14.16 STRATEGY AND INFORMATION

MEDIAN STABLE MATCHINGS

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Consider a one-to-one matching market, where M is the set of men and W is the set of women, and strict preferences for the men and women are given. Let μ and μ' be any two stable matchings. Let $\mu \vee^M \mu'$ be the function from $M \cup W$ to $M \cup W$ that assigns to each man m the more preferred of $\mu(m)$ and $\mu'(m)$, and assigns to each woman w the less preferred of $\mu(w)$ and $\mu'(w)$. Similarly, let $\mu \wedge^M \mu'$ be the function that assigns to each man m the less preferred of $\mu(m)$ and $\mu'(m)$, and assigns to each woman w the more preferred of $\mu(w)$ and $\mu'(w)$. We saw in the matching theory slides (Theorem 5) that $\mu \vee^M \mu'$ and $\mu \wedge^M \mu'$ are again matchings, and moreover are stable.

Now suppose we have any collection $S = \{\mu_1, \ldots, \mu_k\}$ of stable matchings. Define $\sup^M(S)$ to be the function from $M \cup W$ to $M \cup W$ that assigns to each man m the most preferred of $\mu_1(m), \ldots, \mu_k(m)$, and assigns to each woman w the least preferred of $\mu_1(w), \ldots, \mu_k(w)$. We can see that

$$\sup^M(S) = (\cdots((\mu_1 \vee^M \mu_2) \vee^M \mu_3) \vee^M \cdots) \vee^M \mu_k$$

and therefore, by the preceding result, $\sup^M(S)$ is again a stable matching. Similarly, we can define $\inf^M(S)$ to be the function from $M \cup W$ to $M \cup W$ that assigns to each man m the least preferred of $\mu_1(m), \ldots, \mu_k(m)$, and assigns to each woman w the most preferred of $\mu_1(w), \ldots, \mu_k(w)$. Then $\inf^M(S)$ is again a stable matching.

This leads to the following result. The theorem is due to Teo and Sethuraman (1998), but our exposition follows the approach of Klaus and Klijn (2006).

Date: April 27, 2016.

Theorem 1. *Let $\mu_1, \ldots, \mu_l$ be stable matchings, not necessarily distinct, and let k be any integer with $1 \leq k \leq l$. Consider the function $\nu : M \cup W \to M \cup W$ given as follows. For each man m, order the matches $\mu_1(m), \ldots, \mu_l(m)$ from most to least preferred (there may be some repetitions in this list); let $\nu(m)$ be the kth entry in this list. For each woman w, order the matches $\mu_1(w), \ldots, \mu_l(w)$ from most to least preferred, and let $\nu(w)$ be the $(l-k+1)$th entry in this list. Then ν is also a stable matching.*

For a proof, note that $\mu^1 := \sup^M(\{\mu_1, \ldots, \mu_l\})$ has the desired property for $k = 1$, $\mu^2 := \sup^M(\{\mu_1, \ldots, \mu_l\} \setminus \{\mu^1\})$ has the desired property for $k = 2$, and so on.

If $\{\mu_1, \ldots, \mu_l\}$ is the set of stable matchings and l is odd, then applying the theorem for $k = (l+1)/2$ we obtain the *median* matching, in which every agent is assigned to the median partner over all stable matchings. This formally expresses the idea that we can choose a stable matching that balances the interests of men and women. If l is even, then there are two "almost-median" stable matchings, given by $k = l/2$ and $k = l/2 + 1$.