Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control
and Signal Processing

Spring 2005 Final Exam ANSWER BOOKLET

YOUR NAME:

Recitation Hour:

- This is a closed book exam, but you may use SIX $8\frac{1}{2}$" $\times$ 11" sheets of notes (both sides). Calculators are not allowed.
- We would rather see you do 80% of the exam quite well than 100% of the exam quite poorly!
- Be clear and precise in your reasoning and **show all relevant work.**
- **If we can't read it, we can't/won't grade it!** So please write neatly and legibly.
- Verify that this answer booklet has pages numbered up to 26, and that the question booklet has pages numbered up to 7.
- You are to **hand in only this ANSWER booklet**. **No additional pages will be considered in the grading.** You may want to first work things through in the blank areas of the question booklet or on scratch paper, and then neatly transfer to thisr booklet the work you would like us to look at. Let us know if you need additional scratch paper.

Problem	**Your Score**
1 (20 points)	
2 (24 points)	
3 (20 points)	
4 (36 points)	
Total (100 points)	

Problem 1 (20 points)

The input to a particular stable LTI filter with frequency response

$$H(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

is a DT wide-sense stationary (WSS) process $w[\cdot]$ whose power spectral density (PSD) is $S_{ww}(e^{j\Omega}) = 9$ for all Ω. Denote the output of the system at time n by $y[n]$.

1(a) (5 points) Find a first-order difference equation relating the input and output of the system, and also explicitly determine the unit-sample response $h[n]$ of the system. (Do this carefully, as other answers will depend on it!) As a check, explicitly compute $\sum h[n]$ and compare the value you get with what you should expect for the given $H(e^{j\Omega})$. Is the system causal?

1(b) (5 points) Determine the mean $E\{y[n]\} = \mu_y$ and the autocorrelation $E\{y[n+m]y[n]\} = R_{yy}[m]$ of the WSS output process $y[\cdot]$. Your answer for the autocorrelation function should be written out explicitly, not left as an integral or sum. If you've done things correctly, you should find that the variance of $y[n]$ is 12; verify this explicitly.

You may find it helpful to recall the following identity for geometric series:

$$1 + \alpha + \cdots + \alpha^{m-1} = \frac{1 - \alpha^m}{1 - \alpha} \ .$$

1(c) (10 points) Specify completely the linear minimum-mean-square-error (LMMSE) causal one-step predictor for the process $y[\cdot]$. This predictor forms the LMMSE estimator $\widehat{y}[n+1]$ for $y[n+1]$, using all values of $y[k]$ for $k \leq n$. One way to do this is using your input-output equation from (a) to conjecture the form of this predictor, and then to verify your conjecture using the orthogonality condition that characterizes LMMSE estimation. Another way is to design an appropriate causal Wiener filter.

Use **either** of the above approaches to find the predictor, showing the main steps of your calculation. [If you have time at the end of the exam, then for **3 points of extra credit**, find the predictor using the other way too, and check that you get the same answer either way.]

Using **either** of the approaches mentioned above, determine the minimum mean square error (MMSE) associated with the predictor. Could the correct answer for the MMSE be larger than 12?

1(c) continued:

1(c) continued further: You can do the **extra credit** part here and on the next page, if you have time at the end.

1(c) continued still further:

Problem 2 (24 points)

Summarizing the problem statement, the received signal is

$$r[n] = Ap[n] + v[n] ,$$

where A is a random variable that is chosen by the transmitter; the receiver only knows the mean μ_A and variance σ_A^2 of A. Assume A is uncorrelated with the wide-sense stationary (WSS) white-noise process $v[\cdot]$ whose intensity is given to be σ_v^2, i.e., $S_{vv}(e^{j\Omega}) = \sigma_v^2$ for all Ω. Also take $p[\cdot]$ to be a known signal of finite energy

$$\mathcal{E} = \sum_n p^2[n] .$$

With $f[n]$ denoting the unit sample response of the receiver filter, we define

$$B = \sum_{n=-\infty}^{\infty} f[n]r[-n] = A(\sum f[n]p[-n]) + (\sum f[n]v[-n]) = \alpha A + V ,$$

where

$$\alpha = \sum f[n]p[-n] , \qquad V = \sum f[n]v[-n] .$$

Also

$$\mathcal{F} = \sum_n f^2[n] .$$

2(a) (4 points) Determine the mean and variance of V, and the cross-covariance σ_{AV} of A and V. All your answers can be written in terms of σ_v and $\mathcal{F}$.

2(b) (6 points) Determine the mean and variance of B, the cross-covariance σ_{AB} of A and B, and their correlation coefficient ρ_{AB}, all expressed in terms of the problem parameters and the simplified notation introduced above.

2(c) (4 points) Write down the LMMSE estimator of A that uses a measurement of B, i.e., find γ and μ in

$$\widehat{A} = \gamma B + \mu$$

so as to minimize $E[(A - \widehat{A})^2]$. Again, your answers should be expressed in terms of the problem parameters and the simplified notation introduced above.

2(d) (10 points) The minimum mean-square-error (MMSE) associated with the estimator in 2(c) can be written as

$$\sigma_A^2(1 - \rho_{AB}^2) \ .$$

Express this in terms of the problem parameters and the simplified notation above, and note that only α and $\mathcal{F}$ in your expression are affected by how we choose the filter $f[n]$. Use your expression to show *clearly and carefully* that the MMSE is minimized if $\alpha^2/\mathcal{F}$ is made as large as possible

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2(d) continued:

The Cauchy-Schwartz inequality can be used to show that

$$\frac{\alpha^2}{\mathcal{F}} \leq \mathcal{E} \ ,$$

with equality if and only if

$$f[n] = c\, p[-n]$$

for any nonzero constant c, which we can take to be 1 without loss of generality here. Note that with $f[n] = p[-n]$, we get $\alpha = \mathcal{E} = \mathcal{F}$. Using this, rewrite your expressions for the minimum MMSE and for the constants γ and μ in the LMMSE estimator in 2(c), in terms of just μ_A, σ_A, σ_v and $\mathcal{E}$.

