14.12 Game Theory – Final (Answers)
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Instructions. This is an open book exam; you can use any written material. You have two hour and 50 minutes. Each question is 25 points. Good luck!

1. There are two siblings, who have inherited a factory from their parents. The value of the factory is v_i for sibling i, where (v_1, v_2) are independently and uniformly distributed over $[0, 1]$, and each of them knows his or her own value. Simultaneously, each i bids b_i, and the highest bidder wins the factory and pays his own bid to the other sibling. (If the bids are equal, the winner is determined by a coin toss.) Note that if i wins, i gets $v_i - b_i$ and j gets b_i.

   (a) (5 points) Write this as a Bayesian game.

   Answer: $N = \{1, 2\}$; $T_i = [0, 1]$; the CDF is $F(v_j|v_i) = v_j$; $A_i = [0, \infty)$;

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ b_j & \text{otherwise} \end{cases}$$

   (b) (10 points) Compute a symmetric, linear Bayesian Nash equilibrium.

   Answer: See Part c.

   (c) (10 points) Find all symmetric Bayesian Nash equilibrium in strictly increasing and differentiable strategies.

   Answer: We are looking for an equilibrium in which each type v_i bids $b(v_i)$ for some increasing differentiable function. The expected payoff from bidding b_i for a type v_i is

$$U(b_i|v_i) = (v_i - b_i)\, b^{-1}(b_i) + \int_{b^{-1}(b_i)}^{1} b(v_j)\, dv_j.$$

   Hence, the first-order condition for the best response is

$$-b^{-1}(b_i) + (v_i - b_i)/b'\left(b^{-1}(b_i)\right) - b_i/b'\left(b^{-1}(b_i)\right) = 0.$$

   This must be satisfied at $b_i = b(v_i)$:

$$-v_i + (v_i - 2b(v_i))/b'(v_i) = 0.$$

   That is,

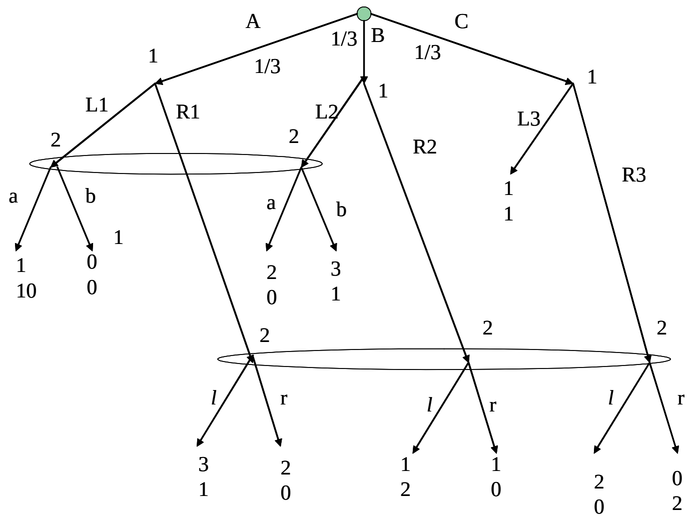
$$v_i = v_i b'(v_i) + 2b(v_i).$$

   The unique solution to this differential equation is

$$b(v_i) = v_i/3.$$

   (This is of course also the unique linear symmetric BNE.)

2. Find a perfect Bayesian Nash equilibrium of the following game:



3. Alice and Bob are bargaining using alternating offers, Alice making offers at $t = 0, 2, 4, \ldots$ and Bob making offers at $t = 1, 3, 5, \ldots$. The set of consumption pairs depends on the date. When Alice makes an offer, the set of possible consumption pairs is $X_A = \{(x, y) : ax + by \leq 1\}$ where $a > 1 > b > 0$ and x and y are the consumptions of Alice and Bob, respectively. When Bob makes an offer that set is $X_B = \{(x, y) : x + y \leq 1\}$. At each date t, proposer offers a pair (x, y) of cosumption from the available set at t, and the responder either accepts the offer, ending the game with payoff vector $(\delta^t x, \delta^t y)$, or rejects the offer, in which case we proceed to the next date. If they never agree, each gets 0.

(a) (20 points) Find a subgame perfect equilibrium of this game.

