14.12 Game Theory

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Homework 1

Due on 9/25/2012

1. Consider a homeowner with Von-Neumann and Morgenstern utility function u, where $u(x) = 1 - e^{-x}$ for wealth level x, measured in million US dollars. His entire wealth is his house. The value of a house is 1 (million US dollars), but the house can be destroyed by a flood, reducing its value to 0, with probability $\pi \in (0, 1)$.

   (a) What is the largest premium P is the homeowner is willing to pay for a full insurance? (He pays the premium P and gets back 1 in case of a flood, making his wealth $1 - P$ regardless of the flood.)

   The homeowner's utility for getting $1 - P$ always is

   $$u(1 - P) = 1 - e^{-(1-P)}$$

   his utility in the outside option is

   $$\pi u(0) + (1 - \pi)u(1) = (1 - \pi)(1 - e^{-1})$$

   The largest premium P he is willing to pay is the P that makes him indifferent between buying and not buying insurance.

   $$\begin{aligned} 1 - e^{-(1-P)} &= (1 - \pi)(1 - e^{-1}) \\ 1 - P &= -ln(1 - (1 - \pi)(1 - e^{-1})) \\ P &= 1 + ln(1 - (1 - \pi)(1 - e^{-1})) \end{aligned}$$

   (b) Suppose there is a local insurance company who has insured n houses, all in his neighborhood, for premium P. Suppose also that with probability π there can be flood in the neighborhood destroying all houses (i.e., either all houses are destroyed or none of them is destroyed). Suppose finally that P is small enough that the homeowner has insured is house. Having insured his house, what is the largest Q that he is willing to pay to get the $1/n$ share of the company? (The value of the company is the total premium it collects minus the payments to the insured homeowners in case of a flood.)

   The company's value is nP with probability $(1-\pi)$ and $nP - n$ with probability π. His utility from buying insurance and not buying stock is

   $$u(1 - P) = 1 - e^{-(1-P)}$$

And his utility from buying stock and insurance is

$$(1-\pi)u(1-P-Q+P)+\pi u(1-P-Q+P-1) = \\ (1-\pi)(1-\exp(-1+Q))+\pi(1-\exp(Q))$$

We find the Q that makes him indifferent between buying and not buying:

$$\begin{aligned} 1-e^{-(1-P)} &= (1-\pi)(1-\exp(-1+Q))+\pi(1-\exp(Q)) \\ \exp(-Q)\exp(P) &= (1-\pi)\exp(0)+\pi\exp(1) \\ Q &= P-\ln(1-\pi+\pi e) \end{aligned}$$

We saw before that it must be true that $\exp(P) \leq \pi\exp(1)+(1-\pi)\exp(0)$ (this is a rearrangement of the indifference condition of part (a)), so we need $\exp(-Q) \geq 1$, so $Q \leq 0$. Thus, the homeowner is never willing to buy the stock.

(c) Answer part (b) assuming now that the insurance company is global. It insured n houses in different parts of the world (all outside of his neighborhood), so that the destruction of houses by flood are all independent (i.e., the probability of flood in one house is π independent of how many other houses has been flooded).

The chance that i houses flood, out of n is

$$\pi^i(1-\pi)^{n-i}C_{n,i}$$

and the company's wealth is $nP-i$. Thus, the buyer's expected utility from buying insurance and stock is

$$1-\sum^{i}\pi^i(1-\pi)^{n-i}C_{n,i}\exp(-(1-P-Q+P-i/n))$$

Setting this equal to his outside option

$$\begin{aligned} 1-e^{-(1-P)} &= 1-\sum^{i}\pi^i(1-\pi)^{n-i}C_{n,i}\exp(-(1-P-Q+P-i/n)) \\ -e^{-Q} &= -\sum^{i}\pi^i(1-\pi)^{n-i}C_{n,i}\exp(-(P-i/n)) \\ Q &= P-\ln(\sum^{i}\pi^i(1-\pi)^{n-i}C_{n,i}\exp(i/n)) \end{aligned}$$

(d) Assume that n is large enough so that $\sum_{k=0}^{n} C_{n,k}e^{k/n}\pi^k(1-\pi)^{n-k} \cong e^{\pi+\pi(1-\pi)/(2n)}$, discuss your answers to above questions (briefly). [Here, $C_{n,k}$ denotes the number of k combinations out of n, and the sum is one minus the expected payoff from the loss due to the payments to the flooded houses.]

Substituting in to our last result, we get

$$Q = P-\pi-\pi(1-\pi)/(2n)$$

The expected value of a share of the company is $P-\pi$. You can see that the player's willingness to pay for a share of the company is the expected value minus $\pi(1-\pi)/(2n)$.

