Fundamental Mathematics for Fluid Mechanics

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Herein **A**, **B**, **C**, and **D** are vectors, and f and g are scalar quantites.

1 Coordinate Systems and Transformations

- Cartesian

$$r_P = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \tag{1}$$
$$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{2}$$

- Cylindrical

$$r_P = r\mathbf{e}_r + \theta\mathbf{e}_\theta + z\mathbf{e}_z \tag{3}$$
$$r_c = \sqrt{x^2 + y^2} \tag{4}$$
$$\theta = \tan^{-1}\frac{y}{x} \tag{5}$$
$$x = r_c \cos\theta \tag{6}$$
$$y = r_c \sin\theta \tag{7}$$
$$\mathbf{e}_r = \cos\theta\,\mathbf{e}_x + \sin\theta\,\mathbf{e}_j \tag{8}$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_x + \cos\theta\,\mathbf{e}_j \tag{9}$$

- Spherical

$$r_P = r\mathbf{e}_r + \theta\mathbf{e}_\theta + \phi\mathbf{e}_\phi \tag{10}$$
$$r_s = \sqrt{x^2 + y^2 + z^2} \tag{11}$$
$$\tag{12}$$

- Earth Centered Coordinates

2 Operators

- Gradient:

$$grad(f) = \nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z \qquad (Cartesian) \tag{13}$$

Figure 1: Coordinates and transformations between Cartesian and cylindrical (left panel) and Cartesian and spherical (right panel).

$$
\begin{aligned}
&= \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \qquad (Cylindrical) && (14)\\
&= \qquad (Spherical) && (15)\\
&= \qquad (Earth) && (16)
\end{aligned}
$$

- Divergence:

$$
\begin{aligned}
div(f) = \nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad (Cartesian) && (17)\\
&= \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} \qquad (Cylindrical) && (18)\\
&= \qquad (Spherical) && (19)\\
&= \qquad (Earth) && (20)
\end{aligned}
$$

- Curl:

$$
\begin{aligned}
curl(f) = \nabla \times \mathbf{A} &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ A_x & A_y & A_z \end{vmatrix} \qquad (Cartesian) && (21)\\
&= \begin{vmatrix} \frac{\mathbf{e}_r}{r} & \mathbf{e}_\theta & \frac{\mathbf{e}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} \qquad (Cylindrical) && (22)\\
&= \qquad (Spherical) && (23)\\
&= \qquad (Earth) && (24)
\end{aligned}
$$

- Laplacian:

$$
\begin{aligned}
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y^2} \qquad (Cartesian) && (25)\\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \qquad (Cylindrical) && (26)\\
&= \qquad (Spherical) && (27)\\
&= \qquad (Earth) && (28)
\end{aligned}
$$

3 Vector Identities

3.1 Vector Products

$$
\begin{aligned}
\mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A} && (29)\\
\mathbf{A} \times \mathbf{B} &= -\mathbf{B} \times \mathbf{A} && (30)\\
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) && (31)\\
&= \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) && (32)\\
&= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} && (33)\\
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) && (34)\\
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}) && (35)
\end{aligned}
$$

3.2 Differentiation with Respect to a Scalar

Herein, $\mathbf{A} = \mathbf{A}(x)$, $\mathbf{B} = \mathbf{B}(x)$, and $f = f(x)$.

$$\frac{d}{dx}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dx} + \frac{d\mathbf{B}}{dx} \tag{36}$$

$$\frac{d}{dx}(f\mathbf{A}) = f\frac{d\mathbf{A}}{dx} + \mathbf{A}\frac{df}{dx} \tag{37}$$

$$\frac{d}{dx}(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \frac{d\mathbf{B}}{dx} + \mathbf{B} \cdot \frac{d\mathbf{A}}{dx} \tag{38}$$

$$\frac{d}{dx}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dx} + \frac{d\mathbf{A}}{dx} \times \mathbf{B} \tag{39}$$

3.3 Identities with the ∇ Operator

$$\nabla(fg) = g\nabla f + f\nabla g \tag{40}$$

$$(\mathbf{A} \cdot \nabla)f = \mathbf{A} \cdot \nabla f \tag{41}$$

$$(\mathbf{A} \times \nabla)f = \mathbf{A} \times (\nabla f) \tag{42}$$

$$\nabla \cdot \sum_n \mathbf{A}_n = \sum_n \nabla \cdot \mathbf{A}_n \tag{43}$$

$$\nabla \times \sum_n \mathbf{A}_n = \sum_n \nabla \times \mathbf{A}_n \tag{44}$$

$$(\mathbf{A} \times \nabla) \cdot \mathbf{B} = \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{45}$$

$$\nabla \cdot f\mathbf{A} = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f \tag{46}$$

$$\nabla \times f\mathbf{A} = f(\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla f) \tag{47}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{48}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{49}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{50}$$

$$(\mathbf{A} \cdot \nabla)\mathbf{A} = \frac{1}{2}\nabla A^2 - \mathbf{A} \times (\nabla \times \mathbf{A}) \tag{51}$$

$$\nabla \times (\nabla f) = 0 \tag{52}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{53}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{54}$$

4 Integration Theorems

- Gauss' theorem or divergence theorem. Herein, $\mathbf{F}$ is a vector function, $\mathcal{V}$ is any volume, and $\mathcal{A}$ is the area that encloses the volume:

$$\int_{\mathcal{A}} \mathbf{F} \cdot \mathbf{e}_n \, d\mathcal{A} = \int_{\mathcal{V}} \nabla \cdot \mathbf{F} \, d\mathcal{V} \tag{55}$$

- Stokes' theorem. Herein, $\mathbf{F}$ is a vector function and $\mathcal{A}$ is any area with $\mathbf{s}$ the line which bounds it and $\mathbf{e}_n$ normal to the area:

$$\oint \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{A}} (\nabla \times \mathbf{F}) \cdot \mathbf{e}_n \, d\mathcal{A} \tag{56}$$

5 Notes

- All identities should be cross-referenced in a couple other books.
- References to proofs of some or all of the identities.
- Introduction, description?
- What are core Fluid mechanics texts, including and excluding GFD.
- Assuming mixed partials equal?
- Spherical, Earth coordinates, description (notation, order).
- $grad()$ versus ∇.
- Align text to right of equations.
- Quick references.
- Include long versions of $curl()$ operator?
- Script version of Laplacian operator, i.e. $curl()$.
- Expand to include things such as Taylor series, Euler's formula, hyperbolic function definitions
- Finish spherical and earth centered coordinates. Look up common notation.
- Define volume integrals in different coordinate systems?

References

[1] Granger, Robert A. "Fluid Mechanics." Dover Publications, Inc. 1995.