Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

MIDTERM, November 9, 2004

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| **Your Full Name:** | |
| **Recitation Instructor & Time :** | |

- This midterm is **closed book**, but **two** sheets of notes are allowed. Calculators will not be necessary and are not allowed.
- Put your name in the space indicated above, and your recitation time next to the name of your instructor.
- Check that you have pages numbered up to 16. This booklet contains spaces for **all** answers.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this booklet the work you would like us to look at. Let us know if you need additional scratch paper. All scratch paper will be collected at the end of the quiz. However, **only** this booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **three problems**, **weighted as indicated on the question booklet**.
- **DO NOT DISCUSS THIS MIDTERM WITH STUDENTS WHO HAVE NOT YET TAKEN IT TODAY!**

| **Problem** | **Your Score** |
|---|---|
| **1 (10 points)** | |
| **2 (40 points)** | |
| **3 (20 points)** | |
| **Total (70 points)** | |

**Problem 1 (10 points)**

Consider the single-input single-output $L$th-order CT LTI state-space system

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}x(t) \ , \quad y(t) = \mathbf{c}^T\mathbf{q}(t) + \mathtt{d}x(t) \ ,$$

whose transfer function is $H(s) = \nu(s)/a(s)$, where $a(s) = \det(s\mathbf{I} - \mathbf{A})$ is the characteristic polynomial of the system.

(You might find it useful to check that your answers to the questions below make sense for the case where $L = 1$.)

(a) (6 points) It turns out that for $\mathtt{d} \neq 0$ the *inverse* system has a state-space representation involving the same state vector $\mathbf{q}(t)$ but input $y(t)$ and output $x(t)$. Determine this state-space representation, i.e., express the quantities $\mathbf{A}_{in}, \mathbf{b}_{in}, \mathbf{c}_{in}^T, \mathtt{d}_{in}$ in the state-space representation below in terms of $\mathbf{A}, \mathbf{b}, \mathbf{c}^T, \mathtt{d}$:

$$\dot{\mathbf{q}}(t) = \mathbf{A}_{in}\mathbf{q}(t) + \mathbf{b}_{in}y(t) \ , \quad x(t) = \mathbf{c}_{in}^T\mathbf{q}(t) + \mathtt{d}_{in}y(t) \ .$$

(b) (4 points) Determine the degree of the polynomial $\nu(s)$ (defined by the expression for $H(s)$ given above) if $\mathtt{d} \neq 0$? (Hint: Either figure out how $\nu(s)$ is related to the characteristic polynomial of $\mathbf{A}_{in}$, or think about the representation of the transfer function $H(s)$ of the original system in modal coordinates.)

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Begin work for Problem 1(a) here:

(6 points)

$\mathbf{A}_{in} =$ $\mathbf{b}_{in} =$

$\mathbf{c}_{in}^T =$ $\mathtt{d}_{in} =$

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Problem 1(b):

(4 points)

Degree of $\nu(s) =$
(Explain reasoning above)

**Problem 2 (40 points)**

Some parts of this problem build on previous parts, so it is important that you proceed carefully, and check answers as you go.

A particular DT WSS random process $\kappa[n]$ has autocorrelation function

$$R_{\kappa\kappa}[m] = 10\delta[m] + 3\gamma\Big(\delta[m-1] + \delta[m+1]\Big) .$$

(a) (5 points) What are the most positive and most negative values that $\gamma$ can take in this instance? Determine the mean and variance of $\kappa[n]$, and also the correlation coefficient between $\kappa[n]$ and $\kappa[n-1]$. (If you have trouble deducing the mean from the autocorrelation function, perhaps you'll see another way to deduce it after you do part (b) below.)

**For the rest of this problem, assume $\gamma = 1$.**

(b) (8 points) Show that $\kappa[n]$ can be generated as the output of an appropriate stable first-order state-space system driven from time $-\infty$ by a (zero-mean) white process $w[\cdot]$ of unit intensity, so $w[n]$ has variance 1. (Hint: First consider what unit sample response or transfer function you would want this system to have.) Explicitly write down this state-space system in the following form:

$$q[n+1] = \alpha q[n] + \beta w[n] \ , \quad \kappa[n] = \xi q[n] + \mathtt{d} w[n] \ ,$$

with appropriately chosen values of the coefficients $\alpha, \beta, \xi, \mathtt{d}$. Be sure to explain your reasoning!

(You might find more than one first-order state-space model that will accomplish the job, but any one of them will suffice as an answer for our purposes.)

(c) (8 points) Suppose we have another first-order state-space system, driven by the colored process $\kappa[n]$ that was produced by the system in (b). Let $p[n]$ denote the state variable of this system, and assume the output $y[n]$ of this system can be measured. The system thus takes the form

$$p[n+1] = ap[n] + b\kappa[n] \ , \quad y[n] = p[n] + v[n] \ .$$

Here $a$ and $b$ are some fixed *nonzero* scalar parameters whose precise values don't matter to us, and $v[n]$ is a (zero-mean) white measurement-noise process with variance $\sigma^2$ and is uncorrelated with $w[\cdot]$. Combine this system description with your result from (b) to *carefully* write down a second-order state-space model with state variables $q[n]$ and $p[n]$, white input $w[n]$, and measured output $y[n]$. Also determine the eigenvalues and eigenvectors associated with the system (i.e., the eigenvalues and eigenvectors of the one-step state transition matrix of this system). As a check, one of the eigenvalues of your model should turn out to be 0; if it isn't, then you have made an error somewhere!

(d) (6 points) Determine what conditions, if any, have to be satisfied by the various coefficients in this problem for the second-order system you derived in (c) to be:

(i) reachable from the input signal $w[n]$?

(ii) observable in the output signal $y[n]$?

For each of the above cases, also specify which modes of this second-order system become unreachable or unobservable when the respective conditions are *not* satisfied.

(e) (9 points) Suppose $w[n]$, $\kappa[n]$ and $v[n]$ cannot be measured, although their properties specified above are known. However, as mentioned before, $y[n]$ is measured. Write down in detail the equations of a second-order observer to propagate estimates $\widehat{q}[n]$ and $\widehat{p}[n]$ of $q[n]$ and $p[n]$ respectively for all $n \geq 0$. Also write down a second-order state-space model describing the evolution of the errors $\widetilde{q}[n] = q[n] - \widehat{q}[n]$ and $\widetilde{p}[n] = p[n] - \widehat{p}[n]$. Pick the observer gains to put both eigenvalues of the error model at 0.

(f) (4 points) For the observer you designed in (e), obtain an expression for the steady-state variance of $\widetilde{p}[n]$, expressed in terms of $a$, $b$ and $\sigma$.

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Begin work for Problem 2(a) here:

(5 points)

Most positive value of $\gamma =$

Most negative value of $\gamma =$

Mean of process $\kappa[n] =$
(Make sure to explain your reasoning above.)

Variance of $\kappa[n] =$

Correlation coefficient between $\kappa[n]$ and $\kappa[n-1]$:

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Begin work for Problem 2(b) here:

Problem 2(b) continued:

Generating $\kappa[n]$ as the output of an appropriate stable first-order state-space system driven from time $-\infty$ by a (zero-mean) white process $w[\cdot]$ of unit intensity:

(8 points)

$\alpha =$ $\beta =$

$\xi =$ $\mathtt{d} =$

Problem 2(c):

Given $p[n+1] = ap[n] + b\kappa[n]$ , $\quad y[n] = p[n] + v[n]$ , write down a second-order state-space model with state variables $q[n]$ and $p[n]$, white input $w[n]$, and measured output $y[n]$:

(8 points)

Eigenvalues: $\lambda_1 =$ $\qquad\qquad\qquad\qquad$ $\lambda_2 =$
(Check that one of the eigenvalues is zero!)

Eigenvectors: $\mathbf{v}_1 =$ $\qquad\qquad\qquad\qquad$ $\mathbf{v}_2 =$

Problem 2(d):

(6 points)

(i) Conditions for reachability:

Unreachable modes when these conditions are violated:

(ii) Conditions for observability:

Unobservable modes when these conditions are violated:

Problem 2(e): (9 points)

Second-order observer to propagate estimates $\widehat{q}[n]$ and $\widehat{p}[n]$ of $q[n]$ and $p[n]$ respectively for all $n \geq 0$:

Second-order state-space model describing the evolution of the errors $\widetilde{q}[n] = q[n] - \widehat{q}[n]$ and $\widetilde{p}[n] = p[n] - \widehat{p}[n]$:

Continue Problem 2(e) on next page $\longrightarrow\longrightarrow$

Problem 2(e) continued:

Observer gains to put both eigenvalues of the error model at 0:

Problem 2(f):

(4 points)

Expression for the steady-state variance of $\widetilde{p}[n]$:

**Problem 3 (20 points)**

A certain zero-mean CT WSS signal $y(t)$ with autocorrelation function $R_{yy}(\tau)$ and corresponding power spectral density (PSD) $S_{yy}(j\omega)$ is transmitted through a channel. The characteristics of the channel and receiver are such that the received signal $x(t)$ is of the form

$$x(t) = by(t) + v(t) \ .$$

The quantity $v(t)$ represents receiver noise, and is a zero-mean WSS noise process with autocorrelation function $R_{vv}(\tau)$ and corresponding PSD $S_{vv}(j\omega)$, and is uncorrelated with $y(\,\cdot\,)$, i.e. $R_{yv}(\tau) = 0$. The quantity $b$ is a random variable that is independent of $y(\,\cdot\,)$ and $v(\,\cdot\,)$, and that takes the value 1 or 0 for all time; it can be thought of as indicating whether the channel works ($b = 1$) or doesn't ($b = 0$). The probability that $b = 1$ is $p$.

(a) (15 points) Compute $S_{yx}(j\omega)$ and $S_{xx}(j\omega)$, then find the frequency response $H(j\omega)$ of a stable and possibly noncausal LTI (Wiener) filter that takes as input the received signal $x(t)$ and produces as output the (linear) minimum mean-square-error (MMSE) estimate $\widehat{y}(t)$ of the transmitted signal $y(t)$, i.e., find the filter that minimizes $E[\{y(t) - \widehat{y}(t)\}^2]$. Express your answer in terms of quantities specified in the problem statement. Check that your filter specializes to what you expect when $p = 1$ and $p = 0$.

(b) (5 points) Find an expression for the PSD $S_{ee}(j\omega)$ of the error $e(t) = y(t) - \widehat{y}(t)$ associated with the optimum filter you designed in (a), again expressing your answer in terms of quantities specified in the problem statement. Again check that your expression reduces to what you expect when $p = 1$ and $p = 0$.

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Begin work for Problem 3(a) here:

Problem 3(a) continued:

(15 points)

$S_{yx}(j\omega) =$

$S_{xx}(j\omega) =$

Wiener filter frequency response $H(j\omega) =$

Checks for $p = 1$ and $p = 0$:

Problem 3(b):

(5 points)

$S_{ee}(j\omega) =$

Checks for $p = 1$ and $p = 0$:

Additional work: