14.126 Game Theory
Final Exam

You have 24 hours from the time you pick up the exam (i.e. you need to return your solutions by the same time next day). You can use any existing written source, but you cannot discuss the content of this exam with others. Questions are equally weighted. Good Luck.

1. Consider two players bargaining about how to divide a dollar, which they can consume only when they reach an agreement. At each t, with probability $p_{t,i}$ player i becomes proposer. Proposer offers a division $x = (x_1, x_2)$ with $x_1 + x_2 \leq 1$, and the other accepts or rejects it. If the offer is accepted, the game ends with payoff vector $\delta^t x$. Otherwise, they proceed to the next date. If they never agree, each gets zero. Show that in any subgame-perfect equilibrium, the expected payoff of player i is

$$V_{0,i} = (1 - \delta) \sum_{t=0}^{\infty} \delta^t p_{t,i};$$

   i.e. all SPE are payoff equivalent. Briefly interpret the equation. [You can make any independence assumption that you find helpful. If you find a SPE with above payoffs and show that it is indeed SPE, you will get half of the points. Try that only if you can't solve the original problem.]

2. Consider a two-player, infinitely repeated game in which players maximize average discounted value of stage payoffs with discount factor $\delta \in (0, 1)$. At each date t, simultaneously each player i invests $x_{i,t} \in \{0, 1\}$ in a public good, $y_t \in \{0, 1\}$, where

$$\Pr(y_t = 1 | x_{1,t}, x_{2,t}) = \begin{cases} 2/3 & \text{if } x_1 + x_2 = 2 \\ 1/2 & \text{if } x_1 + x_2 = 1 \\ r & \text{if } x_1 + x_2 = 0 \end{cases}$$

   with $r \in (1/3, 5/12)$. The stage payoff of player i is $4y_t - x_{i,t}$.

   (a) Assuming that all the previous moves are publicly observable, compute the most efficient symmetric subgame-perfect equilibrium (for each $\delta \in (0, 1)$).

   (b) Assume the previous levels of public goods (i.e., y_s with $s < t$) are publicly observable but individual investments are not. Find the range of δ under which grim trigger strategy profile is a public perfect equilibrium (Grim trigger: select $x_{1,t} = x_{2,t} = 0$ if y has ever been 0 and $x_{1,t} = x_{2,t} = 1$ otherwise.) Compute the expected payoff as $\delta \to 1$.

   (c) In part (b), find the range of δ under which the following is a public perfect equilibrium: start with $x_{1,0} = x_{2,0} = 1$, and for any $t > 0$, select $x_{1,t} = x_{2,t} = y_{t-1}$. Compute the expected payoff as $\delta \to 1$.

   (d) In comparing your answers to parts (a-c), discuss how the structure of efficient public perfect equilibria varies under perfect and imperfect public monitoring.

3. Consider the differentiated Bertrand duopoly. Each firm i has a cost c_i, and the firms set the prices $p_1 \in [c_1, \bar{p}]$ and $p_2 \in [c_2, \bar{p}]$ simultaneously, where $\bar{p}$ is a large number. The demand for the product of firm i is $D_i(p_1, p_2, \theta)$ where θ is a commonly known demand parameter, D_i is continuous, supermodular, decreasing in the price p_i of firm's own price, and increasing in the other firm's price and θ. The profit of firm i is

$$D_i(p_1, p_2, \theta)(p_i - c_i).$$

[You can make any convenient differentiability assumption.]

(a) Assuming that c_1 and c_2 common knowledge, show that there exist pure strategy Nash equilibria $\left(\underline{p}_1, \underline{p}_2\right)$ and $(\bar{p}_1, \bar{p}_2)$ such that for each rationalizable strategy p_i, $\underline{p}_i \leq p_i \leq \bar{p}_i$.

(b) In part (a), show that $\underline{p}_i$ and $\bar{p}_i$ are weakly increasing in θ.

(c) Assume that for each i, c_i is a privately known by i and it is a rational number in $[0, \bar{p}]$. Under this assumption, state analogous results to the ones in parts (a) and (b), and prove them. [Hint: since c_i is restricted to be rational, you don't need to worry about completeness.]

4. Modify the stage game in question 2, defining

$$\Pr(y = 1 | x_1, x_2) = \begin{cases} p & \text{if } x_1 + x_2 = 2 \\ q & \text{if } x_1 + x_2 = 1 \\ r & \text{if } x_1 + x_2 = 0 \end{cases}$$

where $p > q > r$ and $p - q > q - r$, and taking the payoff of player i to be $\theta y - x_i$. The unknown parameter θ is exponentially distributed with parameter $\lambda > 0$. Each player i observes a signal $s_i = \theta + \varepsilon \eta_i$, where ε is a real number and η_i is uniformly distributed on $[-1, 1]$. Assume that (θ, η_1, η_2) are independent. Show that there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, the game is "dominance solvable", and compute the "unique" equilibrium strategy.

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