For this utility function and a lottery that is normally distributed (which is this case for large n), the agent's willingness to pay is always $\mu - \sigma^2/2$

In part b, we found that the agent would never buy into the company that only insures people like him. This is the equivalent of paying for a company to sell himself insurance. However, when the company insures many different people, the company's splits the risk of all of the people. Thus, a $1/N$ share of the company holds less risk as N increases. The agent that buys a share holds a part of everyone's risk, which is preferrable to holding only his own risk.

2. Consider the game in which the following are commonly known. First, Ann chooses between actions a and b. Then, with probability 1/3, Bob observes which action Ann has chosen and with probability 2/3 he does not observe the action she has chosen. In all cases (regardless of whether he has observed Ann chose a, or he has observed Ann chose b, or he has not observed any action), Bob chooses between actions α and β. The payoff of each player is 1 after (a, α) and (b, β) and 0 otherwise.

   (a) Write the above game in extensive form.

   (b) Write the above game in normal form.
   Strategy of Ann is simple: $\{a, b\}$. Strategy of Bob is $s^B = (s^1, s^2, s^3)$, where $s^1 \in \{\alpha, \beta\}$, $s^2 \in \{\alpha, \beta\}$, $s^3 \in \{\alpha, \beta\}$. s^1 denotes the choice of the information set (numbered as 1 in the picture) and s^2 is for node 2, and s^3 is for node 3. Utility of outcomes is in the following table.

	a	b
$\alpha\alpha\alpha$	1,1	0,0
$\alpha\alpha\beta$	1,1	$\frac{1}{3}, \frac{1}{3}$
$\alpha\beta\alpha$	$\frac{2}{3}, \frac{2}{3}$	0,0
$\alpha\beta\beta$	$\frac{2}{3}, \frac{2}{3}$	$\frac{1}{3}, \frac{1}{3}$
$\beta\alpha\alpha$	$\frac{1}{3}, \frac{1}{3}$	$\frac{2}{3}, \frac{2}{3}$
$\beta\alpha\beta$	$\frac{1}{3}, \frac{1}{3}$	1,1
$\beta\beta\alpha$	0,0	$\frac{2}{3}, \frac{2}{3}$
$\beta\beta\beta$	0,0	1,1

3. Consider the following variation of the above game. First, Ann chooses between actions a and b. Then, Bob decides whether to observe the chosen action of Ann or not, by choosing between the actions Open and Shut, respectively. In all cases, Bob then chooses between actions α and β. The payoff of Ann is 1 after (a, α) and (b, β) and 0 otherwise, regardless of whether Bob chooses Open or Shut. The payoff of Bob is equal to the payoff of Ann if he has chosen Shut, and his payoff is equal to the payoff of Ann minus 1/2 if he has chosen Open.

   (a) Write the above game in extensive form.

   (b) Write the above game in normal form.
   Strategy of Ann is simple: $\{a, b\}$. Strategy of Bob is $s^B = (s^1, s^2, s^3, s^4)$, where $s^1 \in$

$\{O,S\}$, $s^2 \in \{\alpha,\beta\}$, $s^3 \in \{\alpha,\beta\}$, $s^4 \in \{\alpha,\beta\}$. s^2 denotes the choice of the node 1 (the left one) and s^3 is for node 2, $s^4$is for the information set on the right side (numbered as 3 in the picture). Utility of outcomes is in the following table.

	a	b
$O\alpha\alpha\alpha$	1, 0.5	0,-0.5
$O\alpha\alpha\beta$	1, 0.5	0,-0.5
$O\alpha\beta\alpha$	1, 0.5	1, 0.5
$O\alpha\beta\beta$	1, 0.5	1, 0.5
$O\beta\alpha\alpha$	0,-0.5	0,-0.5
$O\beta\alpha\beta$	0,-0.5	0,-0.5
$O\beta\beta\alpha$	0,-0.5	1, 0.5
$O\beta\beta\beta$	0,-0.5	1, 0.5
$S\alpha\alpha\alpha$	1,1	0,0
$S\alpha\alpha\beta$	0,0	1,1
$S\alpha\beta\alpha$	1,1	0,0
$S\alpha\beta\beta$	0,0	1,1
$S\beta\alpha\alpha$	1,1	0,0
$S\beta\alpha\beta$	0,0	1,1
$S\beta\beta\alpha$	1,1	0,0
$S\beta\beta\beta$	0,0	1,1

4. Federal government is planning to build an interstate highway between two states, named A and B. The highway costs $C > 0$ to the government, and the value of the highway to the states A and B are $v_A \geq 0$ and $v_B \geq 0$, respectively. Simultaneously, each state $i \in \{A, B\}$ is to bid $b_i \in [0, \infty)$. If $b_A + b_B \geq C$ the highway is constructed. For any distinct $i, j \in \{A, B\}$, state i pays $C - b_j$ to the federal government if $b_j < C \leq b_A + b_B$. (There is no payment otherwise.) The payoff of a state is the value of the highway to the state minus its own payment to the government if the highway is built, and 0 otherwise. (You can focus on the case $v_A + v_B < C$.)

