Lecture notes for 12.086/12.586, Modeling Environmental Complexity

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Contents

1 Percolation theory

Reference: Stauffer and Aharony [1].

1.1 Introduction

We have previously defined complex systems as exhibiting both structure and variability.

To this mix, we now add "connectivity." That is, the structure amidst variability is often a consequence of the way "microscopic" components of a system "connect."

1.1.1 What's percolation?

Percolation theory provides a systematic way of studying connectivity in which "connection" is defined geographically: two locations in space are said to be connected if they are near each other, and unconnected if they are not.

The simplest such systems are defined on a regular lattice. Consider, for example, a square lattice

figure

Occupy a fraction p of the lattice squares by, say, coloring them with a black dot.

Now define

Cluster: a group of occupied (nearest) neighboring sites.

Percolation theory describes how the number and properties of clusters changes with the occupation probability p.

We assume that the occupancy of any particular site is independent of the others.

Then if we have N squares and N is large,

- pN are occupied, and

- $(1-p)N$ are empty.

The richness of the problem is seen by viewing typical clusters on square lattices of varying p.

See, for example, this website.

At $p \simeq 0.6$, one cluster extends from one end of the lattice to the other.

We call this the critical probability p_c, and we shall show that clusters formed at p_c exhibit many of the features of "structured variability."

In particular, the study of such structures will bring us in contact with the following subjects:

- *Scale invariance.* The critical clusters are fractals.

- *Universality.* Scaling properties do not depend on details like lattice geometry.

- *Finite-size scaling.* We show how scale-invariant systems behave in systems of different size, and how that relates to non-scale invariant systems.

- *Renormalization.* We introduce a technique for determining properties of scale-invariant systems.

More generally, the goal is to show how "incipient connectedness" leads to interesting non-trivial properties.

We shall also highlight the ways such incipient connectedness occurs generically, without the need to set a parameter like p to p_c.

1.1.2 Examples

We illustrate some of the pertinent phenomena with two introductory examples on square lattices.

Forest fires   Suppose occupied sites represent trees in a forest, and unoccupied sites empty space.

Light a fire at the top row, and sweep through the lattice row-by-row. Entire sweeps alternate as follows:

- Recently ignited trees ignite neighbors to the bottom and right.

- In the next sweep, recently ignited trees ignite neighbors to the top and left.
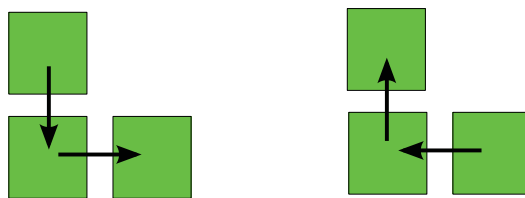
- Trees burn out after one complete sweep.

The simulation continues until all trees are burnt, or all remaining trees cannot be burned.

What is the duration (number of lattice sweeps) of the forest fire as a function of p?

- *Small p*: short, because only a few trees can burn.

- *Large p*: short, because in the limit of a fully occupied forest, each tree immediately burns its neighbor.

- $p \simeq p_c$: very long.

Indeed, there is a (theoretical) singularity at p_c.

Why? Because at $p = p_c$ the largest cluster of neighbors spans the system, but it contains many contorted "dead ends," requiring the fire to wind its way backwards and forwards.

Related examples can be found via Wikipedia.

Fluid invasion We now literally think of a true "percolation" process: the infiltration of a water-filled rock with oil.

Generally water spreads preferentially on solid surfaces, so that it is called the *wetting fluid.*

As a consequence, the penetration of a small pore by the *non-wetting fluid* (oil) is impeded by surface tension. Modeling the pore as a simple capillary tube of radius R, the situation looks like
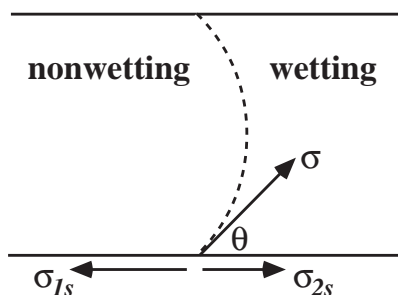
The pressure difference between the non-wetting fluid and the wetting fluid is

$$\Delta p = \frac{2\sigma \cos\theta}{R}.$$

where σ is the surface tension.

Now imagine a large network of capillary tubes of different radii R. If the invading non-wetting fluid is pushed sufficiently slowly, the nonwetting fluid advances along a front of *least resistance*, i.e., the next tube it invades is always the one for which the accessible radius is largest, requiring the smallest Δp.

We can model this on a square lattice by assigning a random number to each site, which we take to be representative of the pressure Δp required to invade that site.

The invader then merely advances at each time step by filling the neighboring site with the smallest Δp.

This model is called *invasion percolation* [2].

As the invading cluster grows to a size L, it is always representative of an incipient spanning cluster in a system of size L. Thus this critical cluster is obtained *dynamically*, rather than by choosing $p = p_c$.

We shall show that this incipient cluster is fractal. What that means is that, if we assigned unit mass to each occupied site, then the total mass M scales not like L^d, where d is the number of dimensions, but rather like

$$M \propto L^D, \qquad D < d.$$

Indeed, when $d = 3$ we have

$$D = 2.5 \quad (d = 3).$$

An important characteristic of such critical phenomena is that they display universality. Here this means that the dimension $D = 2.5$ does *not* depend on the particular type of lattice.

Consequently we expect it to be valid in truly disordered systems such as porous rock.

Indeed, one can conduct the following experiment. Fill a rock with water, and push the water out slowly with oil. As soon as the water reaches the other end, stop the experiment and measure the fraction ϕ of the pore space filled by the invading fluid, oil.

Since $M \propto L^D$, we expect.

$$\phi \propto \frac{L^D}{L^d} = L^{-1/2}.$$

Remarkably, this is precisely
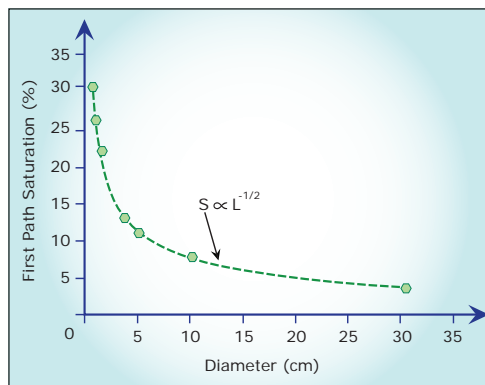what is found [3]:

Figure by MIT OpenCourseWare.

