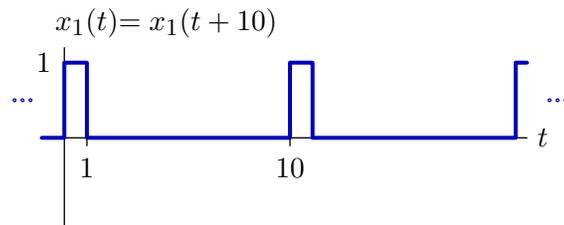# 6.003 Homework #8 Solutions

Problems

1. Fourier Series

Determine the Fourier series coefficients a_k for $x_1(t)$ shown below.

$$x_1(t) = x_1(t + 10)$$

$a_0 = $

$$\frac{1}{10}$$

$a_k = $

$$\frac{1}{\pi k} e^{-j\pi k/10} \sin(\pi k/10)$$ for $k \neq 0$
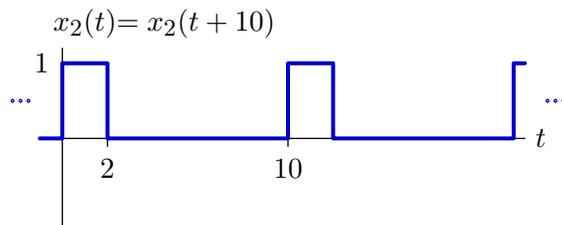
$$a_k = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{10} \int_0^1 1 e^{-j\frac{2\pi}{10}kt} dt = \frac{1}{10} \left. \frac{e^{-j\frac{\pi}{5}kt}}{-j\frac{\pi}{5}k} \right|_0^1 = \frac{1}{j2\pi k} \left(1 - e^{-j\pi k/5} \right)$$

Notice that this expression is badly formed at $k = 0$. We could use l'Hôpital's rule to evaluate this expression, but an easier method (which is also more robust against errors) is to simply evaluate the average value of $x_1(t)$ to find that $a_0 = 1/10$.

This solution could also be written in terms of sinusoids as

$$a_k = \begin{cases} \frac{1}{10} & k = 0 \\ \frac{1}{\pi k} e^{-j\pi k/10} \sin(\pi k/10) & k \neq 0 \end{cases}.$$

Determine the Fourier series coefficients b_k for $x_2(t)$ shown below.

$$x_2(t) = x_2(t + 10)$$

$$b_0 = \boxed{\frac{1}{5}}$$

$$b_k = \boxed{\frac{1}{\pi k} e^{-j\pi k/5} \sin(\pi k/5)} \quad \text{for } k \neq 0$$

$$b_k = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{10} \int_0^2 1 e^{-j\frac{2\pi}{10}kt} dt = \frac{1}{10} \left. \frac{e^{-j\frac{\pi}{5}kt}}{-j\frac{\pi}{5}k} \right|_0^2 = \frac{1}{j2\pi k} \left(1 - e^{-j2\pi k/5} \right)$$

As with the previous part, this expression is badly formed for $k = 0$. We therefore obtain $b_0 = 1/5$ by calculating the average value of $x_2(t)$.

This solution could also be written in terms of sinusoids as

$$b_k = \begin{cases} \frac{1}{5} & k = 0 \\ \frac{1}{\pi k} e^{-j\pi k/5} \sin(\pi k/5) & k \neq 0 \end{cases}.$$

Determine the Fourier series coefficients c_k for $x_3(t)$ shown below.

$$x_3(t) = x_3(t + 10)$$
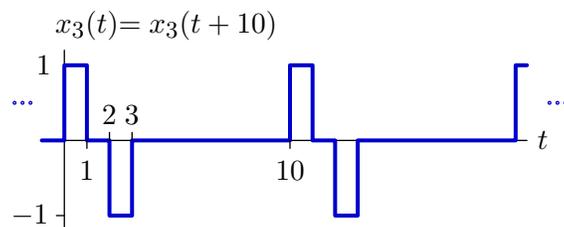
$$c_0 = \boxed{0}$$

$$c_k = \boxed{\tfrac{j2}{\pi k} e^{-j3\pi k/10} \sin(\pi k/5)\sin(\pi k/10)} \quad \text{for } k \neq 0$$

$$x_3(t) = x_1(t) - x_1(t-2)$$

$$\int_T x_1(t-2)e^{-j\frac{2\pi}{T}kt}dt = \int_T x_1(t)e^{-j\frac{2\pi}{T}k(t+2)}dt = e^{-j\frac{2\pi}{T}k2}\int_T x_1(t)e^{-j\frac{2\pi}{T}kt}dt = e^{-j\frac{2\pi}{T}k2}a_k$$
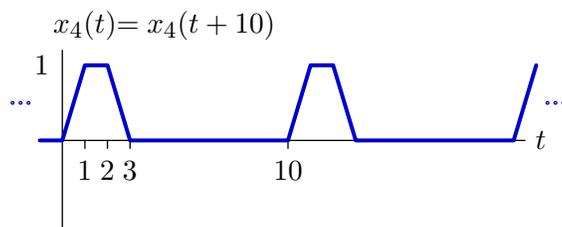
$$c_k = a_k - e^{-j\frac{2\pi}{T}k2}a_k = \left(1 - e^{-j2\pi k/5}\right)\frac{1}{j2\pi k}\left(1 - e^{-j\pi k/5}\right)$$

The average value of $x_3(t)$ is zero, so $c_0 = 0$.

This solution could also be written in terms of sinusoids as

$$c_k = \begin{cases} 0 & k = 0 \\ \frac{j2}{\pi k}e^{-j3\pi k/10}\sin(\pi k/5)\sin(\pi k/10) & k \neq 0 \end{cases}.$$

Determine the Fourier series coefficients d_k for $x_4(t)$ shown below.

$$x_4(t) = x_4(t+10)$$

$d_0 \ = $

$$\frac{1}{5}$$

$d_k \ = $

$$\frac{10}{\pi^2 k^2} e^{-j3\pi k/10} \sin(\pi k/5) \sin(\pi k/10)$$

for $k \neq 0$

$$x_3(t) = \frac{dx_4(t)}{dt}$$

$$c_k = j\frac{2\pi}{T} k d_k$$

$$d_k = \begin{cases} \frac{1}{5} & k = 0 \\ \frac{1}{j\frac{2\pi}{T}k} c_k = -\frac{5}{2\pi^2 k^2}\left(1 - e^{-j2\pi k/5}\right)\left(1 - e^{-j\pi k/5}\right) & k \neq 0 \end{cases}$$

This solution could also be written in terms of sinusoids as

$$d_k = \begin{cases} \frac{1}{5} & k = 0 \\ \frac{10}{\pi^2 k^2} e^{-j3\pi k/10} \sin(\pi k/5) \sin(\pi k/10) & k \neq 0 \end{cases}.$$

2. Inverse Fourier series

Determine the CT signals with the following Fourier series coefficients. Assume that the signals are periodic in $T = 4$. Enter an expression that is valid for $0 \le t < 4$ (other values can be found by periodic extension).

a. $a_k = \begin{cases} jk; & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$

$$x(t) = \boxed{-2\sin(\pi t/2) - 4\sin(\pi t)} \quad \text{for } 0 \le t < 4.$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} = -2je^{-j\frac{2\pi}{4}2t} - je^{-j\frac{2\pi}{4}t} + je^{j\frac{2\pi}{4}t} + 2je^{j\frac{2\pi}{4}2t}$$

$$= -2\sin(\pi t/2) - 4\sin(\pi t)$$

b. $b_k = \begin{cases} 1; & k \text{ odd} \\ 0; & k \text{ even} \end{cases}$

$$x(t) = \boxed{2\delta(t) - 2\delta(t - 2)} \quad \text{for } 0 \le t < 4.$$

$$x(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\frac{2\pi}{T}kt} = \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} e^{j\pi kt/2}$$

Unfortunately, this sum is not easy to close. However, it is closely related to the synthesis formula for an impulse train,

$$x_\delta(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} a_k e^{j\pi kt/2} = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\pi kt/2}.$$

If there were two impulses per period instead of one, then

$$x_{2\delta}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) + \delta\left(t - \frac{T}{2} - kT\right) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j2\pi kt/4} + \frac{1}{T} e^{j2\pi k(t-2)/4}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\pi kt/2}\left(1 + e^{j\pi k}\right) = \sum_{\substack{k=-\infty \\ k \text{ even}}}^{\infty} \frac{2}{T} e^{j\pi kt/2}$$
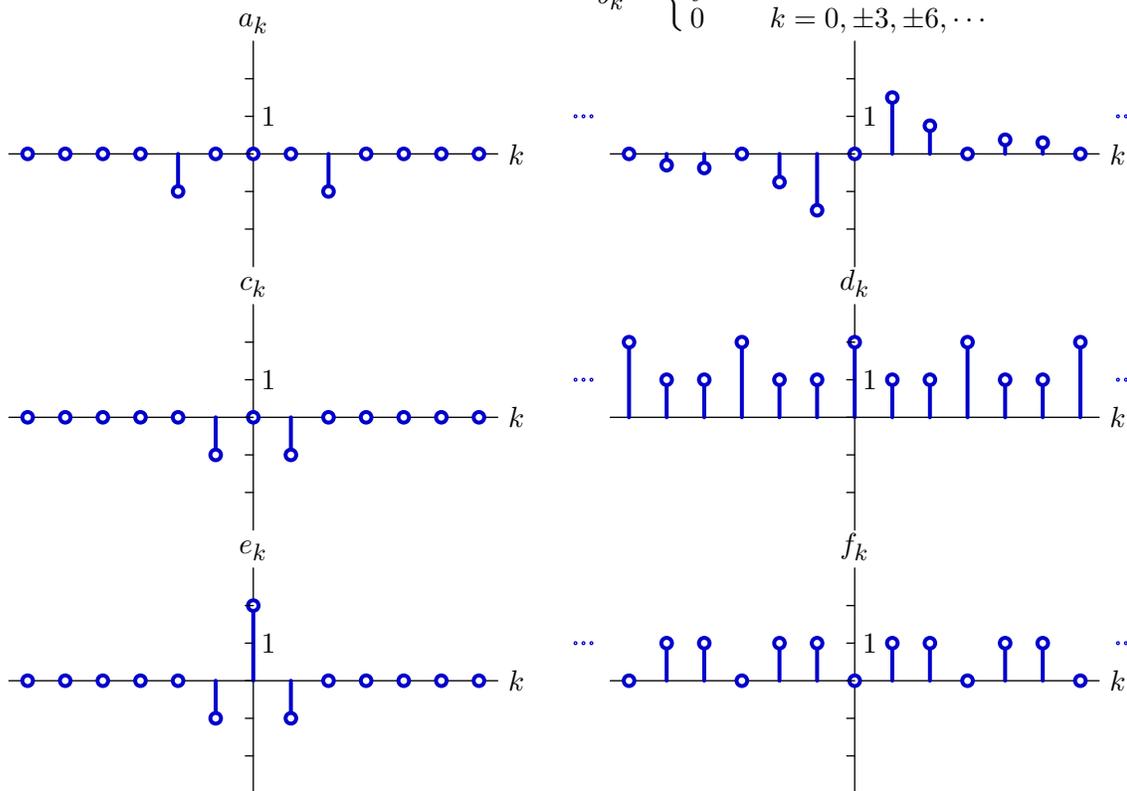
It follows that $x(t) = Tx_\delta(t) - \frac{T}{2}x_{2\delta}(t) = 4x_\delta(t) - 2x_{2\delta}(t)$ so that $x(t)$ is an alternating sequence of impulses

$$x(t) = \sum_{l=-\infty}^{\infty} 2\delta(t - 4l) - 2\delta(t - 2 - 4l)$$

3. Matching

Consider the following Fourier series coefficients.

$$b_k = \begin{cases} \frac{3}{j2k} & k = \pm 1, \pm 2, \pm 4, \pm 5, \pm 7, \ldots \\ 0 & k = 0, \pm 3, \pm 6, \cdots \end{cases}$$

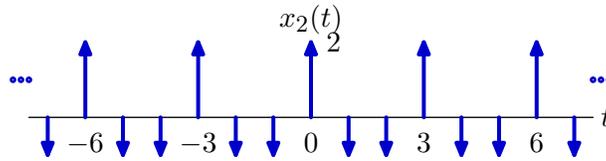**a.** Which coefficients (if any) corresponds to the following periodic signal?

$$x_1(t) = 2 - 2\cos\left(\frac{2\pi}{3}t\right)$$

a_k, b_k, c_k, d_k, e_k, f_k, or **None**: e_k

From the constant 2, it is clear that the zero coefficient is 2. Since $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$, the coefficients for $k = \pm 1$ are -1. Therefore the answer is e_k.

b. Which coefficients (if any) corresponds to the following periodic signal with period $T = 3$?

$x_2(t)$

a_k, b_k, c_k, d_k, e_k, f_k, or **None**: $\quad f_k$

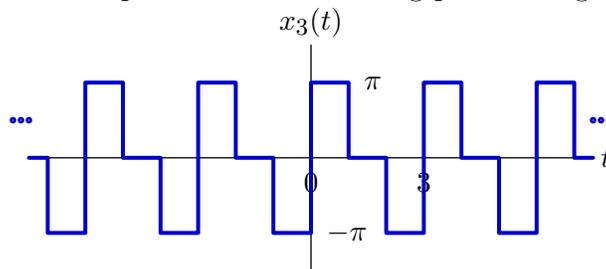
This signal is real and even, so its FS coefficients must be real and even. Also, the signal has no DC value, so the zero coefficient must be zero. This leaves a_k or c_k or f_k. The Fourier series for an impulse train must have infinite extent. Therefore the best candidate is f_k. Solving

$$a_k = \frac{1}{T} \int_T x_2(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{3} \int_{-\frac{3}{2}}^{\frac{3}{2}} \left(2\delta(t) - \delta(t-1) - \delta(t+1)\right) e^{-j\frac{2\pi}{3}kt} dt$$

$$= \frac{1}{3}\left(2 - 2\cos(2\pi k/3)\right) = \begin{cases} 0 & \text{if } k \text{ is evenly divisible by 3} \\ 1 & \text{otherwise} \end{cases}$$

So the answer is f_k.

---

c. Which (if any) set corresponds to the following periodic signal with period $T = 3$?

$x_3(t)$

a_k, b_k, c_k, d_k, e_k, f_k, or **None**: $\quad b_k$


The signal is real and odd, so its FS coefficients must be purely imaginary and odd. Thus the only candidate is b_k. Solving

$$a_k = \frac{1}{T} \int_T x_3(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{3} \int_{-1}^{0} -\pi e^{-j\frac{2\pi}{3}kt} dt + \frac{1}{3} \int_{0}^{1} -\pi e^{-j\frac{2\pi}{3}kt} dt$$

$$= \frac{1}{j2k}\left(e^{-j2\pi kt/3}\Big|_{-1}^{0} - e^{-j2\pi kt/3}\Big|_{0}^{1}\right) = \frac{1}{j2k}\left(2 - e^{j2\pi k/3} - e^{-j2\pi k/3}\right)$$

$$= \frac{1}{jk}\left(1 - \cos(2\pi k/3)\right) = \begin{cases} 0 & \text{if } k \text{ is evenly divisible by 3} \\ 3/j2k & \text{otherwise} \end{cases}$$

So the answer is b_k.

