Chapter 8

Further Applications

This chapter is devoted to exercises that apply the ideas developed in previous chapters to various real-world problems.All of the exercises come from past exams and homework problems. The reader is recommended to solve them before studying the solutions.

Many of the games in this chapter are supermodular, a class of games for which there are powerful general theorems. These theorems could be used to find the rationalizable set relatively easily. I will not use these theorems. Instead, I will explicitly apply the iterated elimination procedure and use the results from the previous chapters. This will hopefully drill the logic of rationalizability better. Moreover, the knowledge of the procedure clarifies the role of knowledge and rationality assumptions, clarifying how sensitive the solution could be to such assumptions.

8.1 Partnership

Consider an employer and a worker. The employer provides the capital $K \geq 0$ (in terms of investment in technology, etc.) and the worker provides the labor $L \geq 0$ (in terms of the investment in the human capital) to produce

$$f(K, L) = K^{\alpha} L^{\beta}$$

for some $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$. They share the output equally. The parties determine their investment level (the employer's capital K and the worker's labor L) simultaneously. The per-unit costs of capital and the labor for the employer and the worker are $r > 0$ and $c > 0$ respectively. The worker cannot put more than some fixed positve $\bar{L}$. The payoffs for the employer and the worker are

$$u_E(K, L) = \frac{1}{2} f(K, L) - rK$$

and

$$u_W(K, L) = \frac{1}{2} f(K, L) - cL,$$

respectively. Everything described up to here is common knowledge.

Exercise 8.1 *Write this formally as a game in normal form.*

Solution: The set of players is $\{E, W\}$ where E stands for the employer and W stands for the worker. The sets of strategies are

$$\begin{aligned} S_E &= [0, \infty) \\ S_W &= \left[0, \bar{L}\right]. \end{aligned}$$

The payoff functions are u_E and u_W.

Exercise 8.2 *Compute the set of Nash equilibria.*

Solution: Since u_E and u_W are strictly concave in own strategies, all Nash equilibria are in pure strategies. Towards finding the Nash equilibria, compute the best-response functions for E and W as

$$K^*(L) = \left(\frac{1}{2}\frac{\alpha}{r}L^\beta\right)^{1/(1-\alpha)} \text{ and } L^*(K) = \min\left\{\left(\frac{1}{2}\frac{\beta}{c}K^\alpha\right)^{1/(1-\beta)}, \bar{L}\right\},$$

respectively. The Nash equilibrium is when the above best responses intersect, i.e., $K = K^*(L)$ and $L = L^*(K)$. Clearly, $(0, 0)$ is a Nash equilibrium. To find the other possible equilibria, first consider the case $L^*(K) < \bar{L}$, the case plotted in Figure 8.1. In that case, the non-zero solution to the above equation is

$$\begin{aligned} \hat{K} &= \left(\frac{1}{2}\left(\frac{\alpha}{r}\right)^{1-\beta}\left(\frac{\beta}{c}\right)^{\beta}\right)^{1/(1-\alpha-\beta)} \\ \hat{L} &= \left(\frac{1}{2}\left(\frac{\beta}{c}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^{\alpha}\right)^{1/(1-\alpha-\beta)}. \end{aligned}$$