The process just described is similar to the upward *secondary migration* of oil from its source to its subsurface reservoir.

We find, therefore, that although a cm-scale laboratory simulation of secondary migration predicts an oil saturation of about 30%, at the 100-m reservoir scale we would expect only 0.3%!.

1.2   Percolation in one dimension

Consider a long chain of lattice sites. Sites are occupied (black) with probability p, and unoccupied with probability $1 - p$.

$$- \bullet - \circ - \bullet - \bullet - \bullet - \bullet - \bullet - \circ$$

Since the occupation of each site is independent of the others,

$$P(n \text{ arbitrary sites are occupied}) = p^n$$

The probability that a fixed site is to the left end of an s-cluster is therefore

$$(1 - p)^2 p^s$$

Each site has an equal probability of being the left end of an s-cluster. Thus in a system of size L, the total number of s-clusters is, ignoring end effects,

$$L(1 - p)^2 p^s$$

Define

$$n_s = \text{number of } s\text{-clusters per site.}$$

Then, obviously,

$$n_s = p^s (1 - p)^2.$$

7

We also have
$$P(\text{arbitrary site is part of } s\text{-cluster}) = n_s s,$$
larger than n_s by a factor of s, since there are s ways an arbitrary site can be situated within an s-cluster.

What is the percolation threshold?

- At $p = 1$, all sites are occupied.

- For $p < 1$, there are necessarily holes as $L \to \infty$.

Thus
$$p_c = 1.$$
Summing over all possible cluster sizes, we obtain the probability that an arbitrary site is a member of *any* cluster:
$$\sum_{s=1}^{\infty} n_s s = p \qquad (p < p_c)$$
where we write $p < p_c$ to eliminate the possibility of an infinite cluster at p_c.

It is instructive to obtain the previous result from the definition of n_s:
$$
\begin{aligned}
\sum_s n_s s = \sum_s p^s (1-p)^2 s &= (1-p)^2 \sum_s p \frac{\mathrm{d}}{\mathrm{d}p} p^s \\
&= (1-p)^2 p \frac{\mathrm{d}}{\mathrm{d}p} \sum_s p^s \\
&= (1-p)^2 p \frac{\mathrm{d}}{\mathrm{d}p} \left(\frac{p}{1-p} \right) \\
&= p.
\end{aligned}
$$
Another quantity of interest is
$$
\begin{aligned}
w_s &= P(\text{arbitrary } occupied \text{ site belongs to an } s\text{-cluster}) \\
&= \frac{P(\text{site belongs to } s\text{-cluster})}{P(\text{site belongs to any cluster})} \\
&= \frac{n_s s}{\sum_s n_s s}.
\end{aligned}
$$

Then, defining
$$S = \text{average cluster size,}$$
we have
$$S = \sum_s w_s s = \sum_s \left(\frac{n_s s^2}{\sum_s n_s s} \right).$$
We can calculate S explicitly, following the same scheme as above:
$$\sum_s n_s s^2 = (1-p)^2 \sum_s s^2 p^s$$
$$= (1-p)^2 \left(p \frac{\mathrm{d}}{\mathrm{d}p} \right)^2 \sum_s p^s.$$

Noting that $\left(p \frac{d}{dp} \right)^2 = p \frac{d}{dp} + p^2 \frac{d^2}{dp^2}$ and using the previous trick to calculate the sums, we obtain
$$S = \frac{1+p}{1-p}.$$
and note that S diverges at $p = p_c = 1$.

Next, we define the *correlation function*
$$g(r) = P(\text{site a distance } r \text{ from occupied site is in same cluster})$$
Obviously,
$$g(0) = 1 \quad \text{and} \quad g(1) = p.$$
Thus
$$g(r) = p^r.$$
Consequently $g(r)$ decays exponentially, like
$$g(r) = e^{-r/\xi},$$
where we have defined
$$\begin{aligned} \xi &= \text{correlation length} \\ &= \frac{-1}{\ln p}. \end{aligned}$$
To determine how ξ behaves near $p = p_c = 1$, write
$$p = 1 - (p_c - p).$$

Then for p close to $p_c = 1$,

$$\ln[1 - (p_c - p)] \simeq -(p_c - p)$$

yielding

$$\xi = \frac{1}{p_c - p}, \qquad p \to p_c.$$

Thus ξ also diverges as $p \to p_c$.

How does ξ vary with S? Comparing both results, we have

$$S \propto \xi, \qquad p \to p_c$$

since as $p \to p_c$ both quantities represent complete occupancy of the lattice.

Summary: In one dimension, the mean cluster size S and the correlation length ξ diverge like

$$(p_c - p)^{-1},$$

a simple power law.

Similar power-law behavior will be evident in higher dimensions.

1.3   Clusters in two dimensions

A two dimensional square lattice serves to illustrate the main difficulties of higher dimensions.

Consider a cluster of size $s = 1$:

$$
\begin{array}{ccc}
 & \circ & \\
\circ & \times & \circ \\
 & \circ &
\end{array}
\qquad P(s = 1) = p(1 - p)^4
$$

Size $s = 2$

$$
\begin{array}{ccc}
 & \circ & \\
\circ & \times & \circ \\
\circ & \times & \circ \\
 & \circ &
\end{array}
\qquad \text{(2 orientations)} \quad P(s = 2) = 2p^2(1 - p)^6
$$

Size $s = 3$:

$$P(s = 3) = 2p^3(1 - p)^8 + 4p^3(1 - p)^7$$

Size $s = 4$:

total: 19

Size $s = 5$: 63.
Size $s = 24 : 10^{13}$!

These objects, called *lattice animals*, can be very complex.

Yet the general form of the probability $P(s = s_0)$ is clear: we need to know the multiplicity of various forms, and the *perimeter t*.

Thus we define

$$g_{st} = \text{number of lattice animals of size } s \text{ and perimeter } t.$$

Then the average number of s-clusters per lattice site in d dimensions is

$$n_s = \sum_t g_{st} p^s (1 - p)^t.$$

In general, there is no exact solution for g_{st}, only special cases on special lattices.

1.4   Percolation on the Bethe lattice

The Bethe lattice—also known as a Cayley tree—is one such special lattice.