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2(d) continued further: As a check on your answers, explicitly verify that your rewritten expressions on the preceding page behave reasonably as the parameters take various extreme values (pick at least three sets of extreme cases to check).

Problem 3 (20 points)

Summarizing the problem statement, suppose under hypothesis H_0 the random variable X is distributed uniformly in the interval $[-2, 2]$, while under hypothesis H_1 it is distributed uniformly in the interval $[-1, 1]$. The Neyman-Pearson (NP) decision rule announces 'H_1' if the likelihood ratio

$$\Lambda(x) = \frac{f_{X|H}(x|H_1)}{f_{X|H}(x|H_0)}$$

exceeds a properly selected threshold η, i.e., if $\Lambda(x) > \eta$; and announces 'H_0' if the likelihood ratio falls below the threshold, i.e., if $\Lambda(x) < \eta$.

3(a) (4 points) Sketch $\Lambda(x)$ as a function of x for $-2 < x < 2$.

3(b) (6 points) For η fixed at some value in each of the following ranges, specify P_D and P_{FA}:

1. η at some value strictly below 0.
2. η at some value strictly between 0 and 2.
3. η at some value strictly above 2.

3(c) (10 points) Suppose we choose $\eta = 2$. What is the probability that we get $\Lambda(X) = 2$ if H_0 holds? And what is the probability we get $\Lambda(X) = 2$ if H_1 holds?

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3(c) continued: With $\eta = 2$, you should see from the above computations that we will never get $\Lambda(x) > \eta$, but we might well get $\Lambda(x) = \eta$ or $\Lambda(x) < \eta$. Suppose we still announce 'H_0' when $\Lambda(x) < \eta$; however, when $\Lambda(x) = \eta$ we shall announce 'H_0' with probability α, and otherwise announce 'H_1'. What are P_{FA} and P_D with this *randomized* decision rule? Explain carefully.

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3(c) continued further: Draw the ROC that you get as α varies from 0 to 1, and also include the three points on the ROC that you computed in 3(b).

Problem 4 (36 points)

4(a) (12 points) A DT zero-mean wide-sense-stationary (WSS) process $e[\cdot]$ has autocorrelation function

$$R_{ee}[m] = \frac{\sin(\pi m/3)}{m} ,$$

for $m \neq 0$, and $R_{ee}[0] = \pi/3$. The process $x[\cdot]$ is defined by the relation

$$x[n] = (-1)^n \, e[n]$$

for all n. Show that $x[\cdot]$ is WSS, and sketch its power spectral density (PSD) $S_{xx}(e^{j\Omega})$ in the region $|\Omega| \leq \pi$. Is $x[\cdot]$ also *jointly* WSS with $e[\cdot]$?

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4(a) continued:

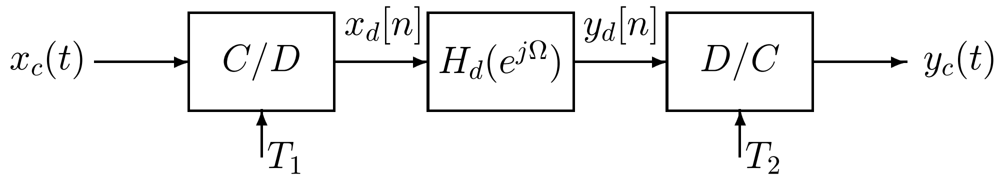
4(b) (12 points) The C/D converter in the figure below is a standard sampling converter, so $x_d[n] = x_c(nT_1)$. The D/C converter is an ideal bandlimited interpolating converter with reconstruction interval T_2. The frequency response of the DT system is

$$H_d(e^{j\Omega}) = \frac{j\Omega}{T_1}$$

for $|\Omega| < \pi$. With $T_2 = T_1$, and if

$$x_c(t) = \cos\left(\frac{5\pi t}{2T_1}\right) ,$$

what is $y_c(t)$? If now we make $T_2 = 2T_1$ but keep everything else the same, what is $y_c(t)$?



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4(b) continued:

4(c) (12 points) The dynamics of a synchronous electric generator are governed by a model of the form

$$\frac{d^2\theta(t)}{dt^2} + \beta\frac{d\theta(t)}{dt} + \alpha\sin\theta(t) = T(t)$$

where $\theta(t)$ is the (relative) angular position of the generator and $T(t)$ is the (normalized) external torque acting on it; the parameter α is positive, but β can be positive or negative. Write a state-space model of this system. Then, assuming $T(t)$ is constant at a positive value $T(t) = \overline{T}$, determine for what values of $\overline{T}$ the system will have:

(i) no equilibrium solutions;

(ii) one equilibrium solution with $\theta(t) = \overline{\theta}$ in the range $[0, 2\pi]$;

(iii) two equilibrium solutions with $\theta(t) = \overline{\theta}$ in the range $[0, 2\pi]$.

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4(c) continued: Write down the two linearized models computed at the two equilibrium solutions you found in part (iii) above, expressing them in the standard form

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t) \; .$$

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4(c) continued further: Determine the reachability of each of the linearized models by checking whether you can obtain an arbitrary closed-loop characteristic polynomial by LTI state feedback.

Additional work: