Homework #5 Solutions

1. 1. As we are looking for symmetric BNE, suppose that in the equilibrium both players play bidding strategy b that each type v_j bids $b(v_j)$ for some increasing differentiable function. From player i's point of view, if he bids b_i, his expected payoff is (given that the other player follows $b_j = b(v_j)$)

$$U\left(b_i | v_i\right) = b_i Pr\left(b_i < b_j\right) + \int_0^{b^{-1}(b_i)} \left\{v_i - b\left(v_j\right)\right\} dv_j$$

$$= b_i \left\{1 - b^{-1}\left(b_i\right)\right\} + \int_0^{b^{-1}(b_i)} \left\{v_i - b\left(v_j\right)\right\} dv_j$$

Taking FOC with respect to b_i, we get

$$\left\{1 - b^{-1}\left(b_i\right)\right\} + b_i \left(\frac{-1}{b'\left(b^{-1}\left(b_i\right)\right)}\right) + \left\{v_i - b\left[b^{-1}\left(b_i\right)\right]\right\} \frac{1}{b'\left(b^{-1}\left(b_i\right)\right)} = 0$$

This must be satisfied at $b_i = b(v_i)$. Thus, we have

$$b'\left(v_i\right)\left(1 - v_i\right) - b\left(v_i\right) + v_i - b(v_i) = 0$$

In particular, we are looking for linear BNE: let $b\left(v_i\right) = \alpha v_i + \beta$. Using this, we have

$$\left(1 - 3\alpha\right) v_i + \left(\alpha - 2\beta\right) = 0$$

So $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{6}$. Thus, the symmetric linear BNE of this game is $b(v_i) = \frac{1}{3} v_i + \frac{1}{6}$.

Comment: the professor made an announcement at the lecture that this game is NOT a second price auction as the winner pays the other sibling's bid to him/her (so if you lose, you get what you did bid). In case you did not interpret the question in this way and reached the conclusion that bidding one's own value is BNE, I gave half points (5 points) assigned for the problem.

2. 2. Writing as a Bayesian game:

$N = \{1, 2, \cdots, n\}$, $T_i = v_i \sim F$, $A_i = b_i \in [0, \infty)$, payoff is $v_i - b_i$ if win the auction and zero otherwise. Types (individual values) are iid with pdf f and cdf F.

Computing the symmetric BNE in increasing differentiable strategies:

Suppose that in the symmetric BNE all players play bidding strategy b that $b_j = b(v_j)$. From player i's point of view, if he bids b_i, his expected payoff is (given that all other players follow $b_j = b(v_j)$)

$$\left(v_i - b_i\right) Pr\left[b_i \geq b_j \ \forall j \neq i\right] = \left(v_i - b_i\right) Pr\left[b^{-1}\left(b_i\right) \geq v_j \ \forall j \neq i\right]$$

$$= \left(v_i - b_i\right) \left[F\left(b^{-1}\left(b_i\right)\right)\right]^{n-1}$$

Taking FOC with respect to b_i, we get

$$- \left[F \left(b^{-1} \left(b_i \right) \right) \right]^{n-1} + \left(v_i - b_i \right) \left(n - 1 \right) \left[F \left(b^{-1} \left(b_i \right) \right) \right]^{n-2} f \left(b^{-1} \left(b_i \right) \right) \frac{1}{b' \left(b^{-1}(b_i) \right)} = 0$$

Substituting $b_i = b(v_i)$ (as at the optimum, player i should also play bidding strategy b), we get

$$- \left[F \left(v_i \right) \right]^{n-1} + \left(v_i - b(v_i) \right) \left(n - 1 \right) \left[F \left(v_i \right) \right]^{n-2} f \left(v_i \right) \frac{1}{b' \left(v_i \right)} = 0$$

We can rewrite this as

$$\frac{d}{dv_i} \left[b(v_i) F(v_i)^{n-1} \right] = (n-1) v_i F(v_i)^{n-2} f(v_i)$$

Integrating both sides and since $b(v_i) F(v_i)^{n-1} = 0$ when $v_i = 0$ (if value is zero you bid zero), we get

$$b(v_i) = \frac{1}{F(v_i)^{n-1}} \int\limits_0^{v_i} (n-1) t F(t)^{n-2} f(t) dt$$

Using integration by parts, we get

$$b(v_i) = v_i - \frac{\int_0^{v_i} F(v)^{n-1} dv}{F(v_i)^{n-1}}$$

Comments: Do not forget to write this as a Bayesian game, as those are easy points at the exam. I did not take points off for this pset grading, but if it were the exam if you missed this part you could have lost 20-40% of the points for this problem.

