6.003 Homework #2 Solutions

Problems

1. Finding outputs

Let $h_i[n]$ represent the nth sample of the unit-sample response of a system with system functional $H_i(\mathcal{R})$. Determine $h_i[2]$ and $h_i[119]$ for each of the following systems:

a. $H_1(\mathcal{R}) = \dfrac{\mathcal{R}}{1 - \frac{3}{4}\mathcal{R}}$

$h_1[2] = \boxed{\dfrac{3}{4}}$ $\qquad$ $h_1[119] = \boxed{\left(\dfrac{3}{4}\right)^{118}}$

$$\frac{\mathcal{R}}{1 - \frac{3}{4}\mathcal{R}} = \mathcal{R}\left(\frac{1}{1 - \frac{3}{4}\mathcal{R}}\right) = \mathcal{R}\left(1 + \frac{3}{4}\mathcal{R} + \frac{9}{16}\mathcal{R}^2 + \cdots\right)$$

$$= \mathcal{R} + \frac{3}{4}\mathcal{R}^2 + \frac{9}{16}\mathcal{R}^3 + \cdots + \left(\frac{3}{4}\right)^{n-1}\mathcal{R}^n + \cdots$$

b. $H_2(\mathcal{R}) = \dfrac{1 - \frac{1}{16}\mathcal{R}^4}{1 - \frac{1}{2}\mathcal{R}}$

$h_2[2] = \boxed{\dfrac{1}{4}}$ $\qquad$ $h_2[119] = \boxed{0}$

$$\frac{1 - \frac{1}{16}\mathcal{R}^4}{1 - \frac{1}{2}\mathcal{R}} = \left(1 - \frac{1}{16}\mathcal{R}^4\right)\frac{1}{1 - \frac{1}{2}\mathcal{R}} = \left(1 - \frac{1}{16}\mathcal{R}^4\right)\left(1 + \frac{1}{2}\mathcal{R} + \left(\frac{1}{2}\mathcal{R}\right)^2 + \cdots\right)$$

$$= 1 + \frac{1}{2}\mathcal{R} + \frac{1}{2^2}\mathcal{R}^2 + \frac{1}{2^3}\mathcal{R}^3 + \left(\frac{1}{2^4} - \frac{1}{2^4}\right)\mathcal{R}^4 + \left(\frac{1}{2^5} - \frac{1}{2^5}\right)\mathcal{R}^5 \cdots$$

$$= 1 + \frac{1}{2}\mathcal{R} + \frac{1}{2^2}\mathcal{R}^2 + \frac{1}{2^3}\mathcal{R}^3$$

c. $H_3(\mathcal{R}) = \dfrac{1}{(1 - \frac{1}{2}\mathcal{R})(1 - \frac{1}{4}\mathcal{R})}$

$h_3[2] = \boxed{2\dfrac{1}{2^2} - \dfrac{1}{4^2} = \dfrac{7}{16}}$ $\qquad h_3[119] = \boxed{2\left(\dfrac{1}{2}\right)^{119} - \left(\dfrac{1}{4}\right)^{119}}$

$$\dfrac{1}{(1 - \frac{1}{2}\mathcal{R})(1 - \frac{1}{4}\mathcal{R})} = \dfrac{2}{1 - \frac{1}{2}\mathcal{R}} - \dfrac{1}{1 - \frac{1}{4}\mathcal{R}}$$

$$= 2\left(1 + \dfrac{1}{2}\mathcal{R} + \dfrac{1}{2^2}\mathcal{R}^2 + \cdots\right) - \left(1 + \dfrac{1}{4}\mathcal{R} + \dfrac{1}{4^2}\mathcal{R}^2 + \cdots\right)$$
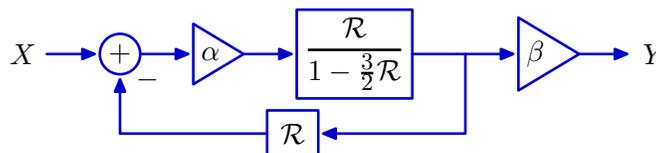
d. $H_4(\mathcal{R}) = \dfrac{1}{(1 - \mathcal{R})^2}$

$h_4[2] = \boxed{3} \qquad h_4[119] = \boxed{120}$

$$\dfrac{1}{(1 - \mathcal{R})^2} = \left(1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots\right)\left(1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots\right)$$

$$\begin{aligned} = \quad &\left(1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots\right) \\ + &\left(\mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \mathcal{R}^4 \cdots\right) \\ + &\left(\mathcal{R}^2 + \mathcal{R}^3 + \mathcal{R}^4 + \mathcal{R}^5 \cdots\right) \\ + &\left(\mathcal{R}^3 + \mathcal{R}^4 + \mathcal{R}^5 + \mathcal{R}^6 \cdots\right) + \cdots \end{aligned}$$

$$= 1 + 2\mathcal{R} + 3\mathcal{R}^2 + 4\mathcal{R}^3 + \cdots$$

2. Feedback

Consider the following system.