Each site on the Bethe lattice has $z > 2$ neighbors. The simplest case is $z = 3$:
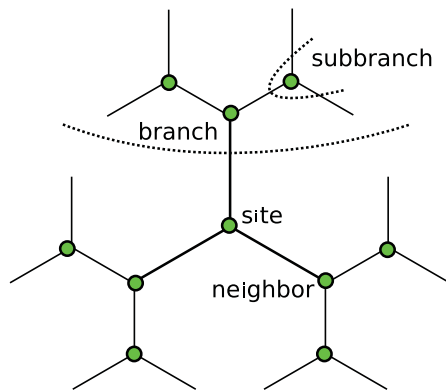
The center is the *origin*, and the exterior sites are said to be on the *surface*.

1.4.1 Infinite dimensionality

The Bethe lattice apparently does not reside in any particular number of dimensions d.

However there is a sense in which the Bethe lattice corresponds to $d = \infty$.

To see this, take the radius of the Bethe lattice to be r. If the Bethe lattice lived in d-dimensions, we'd have

$$\text{volume (\# sites)} \ \propto \ r^d$$
$$\text{surface (\# exterior sites)} \ \propto \ r^{d-1}$$

which implies

$$\text{surface} \propto (\text{volume})^{(d-1)/d} = (\text{volume})^{1-1/d}.$$

On the Bethe lattice,

- The origin is surrounded by $z = 3$ sites.

- The 2nd shell has $z(z-1) = 6$ sites.

- The 3rd has $z(z-1)^2 = 12$ sites, etc

Thus within a distance r from the origin, there are, including the origin, and setting $z_1 = z - 1$,

$$1 + z\left[1 + z_1 + z_1^2 + \ldots z_1^{r-1}\right] = 1 + \frac{z(z_1^r - 1)}{z_1 - 1} \quad \text{volume sites.}$$

But at a distance r there are

$$zz_1^{r-1} \quad \text{surface sites.}$$

Then, taking r large,

$$\lim_{r\to\infty} \frac{\text{surface}}{\text{volume}} = \lim_{r\to\infty} \frac{zz_1^{r-1}}{1 + \frac{z(z_1^r-1)}{z_1-1}} = \frac{z_1 - 1}{z_1} = \frac{z - 2}{z - 1} = \text{const.},$$

where the second equality follows from dividing the numerator and denominator by zz_1^{r-1} and taking the limit. Thus as $r \to \infty$,

$$\text{surface} \propto \text{volume} \quad \Rightarrow \quad d = \infty.$$

This observation does not diminish the interest of the Bethe lattice for percolation, but it does point out its inapplicability to modeling branched $d = 2$ structures such as river networks.

1.4.2 Percolation threshold

For what values of p is the origin connected to infinity? Note that

- $z - 1$ "new" bonds emanate from each site.

- These bonds connect to an occupied site with probability p.

Thus

$$\langle \text{\# occupied new neighbors} \rangle = (z - 1)p.$$

Note that

$$(z - 1)p < 1 \quad \Rightarrow \quad P(\text{connected path}) \to 0 \text{ exponentially.}$$

Thus we expect a connected path when

$$p > p_c = \frac{1}{z - 1} = \frac{1}{2} \quad \text{for} \quad z = 3.$$

Does $p > p_c$ guarantee a connection of the origin to ∞?

1.4.3 The strength $\mathcal{P}$

Define

$$\mathcal{P} = P(\text{arbitrary site—e.g., origin—belongs to infinite cluster}).$$

Note that the arbitrary site may be either occupied or unoccupied. We distinguish between

$$
\begin{aligned}
p &= \text{``concentration''} \\
\mathcal{P} &= \text{``strength''}
\end{aligned}
$$

where the sense of "strength," corresponds, e.g., to a kind of rigidity of the percolation cluster (were it, say, connected by springs).

To calculate $\mathcal{P}$, we define

$$
\begin{aligned}
Q &= P(\text{arbitary occupied site } x \text{ is } not \text{ connected to } \infty \\
&\qquad \text{through one fixed branch}).
\end{aligned}
$$

Take $z = 3$ for simplicity. Then the local geometry looks like
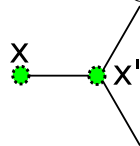
We have

$$P(2 \text{ sub-branches do } not \text{ connect } x' \text{ to } \infty) = Q^2,$$

implying that

$$P(x' \text{ occupied but } not \text{ connected to } \infty) = pQ^2.$$

Therefore

$$
\begin{aligned}
Q &= P(\text{connection broken at } x') + P(\text{connection broken elsewhere}) \\
&= (1 - p) + pQ^2,
\end{aligned}
$$

which has the solutions

$$Q = 1 \quad \text{and} \quad Q = \frac{1 - p}{p}.$$

We identify the first solution with $p < p_c$ and the second with $p > p_c$.

Now note that

$$P(\text{origin is occupied but not connected to } \infty) = p - \mathcal{P}$$
$$= pQ^3.$$

The first equality follows from the observation that only *occupied* sites can be connected to infinity. The second relation arises from the definition of Q with $z = 3$. Equating the two, we have

$$p - \mathcal{P} = pQ^3 \quad \Rightarrow \quad \mathcal{P} = p(1 - Q^3).$$

Then our two solutions for Q correspond to

$$p < p_c, \quad Q = 1 \quad \Rightarrow \quad \mathcal{P} = 0$$

and

$$p > p_c, \quad Q = \frac{1 - p}{p} \quad \Rightarrow \quad \frac{\mathcal{P}}{p} = 1 - \left(\frac{1 - p}{p}\right)^3.$$

Thus the strength $\mathcal{P}$—the probability that the origin belongs to the infinite cluster—rises abruptly at $p_c = 1/2$.
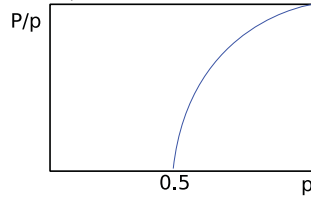
Using our previous analogy between $\mathcal{P}$ and rigidity, one may think of this as a transition from no rigidity, to a kind of floppiness, to strong rigidity.

Qualitatively, the behavior of $\mathcal{P}(p)$ is similar to what one finds in classical phase transitions, such as the spontaneous magnetization that occurs in the transition from paramagnetism to ferromagnetism as the temperature T is lowered through the Curie temperature T_c.

To zoom in on the behavior of $\mathcal{P}$ as $p \to p_c$ from above, we substitute $x = p - p_c$ and expand in powers of x. We obtain, to leading order,

$$\boxed{\mathcal{P} \propto (p - p_c)}, \qquad p \to p_c \downarrow$$

1.4.4　Mean cluster size S

Assume again that $z = 3$. Define

$$T = \text{mean cluster size for 1 branch.}$$

i.e., the average number of sites connected to the origin and belonging to 1 branch.

Note that sub-branches have the same mean cluster size (because all sites in the interior are essentially equivalent).

Therefore each occupied neighbor contributes, on average, T sites to the cluster, so that

$$
\begin{aligned}
T &= P(\text{neighbor empty}) \times 0 \\
&+ P(\text{neighbor occupied}) \times (1 + 2T)
\end{aligned}
$$

where the factor of $2T$ comes from the two sub-branches. We obtain

$$T = p(1 + 2T) \quad \Rightarrow \quad T = \frac{p}{1 - 2p}, \quad p < p_c = 1/2.$$

The mean cluster size S is then the sum of all $z = 3$ branches plus the origin:

$$S = p(1 + 3T).$$

Substituting our solution above for T, we obtain

$$S = \frac{p(1 + p)}{1 - 2p}, \qquad p < p_c = 1/2.$$

We have therefore obtained exact results for $S(p < p_c)$ and $\mathcal{P}(p > p_c)$.

How does the mean cluster size S behave as $p \to p_c$ from below? Near p_c the numerator of $S \sim \text{const.}$, so

$$S \propto \frac{\text{const.}}{\frac{1}{2} - p}$$

and therefore

$$\boxed{S \propto (p_c - p)^{-1}}, \qquad p \to p_c \uparrow$$

Thus S diverges, with exponent -1, near the critical point p_c.

In contrast, we found above that $\mathcal{P}$ vanishes at p_c like $\mathcal{P} \propto (p - p_c)^1$.

Analogous exponents on d-dimensional lattices are more complicated. The power-law approach to criticality, however, is general, and is a defining characteristic of thermal phase transitions.

1.4.5 Cluster numbers $n_s(p)$

Recall the number of s-clusters per site:

$$n_s(p) = \sum_t g_{st} p^s (1 - p)^t,$$

where g_{st} is the number of s clusters with perimeter t.

The perimeter is straightforwardly computed on the Bethe lattice:

- a 1-cluster has z neighbors.

- a 2-cluster has $z + (z - 2)$ neighbors (it removes one neighbor and adds $z - 1$).

- each additional site adds $z - 2$ neighbors.

Thus an s-cluster has

$$
\begin{aligned}
t &= z + (s - 1)(z - 2) \\
&= s(z - 2) + 2
\end{aligned}
$$

Since each s-cluster has the same perimeter, $g_{st} \to g_s$, and the expression for $n_s(p)$ becomes

$$n_s = g_s p^s (1 - p)^{2 + s(z-2)}.$$

We set $z = 3$, and normalize by $n_s(p_c)$ to avoid the prefactor g_s:

$$\frac{n_s(p)}{n_s(p_c)} = \left(\frac{1 - p}{1 - p_c}\right)^2 \left[\left(\frac{p}{p_c}\right)\left(\frac{1 - p}{1 - p_c}\right)\right]^s$$

17

Since $p_c = 1/2$ we replace the product of denominators in the brackets with a factor $a = 4$ and write

$$\frac{n_s(p)}{n_s(p_c)} = \left(\frac{1-p}{1-p_c}\right)^2 \left[1 - a(p - p_c)^2\right]^s, \qquad a = 4.$$

We obtain a further simplification by writing

$$\begin{aligned} c &= -\ln\left[1 - a(p - p_c)^2\right] \\ &\propto (p - p_c)^2, \qquad p \to p_c \end{aligned}$$

so that

$$\frac{n_s(p)}{n_s(p_c)} \propto e^{-cs}, \qquad p \to p_c$$

where we have taken the prefactor constant as $p \to p_c$.

Thus the ratio of cluster numbers decays exponentially with cluster size. In two and three dimensions such an exponential decay turns out to be valid only for large clusters for $p < p_c$.

We next seek the asymptotic behavior of the cluster number $n_s(p)$ near p_c, rather than the ratio.

Recall the expression for the mean cluster size that we derived earlier in Section 1.2:

$$S = \frac{\sum_s n_s s^2}{\sum_s n_s s} \propto \sum_s n_s s^2, \qquad p \to p_c$$

where we obtain the proportionality since the denominator is finite at p_c. [It gives the probability (p) that an arbitrary site belongs to *any* cluster.]

But we already found in Section 1.4.4 that

$$S \propto (p - p_c)^{-1}, \qquad p \to p_c \uparrow$$

Therefore

$$\sum_s s^2 n_s(p_c) = \infty \quad \text{and} \quad \sum_s s^2 n_s(p \neq p_c) \text{ is finite.}$$

What more can we say about $n_s(p_c)$?

Clearly it must decay with increasing s. But how?

If it were to decay exponentially, we would necessarily have

$$S(p_c) \propto \sum_s s^2 n_s(p_c) \propto \sum_s s^2 e^{-\alpha s}, \qquad \alpha > 0.$$

But then $S(p_c)$ would be *finite*, contradicting the above.

Thus we *assume* a power-law decay with exponent τ for large s:

$$n_s(p_c) \propto s^{-\tau}$$

(In studies of phase transitions τ is known as the *Fisher exponent*.)

The assumption of this power-law relation allows us to evaluate S and specify τ. Assuming p is just below p_c,

$$\begin{aligned}
S &\propto \sum_s s^2 n_s(p_c) \frac{n_s(p)}{n_s(p_c)} \\
&= \sum_s s^{2-\tau} e^{-cs} \\
&= \int s^{2-\tau} e^{-cs} \mathrm{d}s
\end{aligned}$$

To evaluate the integral, substitute $z = cs$ so that

$$\begin{aligned}
S &= \int \left(\frac{z}{c}\right)^{2-\tau} e^{-z} \frac{\mathrm{d}z}{c} \\
&= c^{\tau-3} \int z^{2-\tau} e^{-z} \mathrm{d}z
\end{aligned}$$

The exponential decay of the integrand for large z guarantees its convergence, so that

$$S \propto c^{\tau-3} = (p - p_c)^{2\tau-6}$$

since, by definition, $c = (p - p_c)^2$.

But we previously found $S \propto (p - p_c)^{-1}$. Therefore

$$2\tau - 6 = -1 \quad \Rightarrow \quad \tau = 5/2.$$

Finally, using our result $n_s(p) \propto n_s(p_c)e^{-cs}$, we obtain

$$\boxed{n_s(p) \propto s^{-5/2}e^{-cs}}, \quad c = (p - p_c)^2.$$

The boxed relation holds for all p and large s; the second relation is for p near p_c.

This completes our study of the Bethe lattice. The main results are the power laws for

- mean cluster size $S \propto (p_c - p)^{-1}$
- strength $\mathcal{P} \propto (p - p_c)^1$

and the above result for the cluster numbers $n_s(p)$.

1.5 Scaling laws in d-dimensions

We now seek an understanding of percolation that includes the Bethe-lattice solution as a special case.

To do so, we *postulate* the following scaling laws:

Cluster numbers:
$$n_s(p) \propto s^{-\tau}e^{-cs}, \qquad s \to \infty$$
where
$$c \propto |p - p_c|^{1/\sigma}, \qquad p \to p_c$$
Here τ and σ are constants that depend only on d.

We also have the ratio
$$\frac{n_s(p)}{n_s(p_c)} \propto e^{-cs}$$
accompanied, as before, by the critical cluster number

$$n_s(p_c) \propto s^{-\tau}$$

1.5.1 The strength $\mathcal{P}$

We seek

$$
\begin{aligned}
\mathcal{P} &= \text{fraction of sites belonging to infinite cluster} \\
&= \text{(fraction of occupied sites)} - \text{(fraction belonging to finite clusters)} \\
&= p - \sum_{s \text{ finite}} n_s s.
\end{aligned}
$$

At $p = p_c$, $\mathcal{P} = 0$ and therefore

$$
\sum_{s \text{ finite}} n_s(p_c) s = \sum_s n_s(p_c) s = p_c
$$

For the infinite sum to converge, we require $\tau > 2$.

Now approximate $\mathcal{P}$ near p_c by

$$
\mathcal{P} = p_c - \sum_s n_s(p) s + \mathcal{O}(p - p_c), \qquad p \to p_c
$$

Then

$$
\begin{aligned}
\mathcal{P} &\simeq \sum_s [n_s(p_c) - n_s(p)] s \\
&= \sum_s s^{1-\tau}(1 - e^{-cs}) \\
&\simeq \int s^{1-\tau}(1 - e^{-cs}) \mathrm{d}s
\end{aligned}
$$

The passage from the summation to the integral is made possible by noting that the main contribution to the sum comes for large $s \sim 1/c$ as $c \to 0$.

We integrate by parts, using $\int f'g\mathrm{d}s = -\int fg'\mathrm{d}s + (fg)$ where

$$
f(s) = s^{2-\tau} \quad \text{and} \quad g(s) = 1 - e^{-cs}.
$$

The term fg yields

$$
s^{2-\tau}(1 - e^{-cs})\big|_0^\infty = 0 - s^{2-\tau}[cs - (cs)^2/2 + \ldots]\big|^0 = 0
$$

by requiring (as one finds) $\tau < 3$. The remaining integral, after substituting $z = cs$ as in Section 1.4.5, yields $c^{\tau-2} \times$ const. Consequently

$$
\mathcal{P} \propto c^{\tau-2} = (p - p_c)^{(\tau-2)/\sigma}
$$

and therefore

$$\boxed{\mathcal{P} = (p - p_c)^\beta}$$

where we have obtained the scaling relation

$$\boxed{\beta = \frac{\tau - 2}{\sigma}}.$$

Note that the Bethe lattice solution $\beta = 1$, $\tau = 5/2$, and $\sigma = 1/2$ is obtained as a special case.

1.5.2   The mean cluster size S

Near p_c, we have

$$S \simeq \frac{\sum_s n_s s^2}{p_c}, \qquad p \to p_c$$

Converting to an integral,

$$\begin{aligned} S &\propto \int s^2 n_s \mathrm{d}s \\ &\propto \int s^{2-\tau} e^{-cs} \mathrm{d}s \\ &= c^{\tau-3} \int z^{2-\tau} e^{-z} \mathrm{d}z \end{aligned}$$

Then

$$S \propto c^{\tau-3} = |p - p_c|^{(\tau-3)/\sigma}$$

and therefore

$$\boxed{S \propto |p - p_c|^{-\gamma}},$$

where we have obtained the critical exponent for cluster sizes,

$$\boxed{\gamma = \frac{3 - \tau}{\sigma}}.$$

We proceed to generalize these results for the kth moment of n_s, defined by

$$M_k = \sum_s s^k n_s.$$

As above,

$$M_k \propto \sum_s s^{k-\tau}e^{-cs}$$

$$\propto \int s^{k-\tau}e^{-cs}\mathrm{d}s$$

$$\propto c^{\tau-1-k}\int z^{k-\tau}e^{-z}\mathrm{d}z$$

yielding,

$$M_k \propto c^{\tau-1-k} \propto |p-p_c|^{(\tau-1-k)/\sigma}.$$

This is simply a generalization of our previous results for β ($k=1$) and $-\gamma$ ($k=2$).

Thus we see that "everything" depends on two exponents, σ and τ (or β and γ).

These relationships, known as *scaling laws*, originated in the study of thermal phase transitions.

The exponents depend on the dimension d, but the relations between them do not.

Importantly, the exponents do *not* depend on the type of lattice (square, triangular, etc.) This is known as *universality*.

1.6   Fractals

Near p_c, the geometry of clusters turns out to be *fractal*. In this section we define fractals, and show why this special geometry emerges near the critical point.

1.6.1   Cluster radius

Clusters can have complex shapes.

FIGURE

One way of expressing this complexity is to determine

$$R_s = \text{radius of gyration of an } s\text{-cluster},$$

which we express as

$$R_s^2 = \frac{1}{s} \sum_{i=1}^{s} |\vec{r}_i - \vec{r}_0|^2,$$

where

$$\vec{r}_i = \text{position of } i\text{th site in cluster}$$

and

$$\vec{r}_0 = \text{center of mass of an } s\text{-cluster}$$
$$= \frac{1}{s} \sum_{i=1}^{s} \vec{r}_i.$$

The mean-square distance between every pair of sites on the cluster is also related to R_s:

$$\frac{1}{s^s} \sum_{i=1}^{s} \sum_{j=1}^{s} |\vec{r}_i - \vec{r}_j|^2 = 2R_s^2,$$

where the factor of 2 comes from counting each pair twice.

1.6.2 Correlation length

Define the *correlation length*

$$\xi = \text{an average distance between two cluster sites.}$$

Whereas $2R_s^2$ is the mean-square distance between two sites on an s-cluster, ξ^2 is this same distance averaged over all *finite* sizes s.

We obtain that average by noting that, as already discussed,

$$P(\text{site belongs to } s\text{-cluster}) = sn_s.$$

There are (obviously) s sites in each s-cluster. We thus weight the average of R_s^2 by $s \cdot sn_s$, to obtain the squared correlation length

$$\xi^2 = \frac{2 \sum_s R_s^2 s^2 n_s}{\sum_s s^2 n_s}.$$

In Section 1.2 we saw how ξ diverges as $p \to p_c$ when $d = 1$. We expect similar behavior for $d > 1$ and therefore write

$$\xi \propto |p - p_c|^{-\nu}$$

where ν is another exponent.

Given these definitions, we seek how R_s varies with s at $p = p_c$. But we must first take a detour and finally define what we mean by "fractal."

1.6.3 The fractal dimension D

Suppose each occupied site is assigned a unit of mass, while unoccupied sites have no mass.

In a "Euclidean" or "space-filling" geometry, we expect that

$$\text{mass}(s) \propto R_s^d \text{ in } d \text{ dimensions.}$$

Suppose instead that we observe

$$\text{mass} \propto \text{length}^D.$$

Objects are called *fractals* if

$$D < d \text{ and } D \neq \text{ integer.}$$

Now *assume*

$$R_s \propto s^{1/D}, \qquad (p = p_c, \quad s \to \infty)$$

The fractal dimension D is known exactly in $d = 2$ and approximately in $d = 3$:

$$d = 2 : \quad D = 91/48 \simeq 1.9$$
$$d = 3 : \quad D \simeq 2.5$$

Since D and ν both describe the geometry of critical clusters, we expect that they are related. To see how, we recall from Section 1.5.2 that the kth moment of n_s,

$$M_k = \sum_s s^k n_s \propto |p - p_c|^{(\tau - 1 - k)/\sigma}.$$

Since $R_s \propto s^{1/D}$, our expression for ξ^2 above is simplified to

$$\xi^2 = \frac{M_{2+2/D}}{M_2} \propto |p - p_c|^{-2/(D\sigma)}.$$

Comparing to our definition of the correlation-length exponent ν just above in Section 1.6.2, we immediately find

$$2\nu = \frac{2}{D\sigma} \quad \text{or} \quad \frac{1}{D} = \sigma\nu.$$

We have thus related the fractal dimension D to the correlation exponent ν.

1.6.4 Scaling of the infinite cluster at $p = p_c$

Away from p_c, we know the following about the largest cluster:

- For $p < p_c$, the largest cluster is finite.

- For $p > p_c$, the largest cluster is infinite.

What happens at $p = p_c$?

To answer this question, we pose the following question:

How does the size $s_{\max}$ of the largest cluster vary in a finite system of size L?

- For $p < p_c$, we have, from Section 1.5,

$$n_s \propto s^{-\tau} e^{-cs}, \qquad s \to \infty,$$

But the number of clusters of size $s_{\max}$ in a system of size L is just

$$n_{s_{\max}} \sim \frac{1}{L^d} \propto s_{\max}^{-\tau} e^{-cs_{\max}}.$$

Taking logarithms on both sides of the proportionality and noting $\log s_{\max} \ll s_{\max}$, we have

$$\boxed{s_{\max} \propto \log L, \qquad p < p_c}.$$

- For $p > p_c$, we have the obvious relation

$$\boxed{s_{\max} \propto L^d, \quad p > p_c}.$$

- For $p = p_c$, we expect that the radius of gyration $R_{s_{\max}}$ scales with the system size:

$$R_{s_{\max}} \propto L, \quad p = p_c.$$

By the results of the previous section,

$$R_s \propto s^{1/D}, \qquad p = p_c, \quad s \to \infty.$$

Combining these relations, we obtain

$$\boxed{s_{\max} \propto L^D, \qquad p = p_c},$$

Thus the largest cluster at $p = p_c$ has fractal dimension $D < d$, which results from the transition from $s_{\max} \propto \log L$ just below p_c to $s_{\max} \propto L^d$ just above.

This result, which follows here from the scaling assumption $R_s \propto s^{1/D}$ of the previous section, has been confirmed by extensive numerical simulation.

1.6.5   Relating D to d via the correlation length ξ

The above results correspond to p well below p_c, at p_c, and well above it.

Now suppose p is near p_c, with $p > p_c$, and consider the correlation length ξ.

We consider two samples of different size L of a larger system.

- If $L \ll \xi$, we inherit the above results for $p = p_c$:

$$s_{\max} \propto L^D.$$

- If $L \gg \xi$, the probability $\mathcal{P}$ that a site belongs to the infinite (spanning) cluster provides

$$s_{\max} \propto \mathcal{P} L^d \propto (p - p_c)^\beta L^d, \qquad p \to p_c.$$

27

These two expressions should be equal when

$$L \sim \xi \propto |p - p_c|^{-\nu}.$$

where the latter expression derives from our definition of the correlation exponent ν (Section 1.6.2).

Substituting $L \sim |p - p_c|^{-\nu}$ into each equation for $s_{\max}$ and equating the two, we obtain

$$\beta - d\nu = -D\nu \qquad \Rightarrow \qquad \boxed{D = d - \beta/\nu},$$

which relates the difference between the fractal and Euclidean dimensions to the correlation exponent ν. (The various scaling relations in Section 1.5.2 can now be used to obtain further relations that include the dimensionality d.)

1.7 Finite-size scaling

The dependence of $s_{\max}$ on the system size L and the correlation length ξ suggests that there is a general way in which behavior in finite systems reflects the "true" asymptotic scaling one finds in infinite systems.

Consider, for example, the strength $\mathcal{P}$—the probability that a site belongs to the spanning cluster. As we have already seen, in an infinite system $\mathcal{P}$ sharply rises from zero at $p = p_c$:
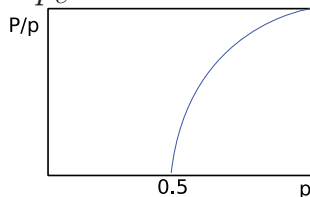
What happens, however, when L is finite?

- When L is small, clusters formed at $p < p_c$ can span the finite system provided that the correlation length $\xi \gtrsim L$.

