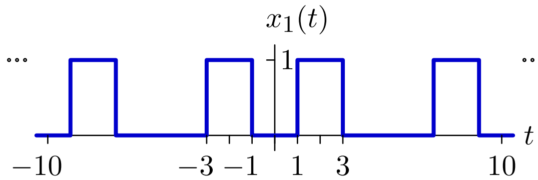
6.003 Homework #9

*Due at the beginning of recitation on **November 9, 2011**.*

Problems

1. Fourier varieties

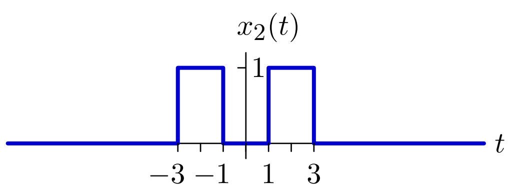
a. Determine the Fourier series coefficients of the following signal, which is periodic in $T = 10$.



$a_0 =$

$a_k =$ for $k \neq 0$

b. Determine the Fourier transform of the following signal, which is zero outside the indicated range.

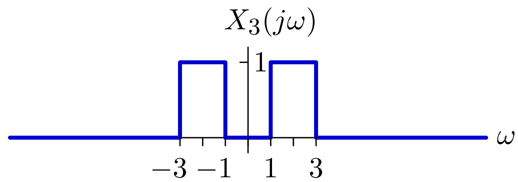


$X_2(j\omega) =$

c. What is the relation between the answers to parts a and b? In particular, derive an expression for a_k (the solution to part a) in terms of $X_2(j\omega)$ (the solution to part b).

$a_k =$

d. Determine the time waveform that corresponds to the following Fourier transform, which is zero outside the indicated range.



$x_3(t) =$ $\boxed{}$

e. What is the relation between the answers to parts b and d? In particular, derive an expression for $x_3(t)$ (the solution to part d) in terms of $X_2(j\omega)$ (the solution to part b).

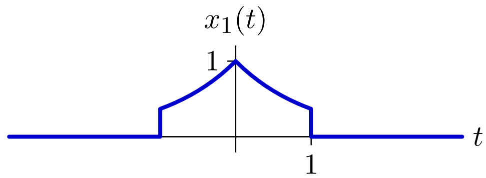
$x_3(t) =$ $\boxed{}$

2. Fourier transform properties

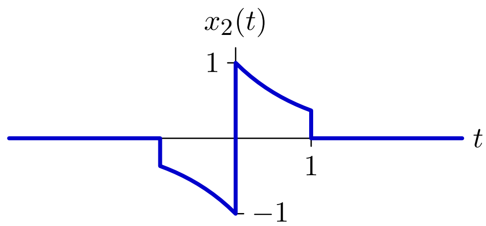
Let $X(j\omega)$ represent the Fourier transform of

$$x(t) = \begin{cases} e^{-t} & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

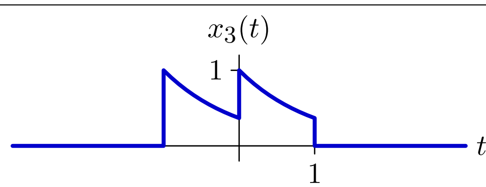
Express the Fourier Transforms of each of the following signals in terms of $X(j\omega)$.



$X_1(j\omega) =$



$X_2(j\omega) =$



$X_3(j\omega) =$

3. Fourier transforms

Find the Fourier transforms of the following signals.

a. $x_1(t) = e^{-|t|}\cos(2t)$

$X_1(j\omega) = \boxed{}$

b. $x_2(t) = \dfrac{\sin(2\pi t)}{\pi(t-1)}$

$X_2(j\omega) = \boxed{}$

c. $x_3(t) = \begin{cases} t^2 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

$X_3(j\omega) = \boxed{}$

d. $x_4(t) = (1 - |t|)\, u(t+1)u(1-t)$

$X_4(j\omega) = \boxed{}$

Engineering Design Problem

4. Parseval's theorem

Parseval's theorem relates time- and frequency-domain methods for calculating the average energy of a signal as follows:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

where a_k represents the Fourier series coefficients of the periodic signal $x(t)$ with period T.

a. We can derive Parseval's theorem from the properties of CT Fourier series.

