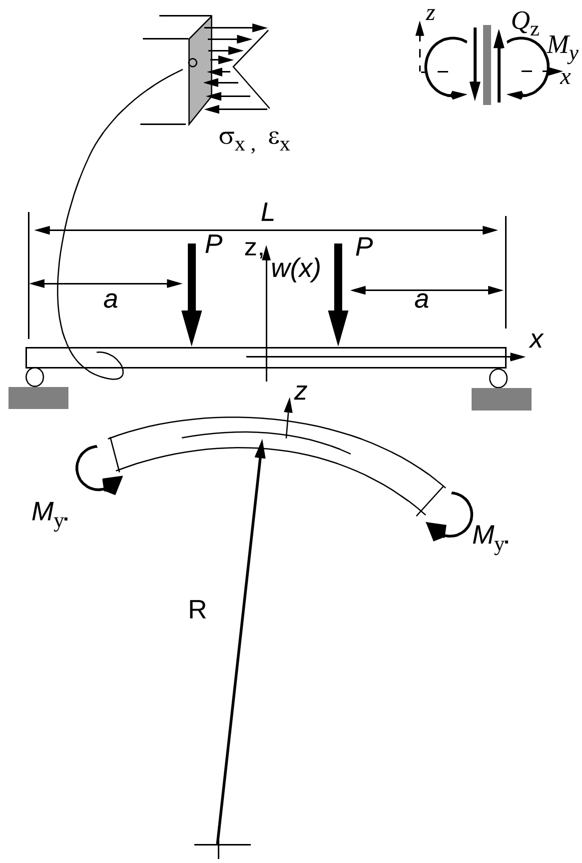


Setting $\phi(L) = 0$ gives the applied end moment in terms of the end force $M_o\big|_{\phi(L) = 0} = \frac{Q_o \cdot L}{2}$

So, in this case, the tip deflection is $\Delta = \frac{Q_o \cdot L^3}{12EI}$ or, expressing Q_o in terms of the displacement,

$$Q_o = K \cdot \Delta \qquad \text{where the stiffness, K, is} \qquad K = \frac{12 \cdot EI}{L^3}$$

Engineering Beam Theory



Engineering beam theory shows that the most significant stress is the normal stress component on an "x face"; σ_x in the example at the right. It is related to the applied loads by

$$\sigma_x = \frac{M_y \cdot z}{I}$$

where z is the distance from the "neutral axis" which, for a doubly symmetric beam, is at the center of the cross-section and I is the moment of inertia of the cross section.

$$I = \int_A z^2 dA$$

For a rectangular cross-section of width b and height h, this is $I = bh^3/12$

The applied loads come in through the bending moment M_y. The convention for positive shear and bending moment is shown in the figure.

The extensional strain, from the stress/strain relations is just $\varepsilon_x = \sigma_x / E$ where E is the Elastic Modulus. In terms of the geometry of deformation, the extensional strain is given by $\varepsilon_x = z / R$ where R is the radius of curvature of the neutral axis. ($1/R$) is the curvature. The "bending stiffness" is defined as the product ***E I*** as it appears in the "moment-curvature" relationship $M_y = (EI) \cdot \left(\frac{1}{R}\right)$ where the curvature, for small deflections, is related to the vertical displacement of the neutral axis by, ($1/R$) = $-d^2w/dx^2$

or . $$-M_y / (EI) = \frac{d^2}{dx^2} w(x)$$

An integration of the differential equation obtained from the moment curvature relation gives, for the case where the beam is loaded as shown, the mid-span deflection

$$w|_{midspan} = -\left(\frac{Pa}{24EI}\right) \cdot (3L^2 - 4a^2)$$