   (a) Write this in the normal form.
   Strategy of player i is choice of $b_i \in [0, \infty)$. Utility from strategy profile x of player i is

$$\begin{aligned} u_i(b_A, b_B) &= v_i - C + b_j \quad if\ b_A + b_B \geq C \\ &= 0 \qquad\quad if\ b_A + b_B < C \end{aligned}$$

   (b) Check if there is a dominant strategy equilibrium, and compute it if there is one.
   There is a unique dominant strategy equilibrium, $(v_A,\ v_B)$. In other words, bidding own value is the dominant strategy equilibrium.
   From player A's point of view, there are three cases. If $b_B \geq C$, then $u_A = v_A$, regardless of what b_A is. If $C - v_A \leq b_B < C$, then A wants to build the highway as $u_A = v_A - C + b_B \geq 0$. Thus, $b_A \geq C - b_B$ is the best response. Lastly, if $b_B < C - v_A$, then A does not want to build the highway as $u_A = v_A - C + b_B < 0$. The best response for this case is $b_A < C - b_B$.

For $b_B = C - v_A + \epsilon$ $(0 < \epsilon < v_A)$, we need $b_A \geq C - b_B = v_A - \epsilon$. As this inequaility has to hold for all $0 < \epsilon < v_A$, we need $b_A \geq v_A$. Similarly, for $b_B = C - v_A - \epsilon$ $(0 < \epsilon < C - v_A)$, we need $b_A < C - b_B = v_A + \epsilon$. As this inequaility has to hold for all $0 < \epsilon < C - v_A$, we need $b_A \leq v_A$. Therefore, the dominant strategy is $b_A = v_A$. For example, if A chooses $b_A = v_A + \delta$ $(\delta > 0)$, when $b_B = C - v_A - \frac{\delta}{2}$, $u_A = -\frac{\delta}{2} < 0$ while $b_A = v_A$ gives $u_A = 0$.

Figure 1: Question 2

HW1-12, Problem 2

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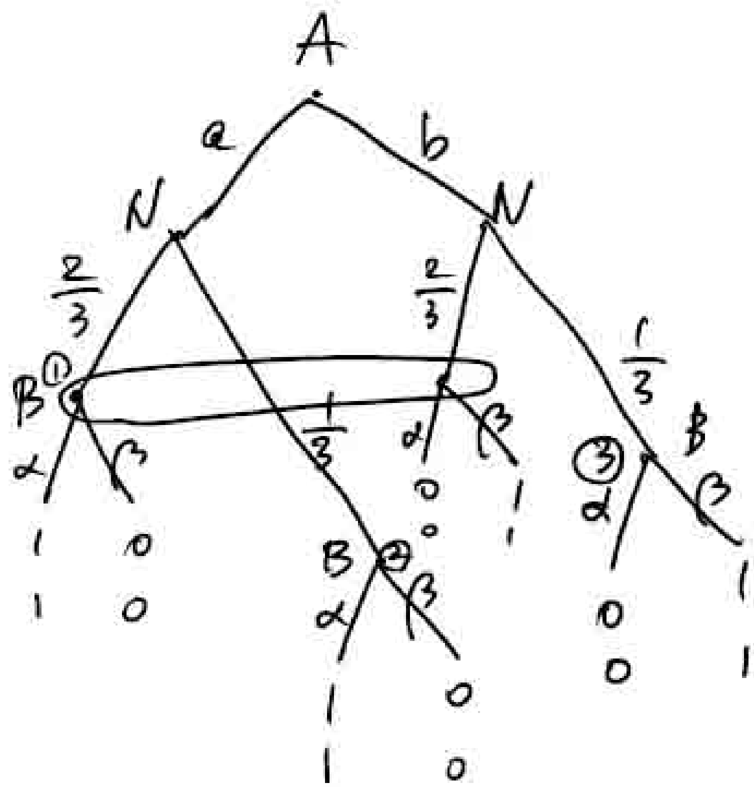

Figure 2: Question 3