

Homework assignment 1

Problem 1

The problem of the onset of turbulence is an area of active research. In class we briefly discussed the original model proposed by Landau and the more recent ideas of Ruelle and Takens. The textbooks by Landau and Lifshitz and Tritton have chapters on the two theories. The article "Hydrodynamic Instabilities and the Transition to Turbulence," published by H.L. Swinney and J.P. Gollub in *Physics Today* (August 1978) is another good review of the issue. Read this material (and whatever else you might find useful) and summarize the differences between Landau's explanation versus the modern view of the onset of turbulence,. Discuss what is the observational evidence which supports one or the other explanation. Your answer should not exceed a couple of pages.

Problem 2

Consider the *cubic map*,

$$x_{n+1} = r x_n(3 - 4x_n^2). \quad (1)$$

The parameter r plays the role of a Reynolds number in that it controls whether the solutions are regular or erratic. The map is defined in the interval $x \in [-1, 1]$ and $r \in [0, 1]$.

Part 1

Write a MATLAB program to draw the bifurcation diagram of the cubic map for $r \in [0, 1]$. How do the solutions change as r becomes larger? Is there a transition to a "chaotic" behavior? Does the bifurcation diagram look similar to the one of the logistic map?

Part 2

Consider the cubic map with $r = 1$. Select a set of different initial conditions x_0 and plot the PDF of this set, i.e. the $P(x_0)$. Now integrate the map for this set of initial conditions, and plot the PDF after n iterations, i.e. the $P(x_n)$. Repeat this exercise for different values of r in the range $[0, 1]$. Do the PDFs asymptote some simple distribution? How do the PDFs change as a function of r ?

Part 3

Following the steps described in class, find the analytical expression for the asymptotic PDF P of the cubic map with $r = 1$. The first step is to find a transformation of variables to reduce the cubic map to a piecewise linear map (follow the steps I showed in class to reduce the logistic map to the tent map). Then write an equation that relates $P_{n+1}(x)$ and $P_n(x)$ for the piecewise linear map; this is the Perron-Frobenius equation. Now find the equilibrium solution of the Perron-Frobenius equation. The last step is to use this solution to express the asymptotic PDF for the cubic map. All these steps are described in detail in the book by Frisch for a map which is a close relative of the logistic map. Does your analytical solution agree with the numerical result of part 2?