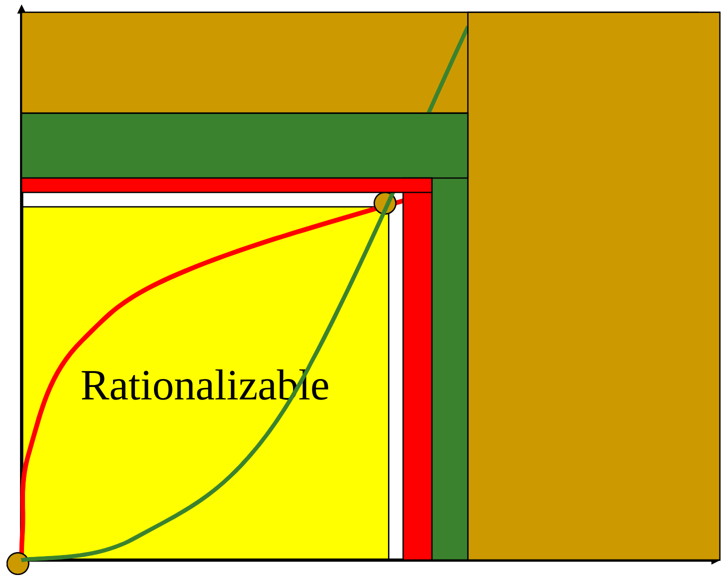


Figure 8.1: Rationalizability and Nash equilibrium in partnership game.

When $\hat{L} \leq \bar{L}$, the only non-zero equilibrium is $\left(\hat{K}, \hat{L}\right)$, yielding the equilibria in Figure 8.1. When $\hat{L} > \bar{L}$, the constraint for the labor binds, and the non-zero equilibrium is $\left(K^*\left(\bar{L}\right), \bar{L}\right)$.

Exercise 8.3 *Find all the rationalizable strategies.*

Solution: Iterated dominance is applied as follows.

Round 1 Since $L \in \left[0, \bar{L}\right]$, any $K > K^*\left(\bar{L}\right)$ is strictly dominated by $K^*\left(\bar{L}\right)$. (The proof is similar to the Cournot duopoly case in the previous chapter, and left as an easy exercise.) Therefore, any such strategy is eliminated for the employer. No other strategy is eliminated at this round because any $K \in \left[0, K^*\left(\bar{L}\right)\right]$ is a best response to some $L \in \left[0, \bar{L}\right]$ and any $L \in \left[0, \bar{L}\right]$ is a best response to some $K \geq 0$. The remaining strategy sets are $\left[0, K^*\left(\bar{L}\right)\right]$ and $\left[0, \bar{L}\right]$.

The subsequent rounds depend on whether $\hat{L} \geq \bar{L}$.

Round 2 ($\hat{L} \geq \bar{L}$) Since $\hat{L} \geq \bar{L}$, as it is clear from Figure 8.1, $L^*\left(K^*\left(\bar{L}\right)\right) = \bar{L}$, and hence any $L \in \left[0, \bar{L}\right]$ is a best response to some $K \in \left[0, K^*\left(\bar{L}\right)\right]$. Moreover, it has been established already that any $K \in \left[0, K^*\left(\bar{L}\right)\right]$ is a best response to some

$L \in [0, \bar{L}]$. Therefore, no strategy is eliminated at this stage, and the elimination procedure stops here. The set of rationalizable strategies is $[0, K^*(\bar{L})]$ for the employer and $[0, \bar{L}]$ for the worker.

Round 2 ($\hat{L} < \bar{L}$) Since $\hat{L} < \bar{L}$, as it is clear from Figure **??**, $L^*(K^*(\bar{L})) = \left(\frac{1}{2}\frac{\beta}{c}(K^*(L))^\alpha\right)^{1/(1-\beta)} < \bar{L}$. Now, since $K \in [0, K^*(\bar{L})]$, any $L > L^*(K^*(\bar{L}))$ is strictly dominated by $L^*(K^*(\bar{L}))$, and is eliminated for the worker. As before, no other strategy is eliminated in this round. The remaining strategy sets are $[0, K^*(\bar{L})]$ and $[0, L^*(K^*(\bar{L}))]$.

Towards an induction, assume that at the end of round $2m$, the remaining strategy sets are $[0, K^* \circ (L^* \circ K^*)^{m-1}(\bar{L})]$ and $[0, (L^* \circ K^*)^m(\bar{L})]$.[1] (This is the case for $m = 1$.)