Assume that X is the unit-sample signal, $x[n] = \delta[n]$. Determine the values of α and β for which $y[n]$ is the following sequence (i.e., $y[0]$, $y[1]$, $y[2]$, ...):

$$0,\ 1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{31}{16},\ \ldots$$

Enter the values of α and β in the boxes below. Enter **none** if the value cannot be determined from the information provided.

$$\alpha = \boxed{\quad\quad \tfrac{1}{2} \quad\quad} \qquad \beta = \boxed{\quad\quad 2 \quad\quad}$$

Express the block diagram as a system functional:

$$\frac{Y}{X} = \frac{\frac{\alpha \mathcal{R}}{1 - \frac{3}{2}\mathcal{R}}}{1 + \frac{\alpha \mathcal{R}^2}{1 - \frac{3}{2}\mathcal{R}}} \beta = \frac{\alpha \beta \mathcal{R}}{1 - \frac{3}{2}\mathcal{R} + \alpha \mathcal{R}^2}.$$

The poles are at

$$\frac{3}{4} \pm \sqrt{\frac{9}{16} - \alpha}\,.$$

Now express $y[n]$ as a weighted sum of geometrics:

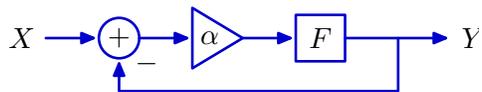$$y[n] = 2 \times 1^n - 2 \times \left(\frac{1}{2}\right)^n \; ; \quad n \geq 0$$

Thus the poles must be at $z = 1$ and $z = \frac{1}{2}$. It follows that α must be $\frac{1}{2}$. Then the system functional is

$$\frac{Y}{X} = \frac{\frac{1}{2}\beta \mathcal{R}}{1 - \frac{3}{2}\mathcal{R} + \frac{1}{2}\mathcal{R}^2} = \frac{\beta}{1 - \mathcal{R}} - \frac{\beta}{1 - \frac{1}{2}\mathcal{R}}$$

and β must be 2.

3. Mystery Feedback

Consider the following feedback system where F is the system functional for a system composed of just adders, gains, and delay elements.

If $\alpha = 10$ then the closed-loop system functional is known to be

$$\left.\frac{Y}{X}\right|_{\alpha=10} = \frac{1+\mathcal{R}}{2+\mathcal{R}}$$

Determine the closed-loop system functional when $\alpha = 20$.

$$\left.\frac{Y}{X}\right|_{\alpha=20} = \boxed{\frac{2+2\mathcal{R}}{3+2\mathcal{R}}}$$

In general

$$\frac{Y}{X} = \frac{\alpha F}{1+\alpha F}$$

If $\alpha = 10$

$$\left.\frac{Y}{X}\right|_{\alpha=10} = \frac{10F}{1+10F} = \frac{1+\mathcal{R}}{2+\mathcal{R}}$$

We can solve for F by equating the reciprocals of these expressions,

$$\frac{1}{10F} + 1 = \frac{2+\mathcal{R}}{1+\mathcal{R}}$$

$$\frac{1}{10F} = \frac{2+\mathcal{R}}{1+\mathcal{R}} - 1 = \frac{1}{1+\mathcal{R}}$$

from which it follows that $10F = 1+\mathcal{R}$. Then if $\alpha = 20$,

$$\left.\frac{Y}{X}\right|_{\alpha=20} = \frac{20F}{1+20F} = \frac{2+2\mathcal{R}}{1+2+2\mathcal{R}} = \frac{2+2\mathcal{R}}{3+2\mathcal{R}}$$

4. Ups and Downs

The unit-sample response of a linear, time-invariant system is given by

$$h[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0, 3, 6, 9, ... \\ 2 & n = 1, 4, 7, 10, ... \\ 3 & n = 2, 5, 8, 11, ... \end{cases}$$

a. Determine a closed-form expression for the system functional for this system.