- Larger L, however, requires larger p associated with larger ξ.

Consequently $\mathcal{P}$ must depend on both p and L.

We expect that, for large systems near p_c,

$$\mathcal{P} = \mathcal{P}[(p - p_c), L], \qquad p \to p_c, \quad L \to \infty$$

We seek a functional dependence on $(p - p_c)$ and L for which the tradeoff between the correlation length ξ and the system size L is evident.

Since $(p - p_c)$ controls the correlation length, we propose the form

$$\mathcal{P} = L^{-A} F\left[(p - p_c)L^B\right], \qquad p \to p_c, \quad L \to \infty$$

We proceed to infer the form of F.

First, note that as $L \to \infty$ for $p > p_c$, we obtain, for p near p_c,

$$\mathcal{P} \propto (p - p_c)^\beta \text{ independent of } L.$$

In other words, as L becomes large, the dependence of $\mathcal{P}$ on L must vanish.

More generally, for large $z = (p - p_c)L^B$,

$$F(z) \propto z^{A/B}, \quad z \to \infty$$

so that the dependence of $\mathcal{P}$ on L vanishes, yielding

$$\mathcal{P} \propto (p - p_c)^{A/B} \quad \Rightarrow \quad \beta = A/B.$$

where the relation to β derives from its definition in Section 1.5.1.

At $p = p_c$, we have seen that

$$s_{\max} \propto L^D, \quad p = p_c.$$

Therefore the probability $\mathcal{P}$ that a site belongs to $s_{\max}$ scales like

$$\mathcal{P} = \frac{s_{\max}}{L^d} \propto L^{D-d}, \quad p = p_c.$$

Since this result is obtained at $p = p_c$, it must be equivalent to

$$\mathcal{P}(0) = L^{-A} F(0)$$

Therefore

$$A = d - D = \beta/\nu$$

where the second equality was derived just above in Section 1.6.5.

Using $\beta = A/B$, we find
$$B = A/\beta = 1/\nu.$$

Substituting into our general form for finite-size scaling, we obtain
$$\mathcal{P} = L^{-\beta/\nu} F\left[(p - p_c)L^{1/\nu}\right],$$

which expresses $\mathcal{P}$ as a function of L and $p - p_c$ in terms of the correlation length exponent ν and the scaling relation $\mathcal{P} \propto (p - p_c)^\beta$.

To understand this result, imagine that we measure $\mathcal{P}$ as a function of $p - p_c$ and L. Finite-size scaling says that plots of
$$\mathcal{P}/L^{-\beta/\nu} \quad \text{vs.} \quad (p - p_c)L^{1/\nu}$$

will all fall on the same curve. It is thus a way of estimating β and ν.

As another type of practical application, consider again the problem of invasion percolation described in Section 1.1.2.

The mass M of the invading cluster at "breakthrough" in a system of size L scales like
$$M \propto L^D$$

The fraction of sites belonging to this cluster is $\mathcal{P}$.

Since the invading cluster at breakthrough corresponds to the largest cluster at $p = p_c$, we have
$$\mathcal{P} = \mathcal{P}(0) \propto L^{-\beta/\nu} = L^{D-d} = L^{-1/2} \text{ for } d = 3.$$

Finally, we expose the crucial role played by the correlation length in the above arguments.

We express finite-size scaling in terms of the correlation length $\xi \propto |p - p_c|^{-\nu}$ so that
$$\mathcal{P} = L^{-\beta/\nu} F_1(L/\xi).$$

More generally, if a quantity Q scales for large $L \gg \xi$ like

$$Q \propto |p - p_c|^\chi \propto \xi^{-\chi/\nu}, \qquad L \gg \xi,$$

we merely replace β with χ above so that

$$Q(L, \xi) = L^{-\chi/\nu} q_1(L/\xi) \propto \begin{cases} \xi^{-\chi/\nu} & L \gg \xi \\ L^{-\chi/\nu} & L \ll \xi \end{cases},$$

where the scaling for small $L \ll \xi$ corresponds also to scaling with respect to any L and $p = p_c$, as in the example of invasion percolation.

1.8   Renormalization

Renormalization is a powerful method for understanding systems that exhibit *self-similarity*.

1.8.1   Self-similarity

Let us clarify the meaning of self-similar. As we have seen above (Section 1.7) in our discussion of finite-size scaling, the behavior of any quantity, say $\mathcal{P}$, with respect to L depends on the size of L compared to the correlation length ξ, i.e.,

$$\mathcal{P}(L, \xi) \propto \begin{cases} \xi^{D-d} & L \gg \xi \\ L^{D-d} & L \ll \xi \end{cases}$$

Thus all systems smaller than the correlation length ξ are *similar* to each other (in an average sense), in that they "see" only the system size L rather than any intrinsic limitation ξ.

More compelling proof of this statement comes from numerical simulations of percolation at $p = p_c$, where one finds outstanding adherence to the power law $s_{\max} \propto L^D$.

At some point, of course, small clusters become sensitive to the lattice constant or pixel size, and large clusters know that they are pushing up against the correlation length ξ. So we expect that self-similarity holds for

$$\text{lattice constant} \ll \text{cluster size} \ll \xi$$

1.8.2   Real-space renormalization

We discuss a particular form of renormalization, called *real-space renormalization.*

Consider, for example, an $L \times L$ square lattice. Replace each site on this lattice with a "super-site" of linear dimension $b \ll \xi$.

Supersites contain some sort of average of the original sites. Consequently information is lost.

From self-similarity, however, we expect that each supersite of b^d micro-sites relates to the others in a way that is similar to the statistical relations between the micro-sites.

To implement such a procedure, we need an averaging rule to determine when a supersite is "occupied."

Once the averaging rule is determined, we then calculate

$$
\begin{aligned}
p'(p) \;\; = \;\; &P(\text{supersite is occupied given microsites} \\
&\text{occupied with probability p)}.
\end{aligned}
$$

Clusters of supersites are then statistically similar to clusters of microsites when p is a fixed point of $p'(p)$, i.e., when $p = p^*$ such that

$$
p'(p^*) = p^*.
$$

We then identify $p_c = p^*$ as the occupancy fraction of microsites that yields self-similar behavior.

1.8.3   Calculation of the correlation-length exponent ν

All of this depends crucially on the correlation length ξ, which must be the same in both the micro-lattice and averaged lattice, i.e.

$$
\xi(\text{micro-sites}) = \xi'(\text{super-sites, in micro-units})
$$

In the original lattice, we have, near p_c,

$$\xi = \text{const} \cdot |p - p_c|^{-\nu}.$$

In the renormalized (averaged) lattice, we must have

$$\xi' = \text{const} \cdot b \cdot |p' - p_c|^{-\nu}.$$

where the constants are the same and b is the averaging length.

Since $\xi = \xi'$, we have

$$b|p' - p_c|^{-\nu} = |p - p_c|^{-\nu}$$

This is the basic statement of renormalization. Taking logarithms on both sides, we solve for the correlation exponent ν:

$$\frac{1}{\nu} = \log\left(\frac{|p' - p_c|}{|p - p_c|}\right) \Big/ \log b = \frac{\log \lambda}{\log b}.$$

To understand λ, we expand $p'(p)$ around its fixed point p^*:

$$
\begin{aligned}
p'(p) &= p'[p^* + (p - p^*)] \\
&= p'(p^*) + (p - p^*)\left.\frac{\mathrm{d}p'}{\mathrm{d}p}\right|_{p=p^*} + \mathcal{O}(p - p^*)^2.
\end{aligned}
$$

Since $p'(p^*) = p^* = p_c$, we have, after dropping higher-order terms,

$$\left.\frac{\mathrm{d}p'}{\mathrm{d}p}\right|_{p=p^*} \simeq \frac{p' - p_c}{p - p_c} = \lambda.$$

Consequently we see that λ gives the rate at which p' changes with respect to p near p_c, thus determining the stability of the fixed point p^* of the *renormalization group transformation*.

1.8.4   One dimension

As a first example, consider one-dimensional percolation (Section 1.2).

We group the sites into cells of b sites.

These cells are "connected" only if each site among the b sites is occupied. Thus

$$p' = p^b.$$

The fixed point $p^* = 1$, which we identify with the percolation threshold $p_c = 1$ deduced earlier.

We also calculate the correlation length ξ. In units of the distance between supersites, we have

$$\xi' = \xi/b.$$

At $p = p'$, $\xi' = \xi$ only if $\xi = 0$ or ∞. We identify the latter with the fixed point $p^* = p_c = 1$.

We calculate the correlation-length exponent ν by first computing

$$\lambda = \left. \frac{\mathrm{d}p'}{\mathrm{d}p} \right|_{p=p^*} = bp^{b-1}|_{p=1} = b$$

which yields

$$\frac{1}{\nu} = \frac{\log \lambda}{\log b} = 1,$$

precisely as we obtained earlier in Section 1.2.

1.8.5   Triangular lattice

In higher dimensions we must identify transformations p' that provide connections. These connections connect lines in $d = 2$, planes in $d = 3$, etc. There is no single way to do this, and no way yields an exact result.

As a rule, we say that a supersite is occupied if contains a cluster that spans b^d sites.

We consider site percolation on a triangular lattice.
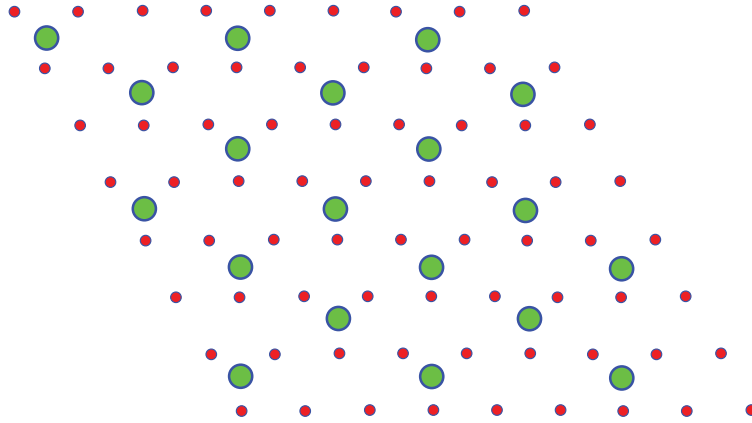
Supersites—the big circles—are located at centers of triangles such that no original sites are shared among supersites.
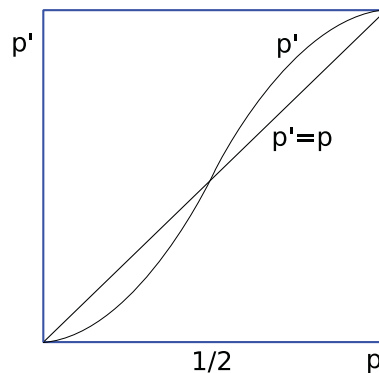
A supersite is considered occupied if occupied microsites span any of the three lattice directions. Thus

$$p' = P(\text{all 3 sites occupied}) + P(\text{2 neighboring sites occupied}).$$

Neighbors turn out to be any pair, and there are three ways of choosing pairs. Thus

$$p' = p^3 + 3p^2(1 - p).$$

Graphically, $p'(p)$ looks like

The fixed points p^* that solve $p'(p^*) = p^*$ are

$$p^* = 0, \ 1/2, \ \text{and } 1.$$

These points occur wherever $p'(p)$ intersects the diagonal line $p' = p$.

If we interpret the mapping $p'(p)$ as a dynamical system, we find that $p^* = 0$ and $p^* = 1$ are *stable* fixed points, and $p^* = 1/2$ is *unstable*. (Recall that $|\mathrm{d}p'/\mathrm{d}p|_{p^*} < 1 \Rightarrow$ stability.)

In other words, successive mappings initiated above $p^* = 1/2$ lead to a fully occupied superlattice, whereas successive mappings below $p^* = 1/2$ lead to a fully empty superlattice.

We thus identify $p_c = 1/2$, in exact agreement with the known value for the triangular lattice.

The rate at which at which perturbations from p_c are unstable under the mapping $p'(p)$ is given by

$$\lambda = \left.\frac{\mathrm{d}p'}{\mathrm{d}p}\right|_{p^*} = 6p - 6p^2|_{1/2} = \frac{3}{2}$$

As in Section 1.8.4, we obtain the correlation-length exponent ν from λ:

$$\nu = \frac{\log b}{\log \lambda} = \frac{\log(3^{1/2})}{\log(3/2)} = 1.355.$$

Here we have used $b^2 = 3$ since 3 microsites contribute to 1 supersite. The result $\nu = 1.355$ compares very well to conjectures that the exact result is $\nu = 4/3$.

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