Round $2m+1$ **($\hat{L} < \bar{L}$)** Write

$$\tilde{K} \equiv K^* \circ (L^* \circ K^*)^{m-1}(\bar{L})$$

and

$$\tilde{L} \equiv (L^* \circ K^*)^m(\bar{L}).$$

Since $\hat{L} < \bar{L}$, as one can see from the figure, $\tilde{K} > \hat{K}$ and $\tilde{L} > \hat{L}$. Hence, $K^*\left(\tilde{L}\right) = K^*\left(L^*\left(\tilde{K}\right)\right) < \tilde{K}$. Any strategy $K > K^*\left(\tilde{L}\right)$ is dominated by $K^*\left(\tilde{L}\right)$ and eliminated in this round. No other strategy is eliminated. The remaining strategy sets are $\left[0, K^*\left(\tilde{L}\right)\right]$ and $\left[0, \tilde{L}\right]$.

Round $2m+2$ **($\hat{L} < \bar{L}$)** Since $\hat{L} < \tilde{L} < \bar{L}$, as in the previous round, $L^*\left(K^*\left(\tilde{L}\right)\right) < \tilde{L}$. Now, since $K \in \left[0, K^*\left(\tilde{L}\right)\right]$, any $L > L^*\left(K^*\left(\tilde{L}\right)\right)$ is strictly dominated by $L^*\left(K^*\left(\tilde{L}\right)\right)$, and is eliminated for the worker. As before, no other strategy is eliminated in this round. The remaining strategy sets are $\left[0, K^*\left(\tilde{L}\right)\right]$ and $\left[0, L^*\left(K^*\left(\tilde{L}\right)\right)\right]$. By substituting $\tilde{L} \equiv (L^* \circ K^*)^m(\bar{L})$ one can check that the formulas in the inductive hypothesis is true for $m+1$.

One can check that as $m \to \infty$, $(L^* \circ K^*)^m(\bar{L}) \to \hat{L}$. Therefore, the set of rationalizable strategies is $\left[0, \hat{K}\right]$ for the employer and $\left[0, \hat{L}\right]$ for the worker.

[1] For any function f, $f^m(x) = f(f \ldots (f(x)))$ where f is repeated m times.

8.2 Coordination in Software Development

[Midterm 1, 2011] There are n software developers, named $i = 1, 2, \ldots, n$. Each software developer i has an "ideal specification" θ_i for his product, but also would like his product to be compatible with the other software, where θ_i is a real number. Simultaneously, each i selects a specification parameter s_i, which is a real number. The payoff of a player i is

$$u_i(s_1, \ldots, s_n) = 100 - (s_i - \theta_i)^2 - \frac{1}{n-1}\sum_{j \neq i}(s_i - s_j)^2.$$

Note that a developers pays two costs: one for being away from his ideal specification, and one for being away from the specification of other softwares (cost of incompatibility). Note also that in the normal-form game, the software developers are the players; each chooses a strategy s_i from real line, and the payoff function of player i is u_i.

Exercise 8.4 *Compute a Nash equilibrium.*

Solution: For each player i, the first-order condition for his best response is

$$2(s_i - \theta_i) - \frac{2}{n-1}\sum_{j \neq i}(s_i - s_j) = 0,$$

which simplifies to

$$2s_i = \theta_i + \frac{1}{n-1}\sum_{j \neq i} s_j.$$

To solve this equation system, sum it up over i and obtain $\sum_i s_i = \sum_i \theta_i$. Substituting this in the above equation, obtain

$$s_i = \frac{n-1}{2n-1}\theta_i + \frac{1}{2n-1}\sum_{j=1}^{n}\theta_j.$$

In equilibrium, a software developer chooses, roughly, the average of his own ideal specification and the average $\sum_{j=1}^{n}\theta_j/n$ of all ideal specifications, including his own.

8.3 Competition in Research and Development

[Midterm 1, 2002] Two start ups, named Firm 1 and Firm 2, are competing for leadership in a software market. The leader wins, and the other loses. Each firm can invest some

$x \in [0.001, 1]$ unit for research and development by paying cost of $x/4$. If a firm invests x units and the other firm invests y units, the former wins with probability $x/(x+y)$. Therefore, the payoff of the former start up will be

$$\frac{x}{x+y} - x/4.$$

All these are common knowledge.