$$H(\mathcal{R}) = \boxed{\frac{1 + 2\mathcal{R} + 3\mathcal{R}^2}{1 - \mathcal{R}^3}}$$

$$\begin{aligned} H(\mathcal{R}) &= 1 + 2\mathcal{R} + 3\mathcal{R}^2 + \mathcal{R}^3 + 2\mathcal{R}^4 + 3\mathcal{R}^5 + \mathcal{R}^6 + 2\mathcal{R}^7 + 3\mathcal{R}^8 + \mathcal{R}^9 + \cdots \\ &= (1 + 2\mathcal{R} + 3\mathcal{R}^2)(1 + \mathcal{R}^3 + \mathcal{R}^6 + \cdots) \\ &= \frac{1 + 2\mathcal{R} + 3\mathcal{R}^2}{1 - \mathcal{R}^3} \end{aligned}$$

b. Enter the poles of the system in the box below.

$$1, \ e^{j2\pi/3}, \ e^{-j2\pi/3}$$

$$\mathcal{H}(z) = \frac{1 + \frac{2}{z} + \frac{3}{z^2}}{1 - \frac{1}{z^3}} = \frac{z^2 + 2z + 3}{z^3 - 1}$$

The poles are at the roots of $z^3 - 1$, i.e., everywhere that $z^3 = 1$. There are three cube roots of 1: 1, $e^{j2\pi/3}$, and $e^{-j2\pi/3}$.

5. Characterizing a system from its unit-sample response

The first 30 samples of the unit-sample response of a linear, time-invariant system are given in the following table.

n	$h[n]$	n	$h[n]$
0	1	15	10761680
1	2	16	32285041
2	7	17	96855122
3	20	18	290565367
4	61	19	871696100
5	182	20	2615088301
6	547	21	7845264902
7	1640	22	23535794707
8	4921	23	70607384120
9	14762	24	211822152361
10	44287	25	635466457082
11	132860	26	1906399371247
12	398581	27	5719198113740
13	1195742	28	17157594341221
14	3587227	29	51472783023662

Determine the poles of this system. Enter the number of poles and list the pole locations below. If a pole is repeated k times, then enter that pole location k times. If there are more than 5 poles, enter just 5 of the pole locations. If there are fewer than 5 poles, leave the unused entries blank.

# of poles:

2

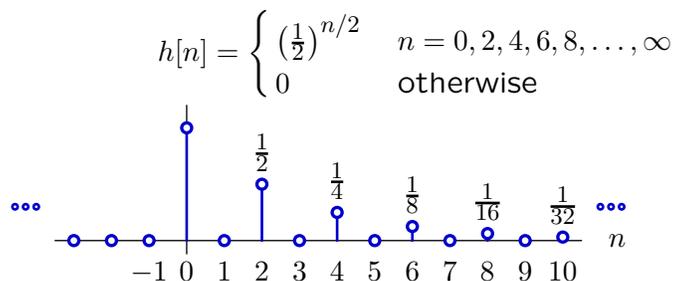locations:

−1	3			

$$\frac{Y}{X} = \frac{1}{1 - 2\mathcal{R} - 3\mathcal{R}^2} = \frac{1}{(1 + \mathcal{R})(1 - 3\mathcal{R})}$$

$$h[n] = \frac{1}{4}(-1)^n + \frac{3}{4}3^n$$

Engineering Design Problems

6. Unit-sample response

Consider a linear, time-invariant system whose unit-sample response $h[n]$ is shown below.

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/2} & n = 0, 2, 4, 6, 8, \ldots, \infty \\ 0 & \text{otherwise} \end{cases}$$

Part a. Is it possible to represent this system with a finite number of poles?

Yes or **No**: | Yes |

If yes, enter the number of poles and list the pole locations below. If a pole is repeated k times, then enter that pole location k times. If there are more than 5 poles, enter just 5 of the pole locations. If there are fewer than 5 poles, leave the unused entries blank.

# of poles: | 2 |

locations:

$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$			

If no, briefly explain why not.

$$H = 1 + \frac{1}{2}\mathcal{R}^2 + \frac{1}{4}\mathcal{R}^4 + \frac{1}{8}\mathcal{R}^6 + \frac{1}{16}\mathcal{R}^8 \cdots = \frac{1}{1 - \frac{1}{2}\mathcal{R}^2}$$

Substitute $\mathcal{R} \to \frac{1}{z}$:

$$H(z) = \frac{z^2}{z^2 - \frac{1}{2}} = \frac{z^2}{(z - \frac{1}{\sqrt{2}})(z + \frac{1}{\sqrt{2}})}$$
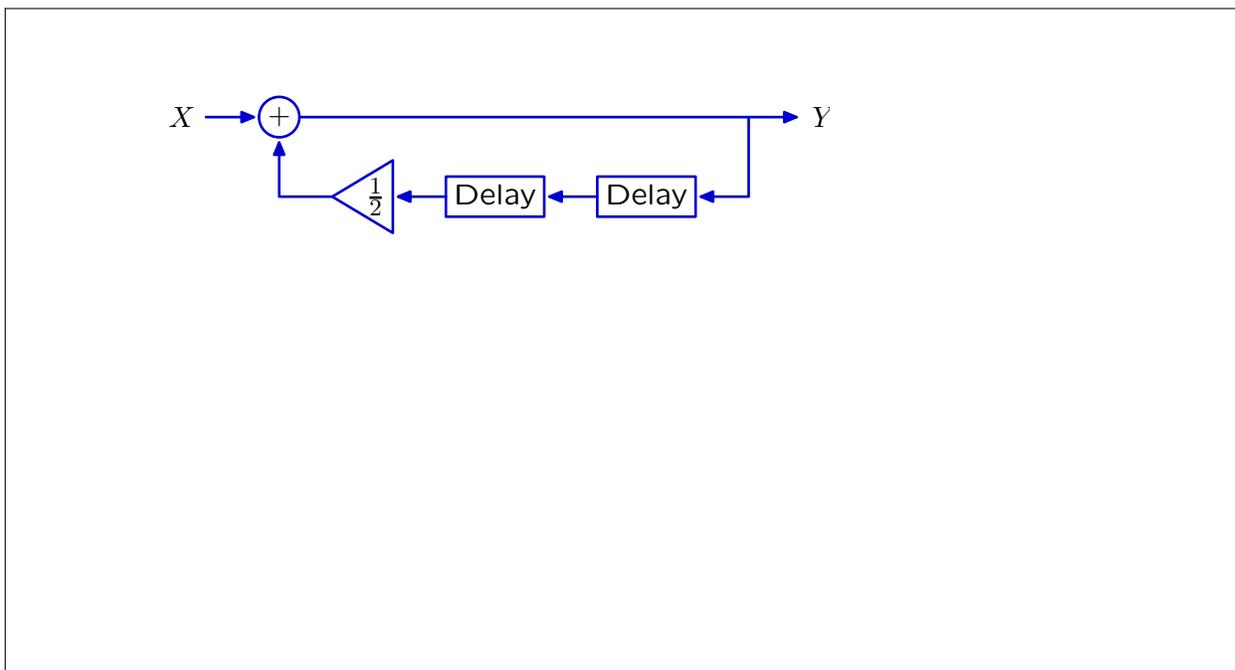
There are two poles:

$$z = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Part b. Is it possible to implement this system with a finite number of adders, gains, and delays (and no other components)?
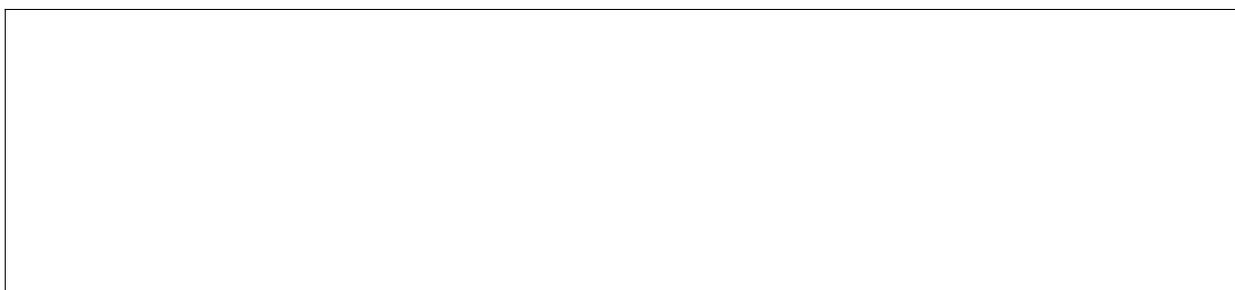
Yes or **No**:   Yes

If yes, sketch a block diagram for the system in the following box.

If no, briefly explain why not.

$$H = \frac{Y}{X} = 1 + \frac{1}{2}\mathcal{R}^2 + \frac{1}{4}\mathcal{R}^4 + \frac{1}{8}\mathcal{R}^6 + \frac{1}{16}\mathcal{R}^8 \cdots = \frac{1}{1 - \frac{1}{2}\mathcal{R}^2}$$

$$y[n] - \frac{1}{2}y[n-2] = x[n]$$

7. Repeated Poles

Consider a system H whose unit-sample response is

$$h[n] = \begin{cases} n+1 & \text{for } n \geq 0 \\ 0 & \text{otherwise}. \end{cases}$$

a. Determine the poles of H.

First find a difference equation. Try first order:

$$y[n] = a_1 y[n-1] + b_0 x[n].$$

Since $x[n] = 0$ for $n \geq 1$, we would need

$$y[n] = a_1 y[n-1] \quad \text{for} \quad n \geq 1$$
$$y[1] = a_1 y[0]: \quad 2 = a_1$$
$$y[2] = a_1 y[1]: \quad 3 = 2a_1$$
$$y[3] = a_1 y[2]: \quad 4 = 3a_1$$
$$y[4] = a_1 y[3]: \quad 5 = 4a_1$$

Clearly there is no solution of this form.

Try second order:

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n].$$

Now

$$y[n] = a_1 y[n-1] + a_2 y[n-2] \quad \text{for} \quad n \geq 1$$
$$y[1] = a_1 y[0] + a_2 y[-1]: \quad 2 = 1a_1 + 0a_2$$
$$y[2] = a_1 y[1] + a_2 y[0]: \quad 3 = 2a_1 + 1a_2$$
$$y[3] = a_1 y[2] + a_2 y[1]: \quad 4 = 3a_1 + 2a_2$$
$$y[4] = a_1 y[3] + a_2 y[2]: \quad 5 = 4a_1 + 3a_2$$

From the first of these equations, we find that $a_1 = 2$. All of the others are satisfied if $a_2 = -1$. The resulting difference equation is

$$y[n] = 2y[n-1] - y[n-2] + x[n]$$

and the corresponding function is

$$\frac{Y}{X} = \frac{1}{1 - 2\mathcal{R} + \mathcal{R}^2} = \frac{1}{(1-R)(1-R)}.$$
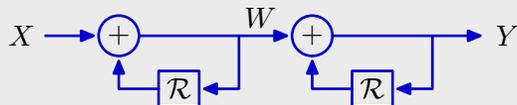
There are two poles, each at $z = 1$.

b. H can be written as the cascade of two identical subsystems, each called G. Determine the difference equation for G.

The functional for G is $\dfrac{1}{1-R}$. Therefore the difference equation for G is

$$y[n] = y[n-1] + x[n].$$

c. Draw a block diagram for H using just adders, gains, and delays. Use the block diagram to explain why the unit-sample response of H is the sequence $h[n] = n + 1$, $n \geq 0$.

> We can develop a block diagram for G directly from the difference equation found in the previous part. Then H is just the cascade of two such systems, as illustrated below:
>
>
>
> Since G is an accumulator, it's unit-sample response is the accumulated values of a unit-sample, which is a unit step signal,
>
> $$w[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$
>
> The output y is then the accumulated values of a unit-step signal,
>
> $$y[n] = \begin{cases} n+1 & n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

d. Because the system functional has two poles at the same location, the unit-sample response of H cannot be expressed as a weighted sum of geometric sequences,

$$h[n] = a_0 z_0^n + a_1 z_1^n ; \quad n \geq 0 .$$

However, h can be written in the previous form if the poles of H are displaced from their true positions by a small amounts (e.g., one pole by $+\epsilon$ and the other by $-\epsilon$). Determine a_0, a_1, z_0, and z_1 as functions of ϵ.

> Consider the following approximation to H:
>
> $$H \approx \frac{1}{\left(1 - (1+\epsilon)\mathcal{R}\right)\left(1 - (1-\epsilon)\mathcal{R}\right)} = \frac{1}{2\epsilon} \left(\frac{1+\epsilon}{1 - (1+\epsilon)\mathcal{R}} - \frac{1-\epsilon}{1 - (1-\epsilon)\mathcal{R}} \right)$$
>
> The corresponding unit-sample response is
>
> $$h[n] = \frac{1+\epsilon}{2\epsilon}(1+\epsilon)^n - \frac{1-\epsilon}{2\epsilon}(1-\epsilon)^n$$
>
> so that $a_0 = (1+\epsilon)/2\epsilon$, $a_1 = -(1-\epsilon)/2\epsilon$, $z_0 = 1+\epsilon$, and $z_1 = 1 - \epsilon$.

e. Compare the results of the approximation in part d for different values of ϵ.

> The following Python program tabulates results
>
> ```
> for e in [0.1,0.01,0.001]:
>     for n in range(11):
>         print "{0:10.5f} {1:10.5f} {2:10.5f} {3:3d}".format(
>             ((1+e)**(n+1))/(2*e),
>             -((1-e)**(n+1))/(2*e),
>             ((1+e)**(n+1))/(2*e)-((1-e)**(n+1))/(2*e),
>             n+1)
>     print
> ```
>
> which are shown in the following table.

```
epsilon = 0.1
  n    a_0 z_0**n  a_1 z_1**n      sum    h[n]
  0        5.50000    -4.50000   1.00000    1
  1        6.05000    -4.05000   2.00000    2
  2        6.65500    -3.64500   3.01000    3
  3        7.32050    -3.28050   4.04000    4
  4        8.05255    -2.95245   5.10010    5
  5        8.85781    -2.65721   6.20060    6
  6        9.74359    -2.39148   7.35210    7
  7       10.71794    -2.15234   8.56561    8
  8       11.78974    -1.93710   9.85264    9
  9       12.96871    -1.74339  11.22532   10
 10       14.26558    -1.56905  12.69653   11

epsilon = 0.01
  n    a_0 z_0**n  a_1 z_1**n      sum    h[n]
  0       50.50000   -49.50000   1.00000    1
  1       51.00500   -49.00500   2.00000    2
  2       51.51505   -48.51495   3.00010    3
  3       52.03020   -48.02980   4.00040    4
  4       52.55050   -47.54950   5.00100    5
  5       53.07601   -47.07401   6.00200    6
  6       53.60677   -46.60327   7.00350    7
  7       54.14284   -46.13723   8.00560    8
  8       54.68426   -45.67586   9.00840    9
  9       55.23111   -45.21910  10.01200   10
 10       55.78342   -44.76691  11.01650   11

epsilon = 0.001
  n    a_0 z_0**n  a_1 z_1**n      sum    h[n]
  0      500.50000 -499.50000   1.00000    1
  1      501.00050 -499.00050   2.00000    2
  2      501.50150 -498.50150   3.00000    3
  3      502.00300 -498.00300   4.00000    4
  4      502.50501 -497.50500   5.00001    5
  5      503.00751 -497.00749   6.00002    6
  6      503.51052 -496.51048   7.00004    7
  7      504.01403 -496.01397   8.00006    8
  8      504.51804 -495.51796   9.00008    9
  9      505.02256 -495.02244  10.00012   10
 10      505.52758 -494.52742  11.00017   11
```
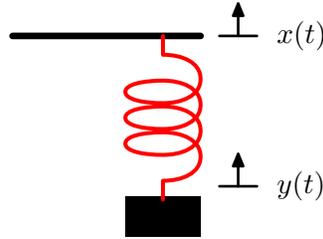
The approximation is increasingly accurate as $\epsilon \to 0$. However, the amplitudes of the fundamental modes also get very large with decreasing ϵ.

8. Masses and Springs, Forwards and Backwards

The following figure illustrates a mass and spring system. The input $x(t)$ represents the position of the top of the spring. The output $y(t)$ represents the position of the mass.

The mass is $M = 1\,\text{kg}$ and the spring constant is $K = 1\,\text{N/m}$. Assume that the spring obeys Hooke's law and that the reference positions are defined so that if the input $x(t)$ is equal to zero, then the resting position of $y(t)$ is also zero.

a. Determine a differential equation that relates the input $x(t)$ and output $y(t)$.

Newton's law says that the force on the mass is equal to its mass times its acceleration,

$$F = M\frac{d^2 y(t)}{dt^2}.$$

Hooke's law says that the force exerted by the spring is proportional to its elongation,

$$F = K(x(t) - y(t)).$$

Combining,

$$M\frac{d^2 y(t)}{dt^2} + Ky(t) = Kx(t).$$

Since $M = K = 1$,

$$\frac{d^2 y(t)}{dt^2} + y(t) = x(t).$$

b. Calculate the step response of the system.

Since this is a linear differential equation with constant coefficients, and since the input is 1 for $t > 0$, the step response will have the following form,

$$s(t) = Ae^{-pt} + B \ ; \quad t > 0.$$

Substituting into the differential equation yields

$$p^2 Ae^{-pt} + Ae^{-pt} + B = 1 \ ; \quad t > 0.$$

Therefore, $B = 1$ and $p = \pm j$ and

$$s(t) = Ce^{jt} + De^{-jt} + 1 \ ; \quad t > 0.$$

The step response is defined as the response to a step when the system is initially at rest. Rest means that $s(t) = 0$ for $t < 0$. If $s(t)$ is zero for $t < 0$, then both $s(t)$ and its time derivative are continuous through zero, since the force from the spring is finite and $F = M\ddot{s}(t)$. Thus

$$s(0) = C + D + 1 = 0$$

and

$$\frac{ds(t)}{dt} = jC - jD = 0$$

and therefore $C = D = -0.5$. The result is

$$s(t) = (1 - 0.5e^{jt} - 0.5e^{-jt})u(t) = (1 - \cos t)u(t).$$

c. The differential equation in part a contains a second derivative (if you did part a correctly). We wish to develop a forward-Euler approximation for this derivatve. One method is to write the second-order differential equation in part a as a part of first order differential equations. Then apply the forward-Eular approximation to the first order derivatives:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} \approx \frac{y[n+1] - y[n]}{T}.$$

Use this approach to find a difference equation to approximate the behavior of the mass and spring system. Determine the step response of the system and compare your results to those in part b.

Start with the differential equation

$$\frac{d^2y(t)}{dt^2} + y(t) = x(t).$$

Then let

$$z(t) = \frac{dy(t)}{dt}$$

so that

$$\frac{dz(t)}{dt} + y(t) = x(t).$$

Make Euler forward appoximations to each of these to get

$$\frac{z[n+1] - z[n]}{T} + y[n] = x[n]$$

and

$$\frac{y[n+1] - y[n]}{T} = z[n].$$

Substitute the latter equation into the former to get

$$\frac{\frac{y[n+2]-y[n+1]}{T} - \frac{y[n+1]-y[n]}{T}}{T} + y[n] = x[n]$$

which is

$$y[n+2] - 2y[n+1] + (1 + T^2)y[n] = T^2 x[n].$$

The following Matlab code solves this difference equation by setting $y(1) = y(2) = 0$ and then recursively finding $y(n+2)$.
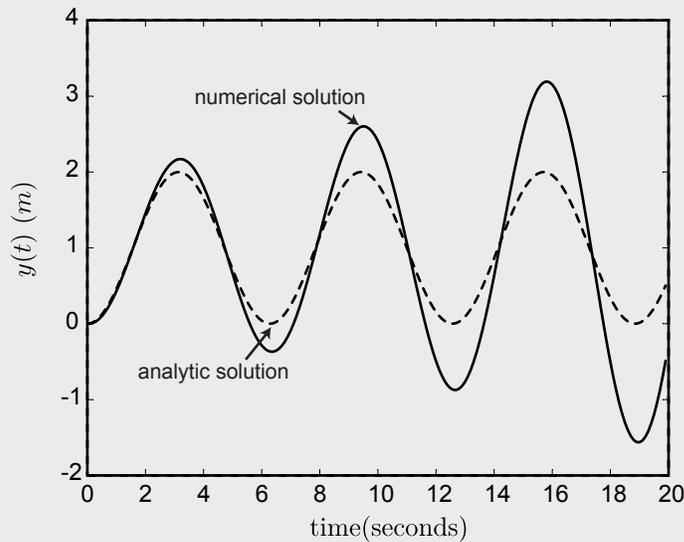
```
T = 0.1;
y(1)=0;
y(2)=0;
t(1)=0;
t(2)=T;
```

```
for n = 1:198
y(n+2)=2*y(n+1)-y(n)-T*T*y(n)+T*T;
t(n+2)=t(n+1)+T;
end
plot(t,y,'-');
```

The following plot shows the result.

Notice that the numerical approximation DIVERGES! The rate of divergence depends on the step size.

d. An alternative to the forward-Euler approximation is the backward-Euler approximation:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} \approx \frac{y[n] - y[n-1]}{T}.$$

Repeat the exercise in the previous part, but using the backward-Euler approximation instead of the forward-Euler approximation.

Start with the pair of first order differential equations

$$z(t) = \frac{dy(t)}{dt}$$

and

$$\frac{dz(t)}{dt} + y(t) = x(t).$$

Make Euler backward appoximations to each of these to get

$$\frac{z[n] - z[n-1]}{T} + y[n] = x[n]$$

and

$$\frac{y[n] - y[n-1]}{T} = z[n].$$

Substitute the latter equation into the former to get

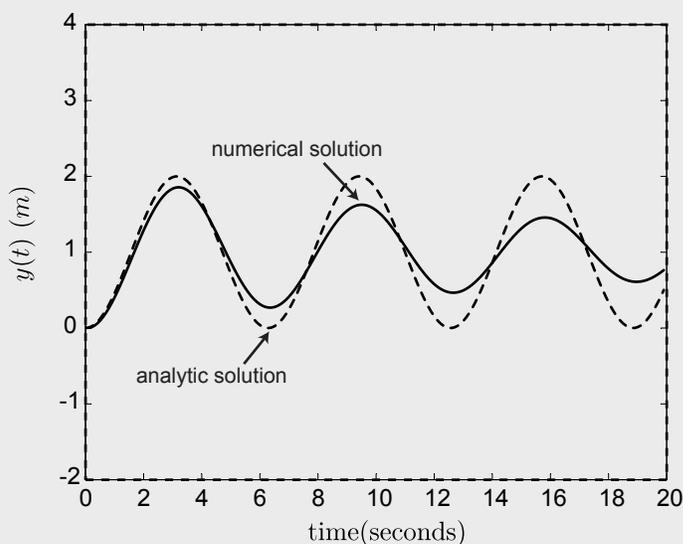$$\frac{\frac{y[n]-y[n-1]}{T} - \frac{y[n-1]-y[n-2]}{T}}{T} + y[n] = x[n]$$

which is

$$(1 + T^2)y[n] - 2y[n-1] + y[n-2] = T^2 x[n].$$

The following Matlab code solves this difference equation by setting $y(1) = y(2) = 0$ and then recursively finding $y(n)$.

```
T = 0.1;
y(1)=0;
y(2)=0;
t(1)=0;
t(2)=T;
a(1)=0;
a(2)=1-cos(T);
for n = 1:198
y(n+2)=(2*y(n+1)-y(n)+T*T)/(1+T*T);
t(n+2)=t(n+1)+T;
a(n+2)=1-cos(t(n+2));
end
plot(t,y,'-',t,a,'-');
```

The following plot shows the result.

This time the numerical solution is a cosine wave with diminishing amplitude. The analytic result is simply an offset cosine. Thus the numerical solutions based on both the Euler forward and Euler backward methods deviate systematically (but oppositely) from the exact solution.

e. The forward-Euler method approximates the second derivative at $t = nT$ as

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n+2] - 2y[n+1] + y[n]}{T^2}.$$

The backward-Euler method approximates the second derivative at $t = nT$ as

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n] - 2y[n-1] + y[n-2]}{T^2}.$$

Consider a compromise based on a centered approximation:

$$\frac{d^2y(t)}{dt^2}\bigg|_{t=nT} = \frac{y[n+1] - 2y[n] + y[n-1]}{T^2}.$$

Use this approximation to determine the step response of the system. Compare your results to those in the two previous parts of this problem.

This time, let

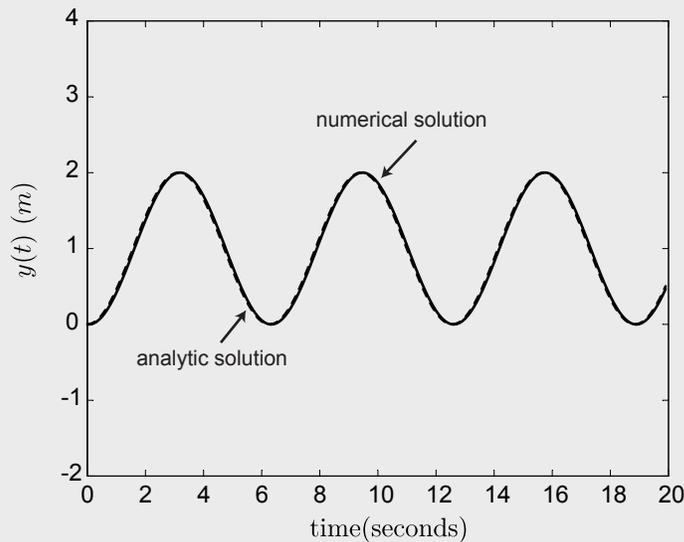$$\frac{\frac{y[n]-y[n-1]}{T} - \frac{y[n-1]-y[n-2]}{T}}{T} + y[n] = x[n]$$

which is

$$y[n+1] - (2 - T^2)y[n] + y[n-1] = T^2x[n].$$

The following Matlab code solves this difference equation by setting $y(1) = y(2) = 0$ and then recursively finding $y(n+1)$.

```
T = 0.1;
y(1)=0;
y(2)=0;
t(1)=0;
t(2)=T;
a(1)=0;
a(2)=1-cos(T);
for n = 1:198
y(n+2)=2*y(n+1)-y(n)-T*T*y(n+1)+T*T;
t(n+2)=t(n+1)+T;
a(n+2)=1-cos(t(n+2));
end
plot(t,y,'-',t,a,'-');
```

The following plot shows the result.

The numerical results using the centered rule are quite similar to the analytic results. Thus the centered rule is significantly better than the Euler forward or Euler backward rule (at least for this problem).

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