Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 1, March 15, 2005

Answer Booklet

Your Full Name:	
Recitation Instructor & Time :	at o'clock

- This quiz is **closed book**, but **two** sheets of notes are allowed. Calculators will not be necessary and are not allowed.
- Put your name in the space indicated above, and your recitation time next to the name of your instructor.
- Check that this answer booklet has pages numbered up to 16. The booklet contains spaces for **all** relevant reasoning and answers.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this booklet the work you would like us to look at. Let us know if you need additional scratch paper. **Only** this booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **two problems**, **weighted as indicated on the question booklet**.
- **DO NOT DISCUSS THIS QUIZ WITH 6.011 STUDENTS WHO HAVE NOT YET TAKEN IT TODAY!**

Problem	Your Score
1 (20 points)	
2 (30 points)	
Total (50 points)	

Problem 1 (20 points)

Suppose $x(t) = y(t)\cos(\omega_o t + \Theta)$, where: $y(t)$ is a wide-sense stationary (WSS) process with mean μ_y and autocovariance function $C_{yy}(\tau)$; ω_o is a known constant; and Θ is a random variable that is independent of $y(\,\cdot\,)$ and is uniformly distributed in the interval $[0, 2\pi]$. Do part (a) below especially carefully, because (b) and (c) depend on it to some extent!

You might find it helpful in one or more parts of the problem to recall that

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)] \ .$$

(a) (8 points) Find the mean $\mu_x(t)$ and autocorrelation function $E[x(t+\tau)x(t)]$ of the process $x(\,\cdot\,)$. Also find the cross-correlation function $E[y(t+\tau)x(t)]$. Explain precisely what features of your answers tell you that: (i) $x(\,\cdot\,)$ is a WSS process; and (ii) $x(\,\cdot\,)$ and $y(\,\cdot\,)$ are jointly WSS.

Begin work for Problem 1(a) here:

$$\begin{aligned} \mu_x(t) &= E[y(t)\cos(\omega_0 t + \theta)] \\ &= E[y(t)]E[\cos(\omega_0 t + \theta)] \qquad \text{since } y(\cdot) \text{ and } \theta \text{ are independent} \\ &= \mu_y \times \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega_0 + \theta)d\theta \\ &= 0 \end{aligned}$$

Note that the factor $\frac{1}{2\pi}$ comes from the pdf of θ.

$$\begin{aligned} E[x(t+\tau)x(t)] &= E[y(t+\tau)y(t)]E[\cos(\omega_0(t+\tau)+\theta)\cos(\omega_0 t+\theta)] \qquad \text{since } y(\cdot) \text{ and } \theta \text{ are independent} \\ &= \frac{R_{yy}(\tau)}{2}E[\cos(\omega_0(t+\tau) + \omega_0 t + 2\theta) + \cos(\omega_0\tau)] \\ &= \frac{R_{yy}(\tau)}{2}\cos(\omega_0\tau) \qquad \text{since } E[\cos(\xi + 2\theta)] = 0 \end{aligned}$$

$$\begin{aligned} E[y(t+\tau)x(t)] &= E[y(t+\tau)y(t)]E[\cos(\omega_0 t + \theta)] \\ &= 0 \end{aligned}$$

Problem 1(a) continued:

Mean $\mu_x(t) = 0$

Autocorrelation function $E[x(t+\tau)x(t)] = \frac{R_{yy}(\tau)}{2}\cos(\omega_0\tau)$

Cross-correlation function $E[y(t+\tau)x(t)] = 0$

$x(\cdot)$ is WSS because: $\mu_x(t) = 0$,doesn't depend on t; and $E[x(t+\tau)x(t)]$ only depends on the lag τ, not t.

$x(\cdot)$ and $y(\cdot)$ are jointly WSS because: $y(t)$ is WSS too, and $E[y(t+\tau)x(t)]$ doesn't depend on, i.e. vary with, t.

1(b) (7 points) Suppose $C_{yy}(\tau) = e^{-|\tau|}$ and $\mu_y \neq 0$. Obtain an expression for the power spectral density (PSD) $S_{yy}(j\omega)$ in this case, and draw a *fully labeled* sketch of it. Also obtain an expression for the PSD $S_{xx}(j\omega)$, and draw a fully labeled sketch of it.