Note that the leader gets 1 and the follower gets 0 revenue. These numbers are multiplied with their respective probabilities, and the final payoff above is obtained after subtracting the cost of research. Note also that in the normal form game, the players are Firm 1 and Firm 2, strategy set of each player is $[0.001, 1]$, and the payoff function is as above.

Exercise 8.5 *Compute all pure strategy Nash equilibria.*

Solution: Firm 1 maximizes

$$\frac{x}{x+y} - x/4$$

over x, and Firm 2 maximizes

$$\frac{y}{x+y} - y/4$$

over y. The best response function of Firm 1 as a function of y is given by

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\left(\frac{x}{x+y} - x/4\right) = \frac{\partial}{\partial x}\left(1 - \frac{y}{x+y} - x/4\right) \\ &= \frac{y}{(x+y)^2} - 1/4, \end{aligned}$$

i.e.,

$$x^*(y) = 2\sqrt{y} - y.$$

Similarly, the best response function of Firm 2 is

$$y^*(x) = 2\sqrt{x} - x.$$

Note that $x^*(y) > y$ whenever $y < 1$. Therefore, the graphs of x^* and y^* intersect each other only at $x = y = 1$ —as shown in Figure 8.2. Therefore, (1,1) is the only Nash equilibrium.

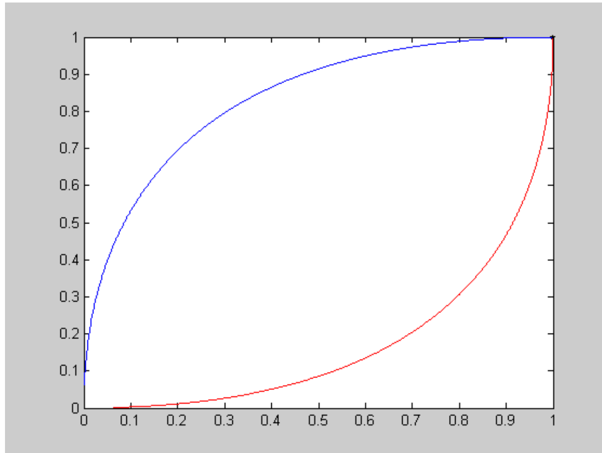


Figure 8.2: Best Response functions in R&D example.

Exercise 8.6 *Compute all rationalizable strategies.*

Solution: $(1,1)$ is the only rationalizable strategy profile. Since $y \geq y_0 \equiv 0.001$, then any strategy $x < x^*(y_0)$ is strictly dominated by $x_1 = x^*(y_0)$, and therefore eliminated. Write also $x_0 = y_0$ and $x_1 = y_1$. Now, the remaining strategy space of each player is $[x_1, 1]$. Note that $x_1 = x^*(.001) > 0.001 = x_0$. Now, similarly, one can eliminate any strategy $x < x_2 \equiv x^*(y_1)$. Applying this iteratively, after nth elimination, the remaining strategy space is $[x_n, 1]$ where

$$x_n = 2\sqrt{x_{n-1}} - x_{n-1}$$

and $x_0 = .001$. It is clear from the figure that $x_n \to 1$ as $n \to \infty$. Hence in the limit we are left with strategy space $\{1\}$.

More formally,

$$x_n = 2\sqrt{x_{n-1}} - x_{n-1} > \sqrt{x_{n-1}} = x_{n-1}^{1/2}.$$

Hence,

$$1 > x_n > x_0^{(1/2)^{n-1}}.$$

Of course, as $n \to \infty$, $(1/2)^{n-1} \to 0$, and hence $x_0^{(1/2)^{n-1}} \to 1$. Therefore, $x_n \to 1$.