Answer: We are looking for a SPE in which Alice always offers (x_A, y_A) and Bob always offers (x_B, y_B), and these offers are accepted. Given that he will get y_B in the next period, Bob must accept an offer (x, y) iff $y \geq \delta y_B$. Therefore, Alice must offer the best pair $(x, y) \in X_A$ for Alice with $y \geq \delta y_B$. That is,

$$ax_A + by_A = 1 \quad (1)$$
$$y_A = \delta y_B. \quad (2)$$

Similarly, Alice accepts (x, y) iff $x \geq \delta x_A$, and Bob offers (x_B, y_B) with

$$x_B + y_B = 1 \quad (3)$$
$$x_B = \delta x_A. \quad (4)$$

(If you came up here, you will get 15.) We need to solve these equation system. By substituing (2) and (4) in (4), we obtain

$$\delta^2 x_A + y_A = \delta.$$

Together with (1), this yields

$$\begin{aligned} x_A &= \frac{1-b\delta}{a-b\delta^2} \\ x_B &= \delta x_A = \frac{\delta(1-b\delta)}{a-b\delta^2} \\ y_B &= 1-x_B = \frac{a-\delta}{a-b\delta^2} \\ y_A &= \delta y_B = \frac{\delta(a-\delta)}{a-b\delta^2}. \end{aligned}$$

(b) (5 points) What happens as $\delta \to 1$? Briefly interpret.

Answer: Clearly, x_A and x_B converge to $x^* = (1-b)/(a-b)$, and y_A and y_B converge to $y^* = (a-1)/(a-b)$. We could find this limit without solving the equations. At $\delta = 1$, the equations (2) and (4) become $x_A = x_B$ and $y_A = y_B$. That is, the soluition converge to the intersection (x^*, y^*) of the boundaries of X_A and X_B. In usual bargaining, the shares converge to equal splitting, and this is interpreted as fairness of the outcome. This example shows that the conclusion is fragile. Take $a = 1+\varepsilon$ and $b = 1-k\varepsilon$ where $\varepsilon \to 0$. Then the available sets are approximately as in the original model. But the limit solution is now $x^* = k/(k+1)$ and $y^* = 1/(k+1)$, i.e. depending on k it can be anywhere on the boundary.

4. There is a seller, who can produce a consumption good. There is also a buyer who would get

$$u(x,p) = \frac{1}{2}x(2\theta - x) - px$$

if he buys x units of good at price p, where $\theta \in [0,1]$. There are n periods: 0, 1, 2, ..., $n-1$. Buyer can trade at only one period. If he buys x units at period t for price p, then his utility is $\delta^t u(x,p)$ and the seller utility is $\delta^t px$ where $\delta \in (0,1)$ is known. In each period t, Seller sets a price p_t, and if he has not traded yet, the buyer decides whether to buy. If he decides to buy, then he also decides how much to buy, x_t, and the game ends. Otherwise, we proceed to next period. If they do not trade at any period, there will be no trade and each gets 0.

(a) (5 points) Assuming θ is commonly known, for $n = 2$, apply backward induction to find a subgame-perfect equilibrium.

(b) (5 points) Assuming θ is commonly known, for arbitrary n, apply backward induction to find a subgame-perfect equilibrium.

Answer: Since the game ends when the consumer buys, he buys the optimal quantity for him i.e. $\max_x u(x,p)$. Compute that the optimal quantity is $x(p) = \theta - p$, and the buyer's payoff is $(\theta-p)^2/2$. Note that if the buyer demands $x(p)$, then the optimal price is $p^* = \theta/2$. The following is the outcome of backward induction. In the last period, the buyer buys at every price p and buys $x(p)$ amount, and the seller offers price $p^* = \theta/2$. At $n-1$, the buyer rejects the prices p with $(\theta-p)^2 < \delta\theta^2/4$, i.e., $p > \bar{p} \equiv \theta - \sqrt{\delta}\theta/2$. Clearly, At any price $p \le \bar{p}$, the buyer accepts the price and buy $x(p)$. Since $\bar{p} > p^*$, the seller offers p^* at $n-1$ too. The behavior at any $t \leq n$ is as in the period $n-1$.

(c) (15 points) Take $n = 2$. Assume that seller does not know θ, i.e., θ is private information of the buyer, uniformly distributed on $[0, 1]$. Find a strategy of the buyer that is played in a perfect Bayesian Nash equilibrium. (Hint: There exist functions $x_0(p_0, \theta)$, $x_1(p_1, \theta)$, and a cutoff $\theta_0(p_0)$, such that given p_0 the types $\theta \geq \theta_0(p_0)$ buy $x_0(p_0, \theta)$ units at $t = 0$ and the other types wait for period 1, when each type θ buys $x_1(p_1, \theta)$ units if he has not traded yet.)

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