7. Potential vorticity and invertibility in the shallow water equations

Consider a layer of strictly incompressible fluid on a level surface, as illustrated in Figure 7.1. The mean depth of the fluid is H with perturbations to the depth denoted by η.

Since the fluid is incompressible, mass continuity gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{7.1}$$

We shall assume that all the motions of interest are hydrostatic, so that integration of the hydrostatic equation gives

$$p = p_0 + \rho g (H + \eta - z), \tag{7.2}$$

where p_0 is the atmospheric pressure and ρ is the fluid density. At the surface of the fluid,

$$w = \frac{d\eta}{dt}. \tag{7.3}$$

The inviscid momentum equations may be written, with the aid of (7.2), as

$$\frac{du}{dt} = -g\frac{\partial \eta}{\partial x} + fv, \tag{7.4}$$

$$\frac{dv}{dt} = -g\frac{\partial \eta}{\partial y} - fu. \tag{7.5}$$

Note from (7.2) that the horizontal pressure gradient acceleration is *independent of depth* in the fluid, as reflected in (7.4) and (7.5). So (7.4) and (7.5) show that if

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at the initial time, u and v are independent of depth, they will remain so forever.
We shall assume that u and v are depth-independent. This means that the mass
continuity equation, (7.1), can be integrated over the local depth of the fluid to give

$$(H + \eta) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{d\eta}{dt} = 0, \tag{7.6}$$

in which we have applied (7.3).

Given that u and v are independent of depth, (7.4) and (7.5) may be expressed
in the alternative form

$$\frac{\partial u}{\partial t} - (f + \zeta)v = -\frac{\partial B}{\partial x}, \tag{7.7}$$

$$\frac{\partial v}{\partial t} + (f + \zeta)u = -\frac{\partial B}{\partial y}, \tag{7.8}$$

where ζ is the relative vorticity,

$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{7.9}$$

and B is the Bernoulli function:

$$B \equiv g\eta + \frac{1}{2}(u^2 + v^2). \tag{7.10}$$

If B is eliminated by cross differentiating (7.7) and (7.8), the result is a vorticity
equation:

$$\frac{d\zeta}{dt} + (f + \zeta) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = 0, \tag{7.11}$$

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showing that vertical stretching is the only source of vorticity in the shallow water system.

Now the horizontal divergence,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

may be eliminated between (7.6) and (7.11) to arrive at the *shallow water potential vorticity equation:*

$$\frac{dq}{dt} = 0, \tag{7.12}$$

with

$$q \equiv \frac{f + \zeta}{H + \eta}. \tag{7.13}$$

If the flow velocity and η are related by some condition, then clearly (7.12), (7.13) constitute a closed system.

One example in which q and $\mathbf{v}$ are strictly related to one another is *steady flow.* First, note that the mass continuity equation, (7.6), can be written in the alternative form

$$\frac{\partial}{\partial x}\left[u(H + \eta)\right] + \frac{\partial}{\partial y}\left[v(H + \eta)\right] = 0, \tag{7.14}$$

in the case of steady flow. Then we may define a *mass streamfunction, ψ,* such that

$$u(H + \eta) \equiv -\frac{\partial \psi}{\partial y},$$
$$v(H + \eta) \equiv \frac{\partial \psi}{\partial x}, \tag{7.15}$$

which clearly satisfies (7.14).

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Now the steady forms of the momentum equations, (7.7) and (7.8), are

$$(f + \zeta)v = \frac{\partial B}{\partial x}, \tag{7.16}$$

$$(f + \zeta)u = -\frac{\partial B}{\partial y}, \tag{7.17}$$

or using (7.15) and (7.13),

$$q\frac{\partial \psi}{\partial x} = \frac{\partial B}{\partial x}, \tag{7.18}$$

$$q\frac{\partial \psi}{\partial y} = \frac{\partial B}{\partial y}. \tag{7.19}$$

At the same time, the steady form of the potential vorticity equation, (7.12), is

$$u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0. \tag{7.20}$$

Multiplying (7.20) by $H + \eta$ and using (7.15) gives

$$-\frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} = 0,$$

or

$$J(q, \psi) = 0, \tag{7.21}$$

where J is the Jacobian operator. The direct implication of (7.21) is that $q = q(\psi)$. Given this conclusion, it is clear from (7.18) and (7.19) that $B = B(\psi)$. Thus if $q(\psi)$ is *known,* then $B(\psi)$ can be found from (7.18) and (7.19), and (7.10) and (7.13) can then be solved for η, u, and v. This is an example of inversion, and for this case it is exact.

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In time-dependent flows, it is necessary to make an approximation to the momentum equations to recover invertibility. *The most commonly used approximation in geophysical flows is to assume that the horizontal divergence, or its time tendency, is small compared to the vertical component of vorticity;* this approximation filters sound waves and inertia-gravity waves from the equations.

We start by differentiating (7.4) with respect to x and (7.5) with respect to y and summing the result to arrive at the *divergence equation:*

$$\frac{dD}{dt} = -g\nabla^2\eta + (f + \zeta)\zeta - \beta u - S^2, \qquad (7.22)$$

where

$$D \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

$$\beta \equiv \frac{df}{dy},$$

and

$$S^2 \equiv \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2.$$

S is the net deformation in the flow.

The *nonlinear balance equation* is obtained by making the approximation

$$\left|\frac{dD}{dt}\right| \ll |(f + \zeta)\zeta|, \tag{7.23}$$

so that

$$g\nabla^2\eta \simeq (f + \zeta)\zeta - \beta u - S^2. \tag{7.24}$$

This relates the *instantaneous* distribution of η to the *instantaneous* distribution of velocity in the system.

Now to be consistent with the approximation (7.23), the velocities that appear on the right side of (7.24) should be dominated by their nondivergent components, allowing us to rewrite (7.24) as

$$g\nabla^2\eta \simeq (f + \nabla^2\psi)\nabla^2\psi + \beta\frac{\partial\psi}{\partial y} - 2\left(\frac{\partial^2\psi}{\partial y\partial x}\right)^2 - \left(\frac{\partial^2\psi}{\partial y^2}\right)^2 - \left(\frac{\partial^2\psi}{\partial x^2}\right)^2, \tag{7.25}$$

where ψ is now the velocity streamfunction, defined such that

$$u = -\frac{\partial\psi}{\partial y},$$
$$v = \frac{\partial\psi}{\partial x}. \tag{7.26}$$

At the same time, the potential vorticity equation (7.12) with (7.13) can be written

$$\frac{\partial q}{\partial t} = -J(\psi, q). \tag{7.27}$$

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The two equations (7.27) and (7.24) constitute a *closed system* for ψ and η. Potential vorticity is just advected around, according to (7.27), and at each time the streamfunction and depth η can be found from (7.25) and

$$q = \frac{f + \nabla^2 \psi}{H + \eta}. \tag{7.28}$$

It should be mentioned here that the approximation (7.23) does *not* imply that $w = 0$ from mass continuity; it is simply a scaling approximation appropriate to the divergence equation. In fact, the system of equations (7.27), (7.28), and (7.25) directly imply a nonzero horizontal divergence, because in general the evolution of the system will give a nonzero charge in $f + \nabla^2 \psi$, the vorticity. But from the vorticity equation, (7.11), vorticity can *only* change if there is nonzero divergence. Thus the divergence and the vertical velocity can be diagnosed by solving (7.11) once the time rate of change of vorticity is known (or by solving (7.6) once the time rate of change of η is known.) This is why the system is referred to as a *quasi-balanced* system: Exact balance is degenerate in the sense that the evolution of the flow cannot be calculated.

In summary, a suitable balance approximation allows one to calculate the evolution of a quasi-balanced flow according to this schema:

1. Prescribe initial flow and/or η that satisfies the balance condition (7.25, in this case).

2. Calculate the distribution of q using its definition (7.28).

3. Find q at the next time step by solving the potential vorticity equation, (7.27).

4. Invert the q distribution using the definition of q and a balance condition (7.28 and 7.25, in this case) to find the velocity and mass distribution. (Note, however, that boundary conditions must be prescribed to carry out the inversion.)

5. Go back to 3.

Note that this scheme is highly analogous to the solution of the vorticity equation for strictly two-dimensional flow:

$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta), \tag{7.29}$$

$$\zeta = \nabla^2 \psi. \tag{7.30}$$

Here again, vorticity is simply advected around by the flow, and the flow is recovered at each time step by inverting (7.30) subject to boundary conditions. The difference is that the system (7.29), (7.30) for two-dimensional flow is *exact,* while quasi-balance systems like (7.25), (7.27), and (7.28) rely on a balance approximation like (7.23). But clearly, the evolution of quasi-balanced flows is strongly analogous to the evolution of two-dimensional flows.

Illustration of balance and invertibility principles: the Rossby adjustment problem

Consider a two-dimensional slab of incompressible fluid, which at time $t = 0$ has a rectangular cross section (Figure 7.2). The fluid is contained on an f plane ($f = $ constant). At time $t = 0$, the fluid is released and allowed to evolve in time. We will approximate the flow as inviscid. What happens?
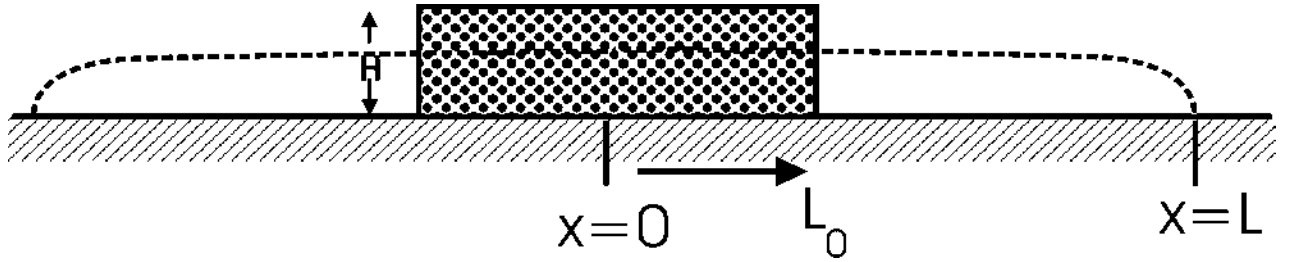
Figure 7.2

Clearly, the fluid will spread horizontally under the influence of gravity, acquiring velocities in the y direction through Coriolis accelerations. Gradually, the Coriolis accelerations acting on these velocities in the y direction will begin to balance the pressure gradient accelerations in the x direction.

We can find steady, two-dimensional solutions of the shallow water equations by assuming that potential vorticity is conserved and by solving the x-momentum equations for steady flow. This is a state toward which the system presumably evolves.

The potential vorticity, from (7.13), is

$$q = \frac{f + \frac{\partial v}{\partial x}}{\eta + H}, \tag{7.31}$$

where v is the velocity in the y direction. At $t = 0$, this is

$$q = \frac{f}{H}, \tag{7.32}$$

and since q is conserved, it must equal this value at all times. Equating (7.31) and (7.32) gives

$$\frac{\partial v}{\partial x} = f\frac{\eta}{H}. \tag{7.33}$$

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The momentum equations in the x direction is

$$\frac{du}{dt} = -g\frac{\partial \eta}{\partial x} + fv. \tag{7.34}$$

In steady equilibrium,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0, \tag{7.35}$$

since the flow is steady and two-dimensional ($\partial/\partial y = 0$) and, we presume, $u = 0$.

Thus the final state is geostrophically balanced:

$$g\frac{\partial \eta}{\partial x} = fv, \tag{7.36}$$

and eliminating v between (7.34) and (7.36) gives an ordinary differential equation

for η:

$$\frac{g}{f}\frac{d^2\eta}{dx^2} - \frac{f}{H}\eta = 0. \tag{7.37}$$

The solution of this equation that satisfies the condition $H + \eta = 0$ at $x = \pm L$

(where L is, by definition, the distance from the symmetry axis to which the fluid

spreads) is

$$\eta = -H\frac{\cosh\frac{x}{L_D}}{\cosh\frac{L}{L_D}}, \tag{7.38}$$

where L_D is the *deformation radius,* defined

$$L_D \equiv \frac{\sqrt{gH}}{f}. \tag{7.39}$$

This is an important scale in all quasi-balanced flows.

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Now we can find the scale L to which the fluid spreads by demanding that mass be conserved:

$$\int_0^L (H + \eta)dx = HL_0. \tag{7.40}$$

The right-hand side of (7.40) is the half-volume of the initial state, while the left side is the half-volume of the final state.

Performing the integration in (7.40) after substituting (7.38) gives a transcendental equation for L:

$$\frac{L}{L_D} - \tanh\left(\frac{L}{L_D}\right) = \frac{L_0}{L_D}. \tag{7.41}$$

Given L_0, this allows us to calculate L. Note that all length scales are now effectively normalized by the deformation radius.

It is instructive to look at solutions to (7.41) in certain limiting cases:

1. $L_0/L_D \gg 1$: The initial horizontal scale is "large." Since $L > L_0$, and $\lim_{x\to\infty} \tanh x = 1$, (7.41) becomes

$$\frac{L}{L_D} - 1 \to \frac{L_0}{L_D},$$

or

$$L \to L_0 + L_D. \tag{7.42}$$

The fluid spreads approximately one deformation radius beyond its initial scale.

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2. $L_0/L_D \ll 1$: The initial horizontal scale is "small." In this case, if we assume that $L/L_0 \ll 1$, the tanh term in (7.41), expanded to second order in L/L_D, is

$$\tanh \frac{L}{L_D} = \frac{L}{L_D} - \frac{1}{3}\left(\frac{L}{L_D}\right)^3 + O\left(\frac{L}{L_D}\right)^5$$

and, therefore,

$$L \to (3L_0)^{1/3} L_D^{2/3}. \tag{7.43}$$

The fluid spreads to a scale that represents a geometric average of the deformation radius and the initial scale, weighted toward the former.

It is worth noting that when $L_0 = L_D$, the solution to (7.41) is $L = 1.96 L_D$, which is much closer to the large-scale asymptotic limit given by (7.42).

This example is one of *geostrophic adjustment* in which an initially highly unbalanced flow adjusts to a balanced state. Note that in the large-scale limit, the mass distribution hardly changes at all, but the v velocity component undergoes a large adjustment near the edges of the block. On the other hand, in the limit of a small-scale initial length, the mass field undergoes a large adjustment while the velocities remain small.

12.803 Quasi-Balanced Circulations in Oceans and Atmospheres
Fall 2009