8.4 Political Competition

[Midterm 1, 2006] Two candidates, Alice and Bob, are running for a political office. Simultaneously, Alice and Bob invest $x_A \geq 0$ and $x_B \geq 0$ for their campaigns, respectively. Alice wins with probability

$$p_A(x_A, x_B) = \frac{x_A}{\alpha + x_A + x_B};$$

Bob wins with probability

$$p_B(x_A, x_B) = \frac{x_B}{\alpha + x_A + x_B},$$

and with remaining probability $\alpha/(\alpha + x_A + x_B)$, a third party candidate wins, where $\alpha > 0$ is a fixed small number. For each party, the value of winning is 1 and the cost of investment is x, so that the expected payoff of Alice and Bob are $p_A(x_A, x_B) - x_A$ and $p_B(x_A, x_B) - x_B$, respectively.

Exercise 8.7 *Compute the Nash equilibria of this game.*

Answer: The first-order condition for x_A being a best response to x_B is

$$\frac{\partial}{\partial x_A}[p_A(x_A, x_B) - x_A] = \frac{\partial}{\partial x_A}\left[\frac{x_A}{\alpha + x_A + x_B} - x_A\right] = \frac{\alpha + x_B}{(\alpha + x_A + x_B)^2} - 1 = 0;$$

i.e.,

$$\alpha + x_B = (\alpha + x_A + x_B)^2. \tag{8.1}$$

Similarly, the first-order condition for Bob is

$$\alpha + x_A = (\alpha + x_A + x_B)^2. \tag{8.2}$$

Comparing the two equations, we find that $x_A = x_B$. Substituting this equality in (8.1), we find that, in equilibrium, x_A solves

$$\alpha + x_A = (\alpha + 2x_A)^2, \tag{8.3}$$

which is equivalent to

$$4x_A^2 + (4\alpha - 1)x_A + \alpha^2 - \alpha = 0.$$

There is only one non-negative solution to this quadratic equation:

$$x^* = \frac{1 - 4\alpha + \sqrt{1 + 8\alpha}}{8} \cong \frac{1}{4}.$$

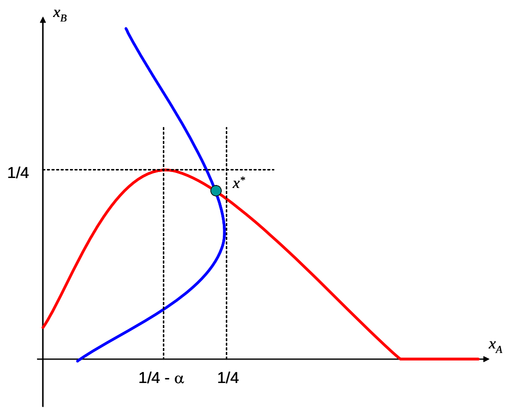
The unique Nash equilibrium is given by $x_A = x_B = x^*$.

Exercise 8.8 *Compute the set of rationalizable strategies.*

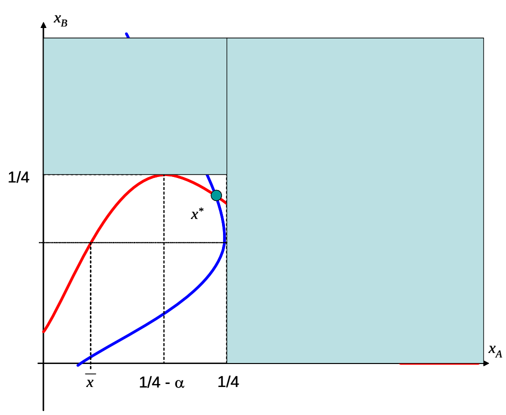
Answer: From (8.1) and (8.2), the best response functions of Alice and Bob are

$$x_A^{BR}(x_B) = \sqrt{x_B + \alpha} - (x_B + \alpha)$$
$$x_B^{BR}(x_A) = \sqrt{x_A + \alpha} - (x_A + \alpha).$$

The best-response functions look like:



