14. The secondary circulation

The quasi-geostrophic system in which pseudo potential vorticity is advected by the geostrophic flow and inverted to obtain the geostrophic flow, geopotential, and temperature perturbation, does not require for its solution explicit knowledge of the secondary circulation, including ω and the ageostrophic flow; but the existence of changing temperature and vorticity fields *implies* the existence of a secondary circulation.

To see this, suppose we have solved the advection and inversion equations for q_p at two different times separated by a small time increment, Δt. Then the quasi-geostrophic vorticity equation (9.3) directly implies that

$$f_0 \frac{\partial \omega}{\partial p} = \mathbf{V}_g \cdot \boldsymbol{\nabla}(\zeta_g + \beta y) + \frac{\zeta_g|_t^{t+\Delta t}}{\Delta t} - \hat{k} \cdot \nabla \times \mathbf{F}. \tag{14.1}$$

This means that *wherever the evolution of the q_p field implies a change of absolute vorticity following the flow, stretching (and/or friction) is implied.*

As an example, suppose that a weak positive q_p anomaly is embedded in a shear flow, as shown in Figure 14.1, which is deliberately placed in a coordinate system moving with the q_p anomaly.

Following a sample of air as it moves with the geostrophic wind (and assuming that the q_p anomaly is not so strong that it appreciably deflects the background geostrophic trajectories), the absolute vorticity first increases and then decreases,
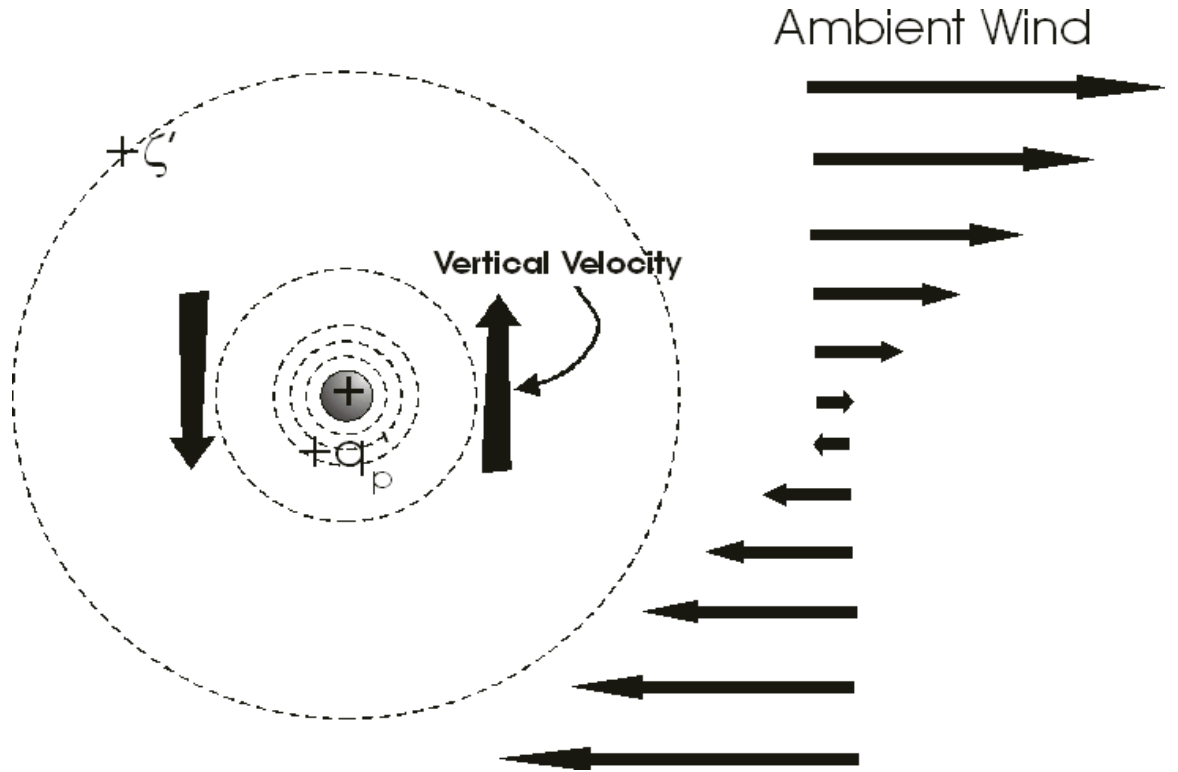
Figure 14.1

implying stretching and then shrinking of vertical columns. From this, the distribution of ω can be inferred.

Alternatively, one can solve a diagnostic equation for ω by eliminating the local time tendency terms from the quasi-geostrophic vorticity and thermodynamic equations. To do this, operate on (9.4) by $\mathcal{S}$ times the ∇^2 operator, and on (9.3) by $\partial/\partial p$ and subtract one from the other. The result may be written

$$\left(f_0^2 \frac{\partial^2}{\partial p^2} + \mathcal{S}\nabla^2 \right) \omega = f_0 \frac{\partial}{\partial p} \left[\mathbf{V}_g \cdot \boldsymbol{\nabla}\eta_g \right] - \nabla^2 \left[\mathbf{V}_g \cdot \boldsymbol{\nabla} \frac{\partial\varphi}{\partial p} \right] - \frac{\overline{\alpha}}{\overline{\theta}} \nabla^2 \dot{Q} - f_0 \frac{\partial}{\partial p} (\hat{k} \cdot \boldsymbol{\nabla} \times \mathbf{F}).$$

$$(14.2)$$

This is the "classical" form of the *quasi-geostrophic ω equation.* Vertical velocity is

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associated with vertically varying geostrophic advection of vorticity and horizontal Laplacians of the geostrophic temperature advection, as well as with friction and heating. The non-Galilean invariant parts of the right side of (14.1) cancel, so that this is perhaps not the most useful form of the equation. A somewhat improved form may be obtained by noting from (9.11) that

$$\eta_g = q_p - f_0 \frac{\partial}{\partial p}\left(\frac{1}{\mathcal{S}}\frac{\partial \varphi}{\partial p}\right). \tag{14.3}$$

Substitution of this into (14.1) results, after some rearrangement of terms, in

$$\left(f_0^2 \frac{\partial^2}{\partial p^2} + \mathcal{S}\nabla^2\right)\left(\omega - \frac{1}{\mathcal{S}}\mathbf{V}_g \cdot \nabla\theta'\right)$$
$$= f_0 \frac{\partial}{\partial p}\left(\mathbf{V}_g \cdot \nabla q_p\right) - \frac{\overline{\alpha}}{\overline{\theta}}\nabla^2 \dot{Q} - f_0 \frac{\partial}{\partial p}\left[\hat{k} \cdot \nabla \times \mathbf{F}\right]. \tag{14.4}$$

The adiabatic quantity on the right side of (14.4) is just the variation with altitude of the geostrophic advection of pseudo potential vorticity, while the elliptic operator on the left now acts on the sum of ω and the geostrophic temperature advection. Unfortunately, the boundary conditions in inverting the elliptic operator are now inhomogeneous, since $\mathbf{V}_g \cdot \nabla\theta'$ may be nonzero on the boundaries. This can be "fixed" by the method of "bound charge," as before, so that (14.3) may be rewritten

$$\left(f_0^2 \frac{\partial^2}{\partial p^2} + \mathcal{S}\nabla^2\right)\left(\omega - \frac{1}{\mathcal{S}}\mathbf{V}_g \cdot \nabla\theta'\right)$$
$$= f_0 \frac{\partial}{\partial p}\left[\mathbf{V}_g \cdot \nabla\left(q_p + \delta(p_0 - p)\frac{f_0}{\mathcal{S}}\theta' - \delta(p - p_t)\frac{f_0}{\mathcal{S}}\theta'\right)\right] \tag{14.5}$$
$$- \alpha^2 \nabla^2 Q - f_0 \frac{\partial}{\partial p}\left[\hat{k} \cdot \nabla \times \mathbf{F}\right],$$

where it is understood that $\omega - \frac{1}{\mathcal{S}}\mathbf{V}_g \cdot \nabla\theta'$ is forced to be zero on horizontal boundaries when inverting (14.5). Once again we see that the system is forced by changes in q_p, and θ' at the boundaries.

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