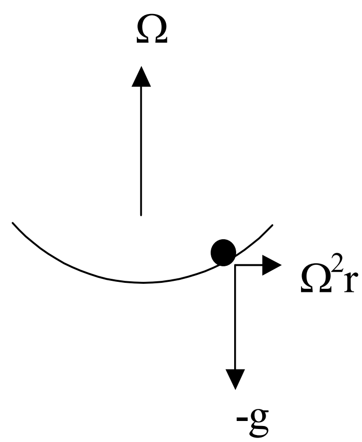
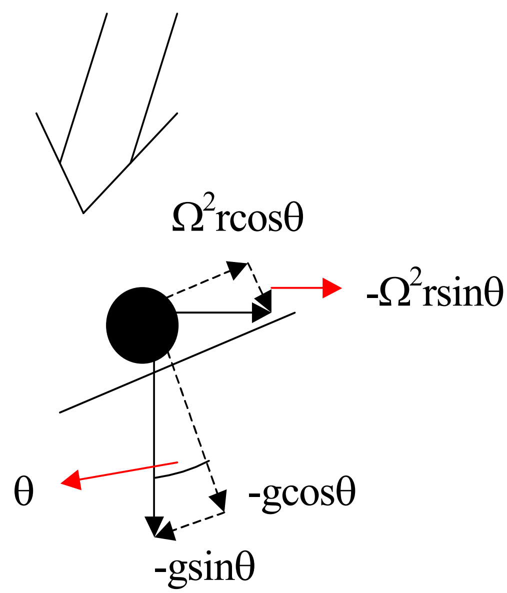
Rotation 101 (some basics)

Most students starting off in oceanography, even if they have had some fluid mechanics, are not familiar with viewing fluids in a rotating reference frame. This is essential in order to understand large scale, low frequency dynamics of the ocean. So we must return to some basic about rotating dynamics.

Consider a dish that is curved as shown and which is rotating at a rate Ω about a vertical axis. Only gravity, also vertical, is acting. A particle of mass m is resting on the surface. (1) What shape does the surface have to be in order that there are *no net forces* on the particle: can it be at rest in the rotating frame? (2) What is the apparent gravity acting normal to the surface?



In a rotating frame, there is a centrifugal force acting outward from the axis of rotation and gravity acting downward. We decompose these two forces into ones acting parallel (a) and normal (b) to the surface

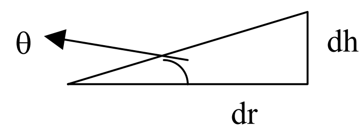


$$-mg\sin\theta + m\Omega^2 r\cos\theta = 0$$

$$-mg\cos\theta - m\Omega^2 r\sin\theta = -m\hat{g}$$

where $\hat{g}$ is the apparent gravity. For the case in which $\theta < \pi/2$ the first relation (balance parallel to surface) becomes

$$-g\tan\theta+\Omega^2 r=0,\ or\ \ \frac{dh}{dr}=\frac{\Omega^2}{g}r,\ \ recognizing\ that\ \tan\theta\equiv\frac{dh}{dr}$$



So the answer to part (1) of the problem is that the shape is a parabola, the solution being

$$h=h_0+\frac{\Omega^2}{2g}r^2$$

where h_0 is a constant. The answer to part (2) requires the use of some trigonometric identities as well as using the above answer for $\tan\theta$. It can be *easily* shown (this is often written and it is usually not *that* easy) that

$$\hat{g}=g[1+(\Omega^2 r/g)^2]/[1+(\Omega^2 r/g)^2]^{1/2}=g[1+(\Omega^2 r/g)^2]^{1/2}$$

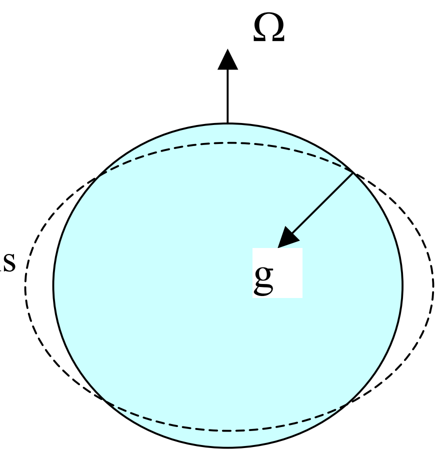
Centrifugal force can be subsumed into a revised definition of "gravity" which is acting on a curved, parabolic surface. For a rotating surface in 2-d this varies with radius, increasing outward. So if you want to weigh less, move to the center of the surface!

Many rotating surfaces on which things are placed and on which laboratory experiments are done already have this parabolic shape. If one saw such a surface in the lab and wanted to know what was the correct rotation rate in which a particle would be at rest, one could (hard way) figure out the quadratic term of the surface or (easy way), drop a marble on the non-rotating surface and figure out it's period of oscillation. This will now be shown. Newton's second law for forces acting in the radial direction parallel to the surface is

$$\frac{d^2 r}{dt^2} = -g\frac{dh}{dr} = -g(\Omega^2 r / g) = -\Omega^2 r$$

This reflects the force balance that makes a particle roll down the slope and accelerate under the action of gravity. But we know what the expression for the slope (*dh/dr*) must be from before and inserting that above we see that r = a $\sin(\Omega t)$ + b $\cos(\Omega t)$ is a solution, where a and b are arbitrary constants that can be used to satisfy the initial conditions. So the frequency of oscillation, Ω is the same as the period of rotation used to construct the surface!

Now consider a rotating, self-gravitating sphere shown at the right. Gravity acts toward the center of the sphere but since it is rotating, there is a centrifugal force acting outward from the rotation axis on the fluid. What shape does it take if there is no motion in the rotating frame? What does rotation do to apparent gravity? We will discuss this is class, but not actually solve this problem…



Coriolis Acceleration/Force

We have examined what shape a rotating surface must take in order for a particle to reside on it without motion: it is parabolic. Now consider what happens when, in this rotating frame of reference, there is motion. The example we will use is taken from a program (coriolis.m) written by Jim Price to illustrate what happens when a particle, say a hockey puck, is given an initial velocity and afterwards moves without any external forces (other than gravity & centrifugal forces, that is). So the surface must be relatively slick (ice) so that friction is weak and can be ignored. We will look at the motion of the particle in two reference frames: one an inertial frame (non-rotating) and the other the reference frame rotating with angular velocity Ω about an axis parallel with gravity. In the inertial frame, the particle will move in the direction given by the initial velocity. But as we have already shown, it will be stopped by the parabolic surface and will oscillate with frequency Ω. In the rotating frame, it will be seen to move in a circle, coming back to its initial position after half a rotation cycle: its frequency of oscillation is 2Ω. This circular motion is caused by the reference frame rotation under the particle. In this rotating frame, we can account for this motion by adding an acceleration to the particle equal to **2Ω** x **v**, where the acceleration is at right angles to the both the relative velocity vector **v**, and the rotation vector **Ω**. Here we use bold face as indicating a vector, which is also denoted by an arrow over the letter, as below. This acceleration, if multiplied by the mass of the particle, is equivalent to a force directed to the right of the particle if **Ω** is as shown (counter-clockwise), and to the left of the particle motion if **Ω** is in the opposite direction. This fictitious force is one of the most important ones in oceanography! It arises solely because we (and the oceans) are confined to a rotating earth.

Try running the m-file "coriolis.m" written by Jim Price. It illustrates the views of a particle on a parabolic surface seen in an inertial (non-accelerating) reference frame and in the reference frame of the rotating system. In the former, a particle will undergo an oscillation about the center of the paraboloid, with a frequency Ω, as we showed above. When it is halfway through its oscillation, the rotating reference frame is now halfway rotated about the axis of rotation, so the particle is at the same position in the rotating frame as it had initially! In this reference frame, the frequency of oscillation is 2Ω, or twice that in the inertial frame. Notice also how the particle appears to move in a circle in a clockwise fashion due to the 'fictitious' coriolis force. In the rotating frame, if (x,y) is the position of the particle and (u,v) its velocity, Newton's second law can be written:

$$m[\frac{d\vec{v}}{dt}+2\vec{\Omega}\times\vec{v}]=\sum_{i=1}\vec{F}_i$$

where the external forces are represented on the right hand side. We will now consider one special type of "external" force: friction. Friction acts to impede the motion of the hockey puck. We have all observed that friction will eventually act to bring moving things to rest. The simplest form that friction can take reflects the fact that it acts in proportion to the mass of the particle and is opposite to the velocity of the particle. It acts only on the point(s) of contact between the particle and the surface, but can be most simply represented by a linear drag force acting on the whole particle. So we will modify the above equation to the following:

$$m[\frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v}] = -mr\vec{v} + \sum_{i=2} \vec{F}_i$$

where "r" is a coefficient of friction proportional to "stickiness" of the contact between the particle and the surface.

Now we will add another "force". Suppose that there are bumps in the surface: it is not flat. [Recall that we have already accounted for a slow curvature due to the centrifugal force.] If the topography of the surface is represented by h, then this force is just $-g\nabla h$. Where the symbol ∇ is the gradient operator which is a vector shorthand the slope of the surface in the x- and y- directions: $(\partial_x,\partial_y)h$. This was already introduced earlier when we figured out what the equilibrium shape of a rotating surface must take. So our equation of motion then becomes:

$$m[\frac{d\vec{v}}{dt} + 2\vec{\Omega} \times \vec{v}] = -mr\vec{v} - mg\nabla h + \sum_{i=3} \vec{F}_i$$

Any remaining forces on the hockey puck ($i \geq 3$) are left for the present and simply written as $\boldsymbol{F}$. If we write the above vector equation in its component form, it becomes:

$$\frac{du}{dt} - fv = -g\frac{\partial h}{\partial x} - ru + \frac{F^x}{m},$$

$$\frac{dv}{dt} + fu = -g\frac{\partial h}{\partial y} - rv + \frac{F^y}{m}$$

where $f \equiv 2\Omega$, and we have divided through by the mass, m.

We will illustrate some of the dynamics of the above balances with a homework problem: **pucks_on_ice**. We use ice to illustrate, in part, what permanent “bumps” on the surface will do particles. In fact, in a fluid, there are no permanent “bumps” on a “free surface” unless they are maintained by forces. We will later see that these can be maintained by something called the “geostrophic” force balance, in which steady flows are required to balance permanent bumps or dimples on the free fluid surface.