Remember from the class that, since the utility functions are strictly concave (or "single-peaked"), if $x_B > x_B^{BR}(x_A)$ for each x_A, then x_B is strictly dominated. For example, any $x_B > 1/4$ is strictly dominated by $x_B = 1/4$. Similarly, any $x_A > 1/4$ is strictly dominated by $x_A = 1/4$. Hence, in the first round, we eliminate all such strategies:



In the next round we eliminate the strategies with $x_A < x_A^{BR}(0) = \sqrt{\alpha} - \alpha x_1$.This is because all such strategies are now dominated by x_1. We continue this elimination iteratively. All we need to know where the process stops. The answer is actually easy. It will stop at x^*, and $x_A = x_B = x^*$ are the only rationalizable strategies.

Here is a mathematical proof: Since the game is symmetric, the set of rationalizable strategies is the same for both players; call that set R. Recall that no rationalizable strategy is strictly dominated when we restrict the other player's strategies to be rationalizable. That is, for each $x \in R$, there exists $y \in R$ such that $x = x_A^{BR}(y) = x_B^{BR}(y)$. Suppose that $\min R < 1/4 - \alpha$. Now, $\min R = x_B^{BR}(y)$ for some $y \in R$. Since x_B^{BR} is "single-peaked", either $y = \min R$ or $y = \max R$. But, since $x^* \leq \max R \leq 1/4$, as in the figure, $x_B^{BR}(\max R) > 1/4 - \alpha$, showing that $y \neq \max R$. Hence, $y = \min R$, i.e., $\min R = x_B^{BR}(\min R)$. But this is a contradiction because it implies that $(\min R, \min R)$ is a Nash equilibrium. Therefore, $\min R \geq 1/4 - \alpha$. Then, x_B^{BR} is strictly decreasing on R. That is, $\min R = x_B^{BR}(\max R)$ and $\max R = x_A^{BR}(\min R)$, i.e., $(\max R, \min R)$ is a Nash equilibrium, showing that $\max R = \min R = x^*$.

8.5 Exercises

1. [Homework 1, 2002] In the partnership above compute the rationalizable strategies for the case

   (a) $\alpha = \beta = 1/2$, $c > 1/4$, $r > 1/4$;

   (b) $\alpha = \beta = 1/2$, $c = r = 1/4$;

   (c) $\alpha = \beta = 1/2$, $c < 1/4$, $r < 1/4$.

2. [Homework 2, 2006] Alice and Bob seek each other. Simultaneously, Alice puts effort s_A and Bob puts effort s_B to search. The probability of meeting is $s_A s_B$; the value of the meeting for each of them is v, and the search costs s_A^3 to Alice and s_B^3 to Bob.

   (a) Find the Nash equilibria of this game.

   (b) How do the search efforts in equilibrium change when we increase v?

   (c) Take $v = 1$ and compute all rationalizable strategies.

3. [Homework 2, 2011] There are 3 partners, namely 1,2, and 3. Simultaneously, each partner i puts effort $x_i \in [0,1]$, producing output level of $x_1x_2x_3$ and costing cx_i^2 to i. The partners share the output equally; the payoff of i is $x_1x_2x_3/3 - cx_i^2$.

   (a) Write this game formally in normal form.

   (b) For $c > 1/6$, compute the sets of rationalizable strategies and Nash equilibria.

   (c) For $c \in (0, 1/6)$, compute the sets of rationalizable strategies and Nash equilibria.

4. [Midterm 1 Make Up, 2002] Consider a two player game in which each player's strategy is a real number $x \in [0,1]$. A player's payoff is

$$-(x - y/2 - 1/4)^2$$

   where x is his own strategy and y is the strategy chosen by the other player.

   (a) Find all Nash equilibria.

   (b) Compute all rationalizable strategies.

5. In the software development game in Section 8.2, compute the set of rationalizable strategies for the case

   (a) the set of strategies is all real numbers;

   (b) the set of strategies is $[0,1]$ and each $\theta_i \in (0,1)$.

6. Redo the analysis in Section 8.4 for the easier case of $\alpha = 0$.