Begin work for Problem 1(b) here:

$$F\{C_{yy}(\tau)\} = \frac{1}{1+j\omega} + \frac{1}{1-j\omega} = \frac{2}{1+\omega^2}$$

$$F\{R_{yy}(\tau)\} = F\{C_{yy} + \mu_y^2\} = \frac{2}{1+\omega^2} + \mu_y^2 2\pi\delta(\omega)$$

$$\begin{aligned} F\{R_{xx}(\tau)\} &= F\{\frac{R_{yy}(\tau)}{2}\cos(\omega_0\tau)\} \\ &= \frac{1}{2\pi}F\{\frac{R_{yy}(\tau)}{2}\} * (\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)) \\ &= \frac{1}{4}(S_{yy}(j(\omega-\omega_0)) + S_{yy}(j(\omega+\omega_0))) \end{aligned}$$

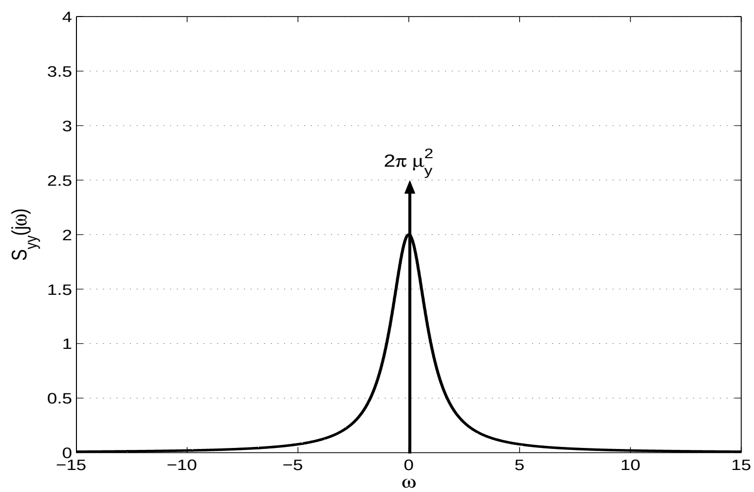

Figure 1: Plot of $S_{yy}(j\omega)$.

Problem 1(b) continued:

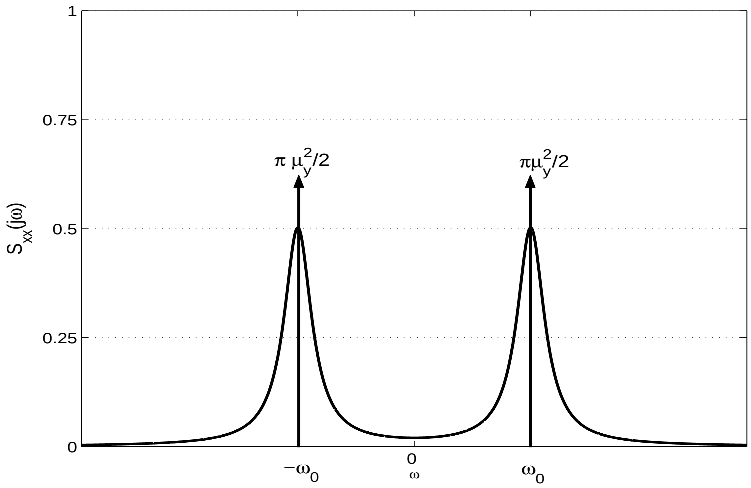

Figure 2: Plot of $S_{xx}(j\omega)$.

$S_{yy}(j\omega) = \frac{2}{1+\omega^2} + 2\pi\mu_y^2\delta(\omega)$

Fully labeled sketch:

$S_{xx}(j\omega) = \frac{1}{4}\left(S_{yy}(j(\omega-\omega_0)) + S_{yy}(j(\omega+\omega_0))\right)$

Fully labeled sketch:

1(c) (5 points) With the properties of $y(t)$ specified as in (b), is $y(t)$ ergodic in mean value? Be sure to give a reason for your answer! A somewhat harder question: is $x(t)$ ergodic in mean value? Again, describe your reasoning. If you are able to evaluate either of the following integrals on the basis of your answers here, please do so:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, dt \ ,$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\cos(w_o t + \Theta)\, dt \ .$$

Begin work for Problem 1(c) here:

Since $C_{yy}(\tau) \to 0$ as $|\tau| \to \infty$, y is ergodic in mean value (this is a sufficient condition), i.e.:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y(t)\, dt = \mu_y$$

$$C_{xx}(\tau) = R_{xx}(\tau) = \frac{R_{yy}(\tau)}{2}\cos(\omega_0 \tau) \qquad \text{since } \mu_x = 0$$

This no longer $\to 0$ as $|\tau| \to \infty$, so we have to use the more refined criterion derived on homework, and check whether: $\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})C_{xx}(\tau)\, d\tau = 0$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} (1-\frac{|\tau|}{2T})C_{xx}(\tau)\, d\tau = \lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} (1-\frac{|\tau|}{2T})\frac{1}{2}(e^{-|\tau|}+\mu_y^2)\cos(\omega_0\tau)\, d\tau$$

We have:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} (1-\frac{|\tau|}{2T})e^{-|\tau|}\cos(\omega_0\tau)\, d\tau \le \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} e^{-|\tau|} d\tau = 0$$

and:

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-2T}^{2T} (1-\frac{|\tau|}{2T})\frac{1}{2}\mu_y^2\cos(\omega_0\tau)\, d\tau = 0$$

because the integral oscillates between values bounded by $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\mu_y^2 \cos\theta d\theta$, which is finite.