1. Let $y(t) = |x(t)|^2$. Find the Fourier series coefficients b_k of $y(t)$.
[Hint: $|x(t)|^2 = x(t)x^*(t)$.]

2. Use the result from the previous part to derive Parseval's theorem.

b. Let $x_1(t)$ represent the input to an LTI system, where

$$x_1(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} e^{jk\frac{\pi}{4}t}$$

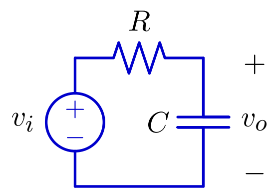
for $0 < \alpha < 1$. The frequency response of the system is

$$H(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & \text{otherwise.} \end{cases}$$

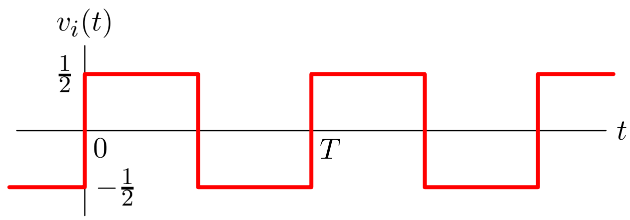
What is the minimum value of W so that the average energy in the output signal will be at least 90% of that in the input signal.

5. Filtering

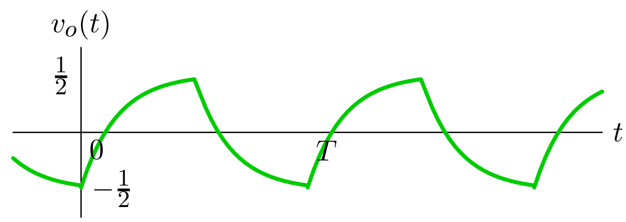
The point of this question is to understand how the magnitude of a filter affects the output and how the angle of a filter affects the output. Consider the following RC circuit as a "filter."



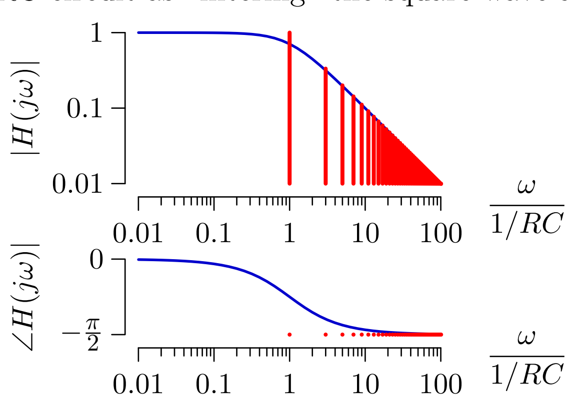
Assume that the input $v_i(t)$ is the following square wave.



If the fundamental frequency of the square wave ($\frac{2\pi}{T}$) is equal to the cutoff frequency of the RC circuit ($\frac{1}{RC}$) then the output $v_o(t)$ will have the following form.



We can think of the RC circuit as "filtering" the square wave as shown below.



The RC filter has two effects: (1) The amplitudes of the Fourier components of the input (vertical red lines in upper panel) are multiplied by the magnitude of the frequency response ($|H(j\omega)|$). (2) The phase of the Fourier components (red dots in lower panel) are shifted by the phase of the frequency response ($\angle H(j\omega)$).

a. Determine (using whatever method you find convenient) the output that would result if $v_i(t)$ were passed through a filter whose magnitude is $|H(j\omega)|$ (as above) but whose phase function is 0 for all frequencies. Compare the result with $v_o(t)$ above.

b. Determine (using whatever method you find convenient) the output that would result if $v_i(t)$ were passed through a filter whose phase function is $\angle H(j\omega)$ (as above) but whose magnitude function is 1 for all frequencies. Compare the result with $v_o(t)$ above.