4. Input/Output Pairs

The following signals are periodic with period $T = 1$.

$$x_1(t) = x_1(t+1)$$


$$x_2(t) = x_2(t+1)$$

$$x_3(t) = x_3(t+1)$$
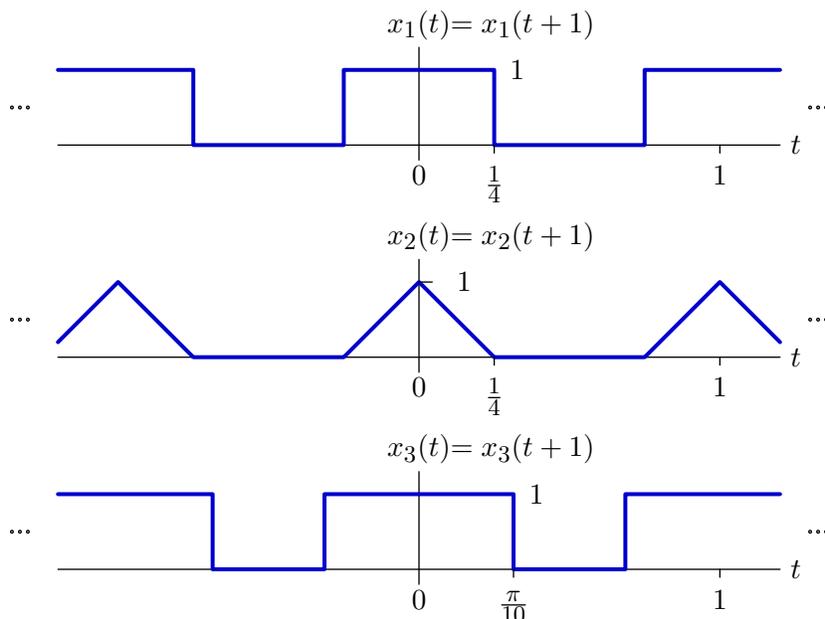
Determine if the following systems could or could not be linear and time-invariant (LTI).

We can use the "filter" idea as follows. First calculate the Fourier series coefficients. Then ask if each Fourier series coefficient in the output is a scaled version of the corresponding coefficient in the input.

$$x_1(t) \leftrightarrow a_k = \frac{1}{1}\int_{\frac{-1}{4}}^{\frac{1}{4}} e^{-j\frac{2\pi}{1}kt}dt = \frac{\sin\frac{\pi k}{2}}{\pi k} = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{\pi k} & |k| = 1,5,9,13,\ldots \\ -\frac{1}{\pi k} & |k| = 3,7,11,15,\ldots \\ 0 & |k| = 2,4,6,8,\ldots \end{cases}$$
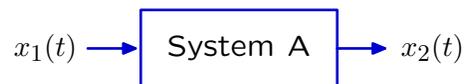
$$x_2(t) = y(t) \underset{T}{*} y(t)$$
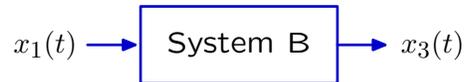
where $y(t)$ is the following signal:

$$y(t) = y(t+1)$$
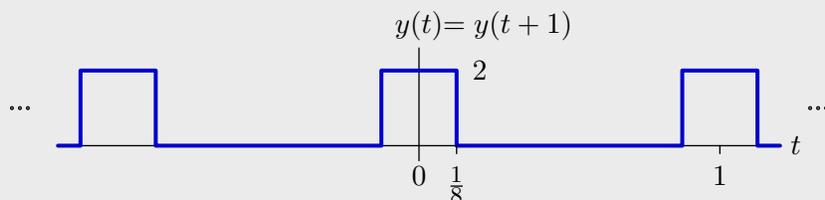
$$y(t) \leftrightarrow d_k = \frac{1}{1}\int_{\frac{-1}{8}}^{\frac{1}{8}} 2e^{-j\frac{2\pi}{1}kt}dt = \frac{2\sin\frac{\pi k}{4}}{\pi k}$$