3. (a) If $\theta < 0$, the only equilibrium is to play Stag.

   If $\theta = 0$ there are equilibria where any number of players between 0 and $k-2$ play Stag, and where everyone plays stag.

   For $0 < \theta \leq v$, the equlibria are everyone plays Stag and everyone plays Rabbit. In addition, if $\theta = v$ there are equilibria where any number of players between k and n play Stag, and the rest play Rabbit.

   If $\theta > v$, then everyone plays Rabbit is the only equilibrium.

   (b) There will be one monotone BNE. The value of Stag is decreasing in x_i, and the value of Rabbit is increasing in x_i. For a given x^*, there will be one $\hat{x}$ that is indifferent between Stag and Rabbit. This $\hat{x}$ must equal x^* for this to be a NE. $\hat{x}$ is monotone increasing x^*, so there will be one BNE.

   To find that BNE, consider some x^*. If $x_i = x^*$, then he is indifferent to Stag and Rabbit. He has expected value of x^* when playing Rabbit. His expected value of playing Stag is $vP(x_j < x^*$ for k players). By the hint, the probability that k players have lower x_j than our x^* is $1 - k/n$, so our expected value simplifies to $v(1 - k/n)$. Thus, for $x_i = x^*$ to be indifferent, it must be $x^* = v(1 - k/n)$.

4. (a) In equilibrium, A picks $p = 7000$ after Peach and $p = 2000$ with probability $1/4$, $p = 7000$ with probability $3/4$ after Lemon. After seeing 2000, Bob believes *Lemon* with probability 1, and after 7000 Bob believes Lemon with probability $3/7$ and Peach with $4/7$. These beliefs are derived from bayesian updating. After 2000 Bob accepts, and after 7000 Bob accepts with probability $1/6$. Bob is indifferent between accept and reject after 7000, because he has expected utility 0. And Alice is indifferent between 2000 and 7000 after Lemon, because he has expected utility 2000 in either case.

To see consistency, consider $\theta^m = ((1-\epsilon)7000+\epsilon 2000, (0.25)2000+0.757000; 1/6 accept + 5/6 reject, (1 - \epsilon)accept + \epsilon reject)$, where the actions specified are for player A after peach, player A after Lemon, Player B after seeing 7000 and after seeing 2000. Then beliefs for player 2 are $b(Lemon|2000) = 0.75/(0.75 + \epsilon) \to 1$ and $b(Lemon|7000) = 0.75/(0.75 + (1 - \epsilon)) \to 3/7$.

(b) One equilibrium is A picks $p = 5000$ in either state. After any $p \leq 5000$, B believes that it is lemon with probability $1/2$ and accepts. After any $p > 5000$, B believes that the state is Lemon with probability 0.95 and rejects. For this to be consistent, we must have σ^m, β^m that converge to σ and b. Define σ^m to be:

Type Lemon plays
$p = 5000$ with probability $1 - 2\epsilon$
uniform(0,5000) with probability ϵ
5000+X, where $x \sim$ exponential distribution(1) with probability ϵ

Type Peach plays
$p = 5000$ with probability $1 - 20\epsilon$
uniform(0,5000) with probability ϵ
$5000 + Y$, where $X \sim$ exponential (1) with probability $19\,\epsilon$

$\beta^m(Lemon|p < 5000) = 1/2$, and $\beta^m(Lemon|p > 5000) = 0.95$, $\beta^m(Lemon|p = 5000) = (1 - 2\epsilon)/(2 - 22\epsilon)$. With $\epsilon \to 0$. In the limit, $p = 5000$ and $b = \lim_{\epsilon \to 0} \beta^m$.

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