$$x_2(t) \leftrightarrow b_k = Td_k^2 = 1 \times \frac{4\sin^2\frac{\pi k}{4}}{\pi^2 k^2} = \begin{cases} \frac{1}{4} & k = 0 \\ \frac{2}{\pi^2 k^2} & |k| = 1,3,5,7,9,11,13,\ldots \\ \frac{4}{\pi^2 k^2} & |k| = 2,6,10,14,\ldots \\ 0 & |k| = 4,8,12,16,\ldots \end{cases}$$
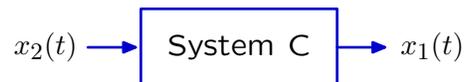
$$x_3(t) \leftrightarrow c_k = \frac{1}{1}\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} e^{-j\frac{2\pi}{1}kt}dt = \frac{\sin\frac{2\pi^2 k}{10}}{\pi k}$$
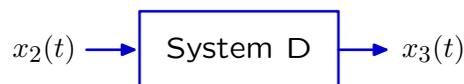
$$x_1(t) \longrightarrow \boxed{\text{System A}} \longrightarrow x_2(t)$$

The Fourier series coefficients at $k = 2, 6, 10, \ldots$ are zero in x_1 but these are not zero in x_2. Therefore the system could not be LTI.

$$x_1(t) \longrightarrow \boxed{\text{System B}} \longrightarrow x_3(t)$$

The Fourier series coefficients at $k = 2, 4, 6, 8, 10, \ldots$ are zero in x_1 but these are not zero in x_3. Therefore the system could not be LTI.
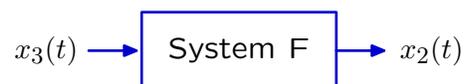
$$x_2(t) \longrightarrow \boxed{\text{System C}} \longrightarrow x_1(t)$$

All of the nonzero Fourier coefficients in x_1 are also present in x_2. Therefore the system could be LTI.

$$x_2(t) \longrightarrow \boxed{\text{System D}} \longrightarrow x_3(t)$$

The Fourier series coefficients at $k = 4, 8, 12, 16, \ldots$ are zero in x_2 but these are not zero in x_3. Therefore the system could not be LTI.
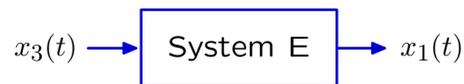
$$x_3(t) \longrightarrow \boxed{\text{System E}} \longrightarrow x_1(t)$$

All of the nonzero Fourier coefficients in x_1 are also present in x_3. Therefore the system could be LTI.

$$x_3(t) \longrightarrow \boxed{\text{System F}} \longrightarrow x_2(t)$$

All of the nonzero Fourier coefficients in x_2 are also present in x_3. Therefore the system could be LTI.

Enter a list of the systems that could **NOT** be LTI. If your list is empty, enter **none**.

answer = $\boxed{\qquad \text{A, B, D} \qquad}$
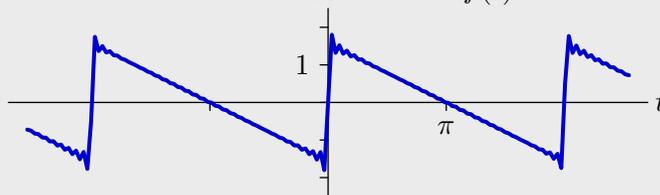
Engineering Design Problems

5. Overshoot

a. What function $f(t)$ has the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}?$$

You can evaluate the sum analytically or numerically. Either way, guess a closed form for $f(t)$ and then sketch it.

Here is a plot of the sum of the first 30 terms of $f(t)$.

It shows that $f(t)$ is a sawtooth waveform with period $T = 2\pi$ and

$$f(t) = \frac{\pi - t}{2}$$

for $0 < t < 2\pi$.

b. Confirm your conjecture for $f(t)$ by finding the Fourier series coefficients f_n for $f(t)$. Compare your result to the expression in the previous part. What happens to the cosine terms?

Let $g(t) = \frac{df(t)}{dt}$. Then $g(t) = -\frac{1}{2} + \pi\delta(t)$ for $-\pi < t < \pi$. The Fourier series coefficients of $g(t)$ are then

$$a_k = \begin{cases} 0 & k = 0 \\ \frac{1}{2} & k \neq 0 \end{cases}$$

For $k \neq 0$, we can find the Fourier series coefficients for $f(t)$ by dividing those for $g(t)$ by jk. The function $f(t)$ has no average value, so the Fourier coefficient for $k = 0$ is zero. Thus the Fourier series coefficients for $f(t)$ are

$$b_k = \begin{cases} 0 & k = 0 \\ \frac{1}{j2k} & k \neq 0 \end{cases}$$

Thus the Fourier series is

$$f(t) = \sum_{k=1}^{\infty} \frac{1}{j2k} \left(e^{jkt} - e^{-jkt} \right) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(kt)$$

The cosine terms are all zero because the function is odd.

c. Define the partial sum

$$f_N(t) = \sum_{n=1}^{N} \frac{\sin nt}{n},$$

Plot some $f_N(t)$'s. By what fraction does $f_N(t)$ overshoot $f(t)$ at worst? Does that fraction tend to zero or to a finite value as $N \to \infty$? If it is a finite value, estimate it.

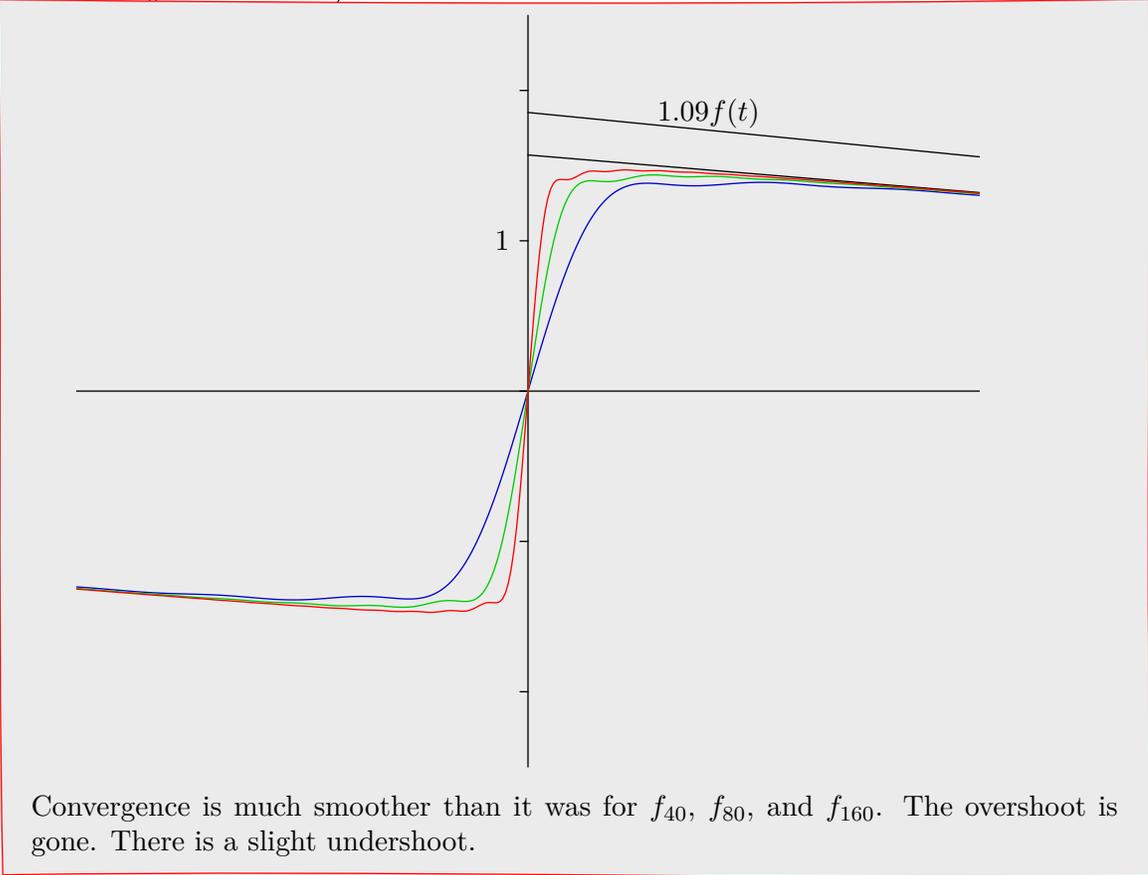The following plots show the partials sums of 40, 80, and 160 terms. It appears that the maximum value is approximately 1.09 times the value of $f(t)$, i.e., the overshoot is approximately 9%, as in the Gibb's overshoot to a square wave.
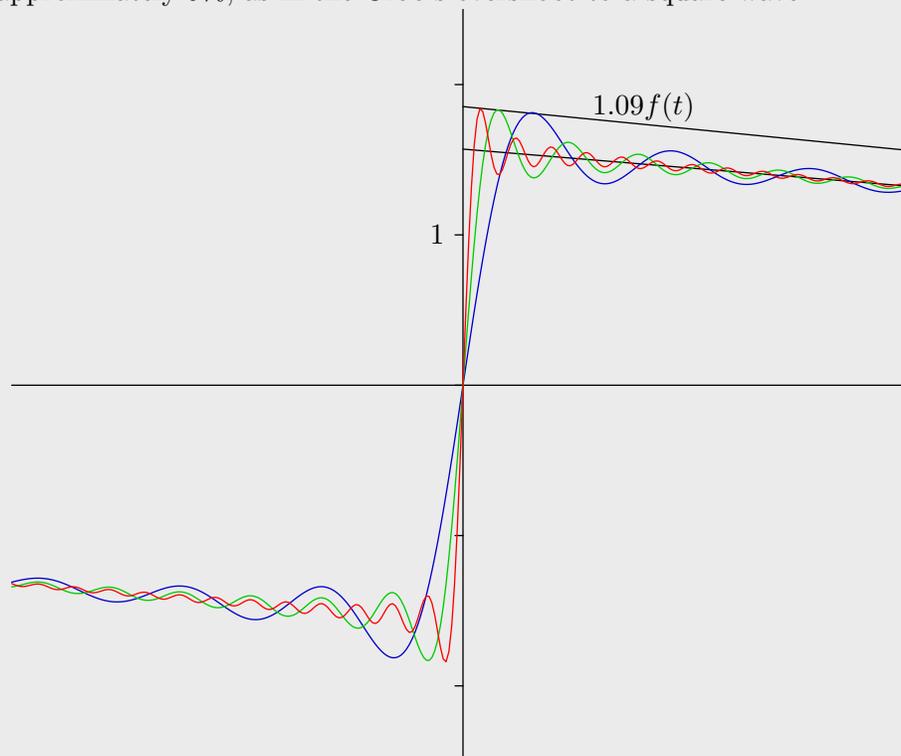
It appears that the maximum value is approximately 1.09 times the value of $f(t)$, i.e., the overshoot is approximately 9%, as in the Gibb's overshoot to a square wave.

d. Now define the average of the partial sums:

$$F_N(t) = \frac{f_1(t) + f_2(t) + f_3(t) + \cdots + f_N(t)}{N}$$

Plot some $F_N(t)$'s. Compare your plots with those of $f_N(t)$ that you made in the previous part, and qualitatively explain any differences.

The following plots show F_{40}, F_{80}, and F_{160}.

Convergence is much smoother than it was for f_{40}, f_{80}, and f_{160}. The overshoot is gone. There is a slight undershoot.

6.003 Signals and Systems
Fall 2011