Problem 1(c) continued:

Is $y(t)$ ergodic in mean value? (Explain!): Yes, $C_{yy}(\tau) \to 0$, so $\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}(1 - \frac{|\tau|}{2T})C_{yy}(\tau)\,d\tau = 0$

Is $x(t)$ ergodic in mean value? (Explain!): Yes, $\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}(1 - \frac{|\tau|}{2T})C_{xx}(\tau)\,d\tau = 0$

If possible, evaluate the following integrals:

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t)\,dt \;= \mu_y$$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t)\cos(w_o t + \Theta)\,dt \;= \mu_x = 0$$

Problem 2 (30 points)

Let $y[n]$ be a wide-sense stationary (WSS) process with autocorrelation function

$$R_{yy}[m] = 9\Big(\delta[m] - \alpha\delta[m-1] - \alpha\delta[m+1]\Big) .$$

where α is **positive**, $\alpha > 0$.

(a) (3 points) What is the maximum value α can take? Explain your reasoning. If α is increased towards its maximum value, does the power of the signal shift to lower or higher frequencies?

Begin work for Problem 2(a):

We need $S_{yy}(e^{j\Omega}) = 9(1 - 2\alpha\cos(\Omega)) \geq 0$, so $\alpha \leq \frac{1}{2}$ (and > 0, given).
As α increases, power shifts to higher frequencies.

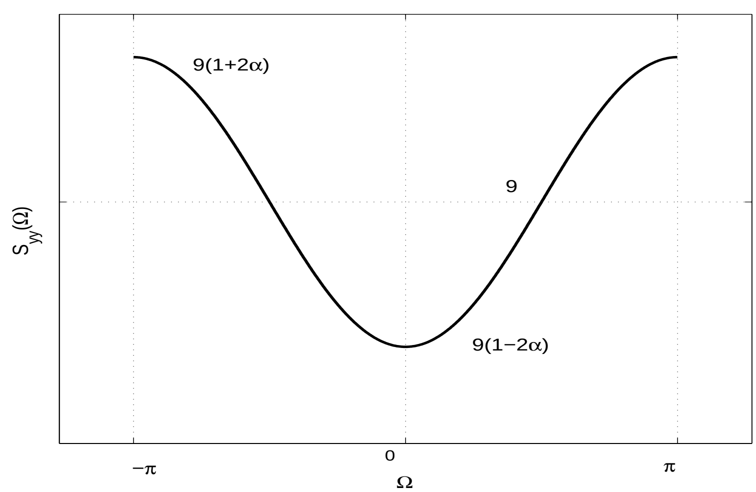

Figure 3: Plot of $S_{yy}(e^{j\Omega})$.

Largest $\alpha = \frac{1}{2}$

Reasoning: $S_{yy}(e^{j\Omega}) = 9(1 - 2\alpha\cos(\Omega)) \geq 0$

For increasing α, signal power shifts to **higher** frequencies: See plot of $S_{yy}(e^{j\Omega})$.

(2b) (4 points) Determine the following (expressed in terms of α, if necessary):

(i) $E\{y[n]\}$ and $E\{y^2[n]\}$;

(ii) the correlation coefficient ρ between $y[4]$ and $y[5]$.

Work for Problem 2(b):

$E[y[n]] = 0$ since $S_{yy}(e^{j\Omega})$ has no impulse at $\Omega = 0$.

$$E[y^2[n]] = R_{yy}[0] = 9$$

$$\begin{aligned}\rho &= \frac{cov(y[4], y[5])}{\text{std.dev.}(y[4])\,\text{std.dev.}(y[5])} \\ &= \frac{R_{yy}[1]}{(\sqrt{R_{yy}[0]})^2} \\ &= -\alpha\end{aligned}$$

($C_{yy} = R_{yy}$, since $\mu_y = 0$)

$E\{y[n]\} = 0$

$E\{y^2[n]\} = 9$

Correlation coefficient $\rho_{y[4],y[5]} = -\alpha$

(2c) (4 points) Suppose we are told that we will be given the measurement $y[4]$, and we want to find the linear minimum mean-square-error (LMMSE) estimator of $y[5]$ in terms of $y[4]$. Find the estimator, and determine the associated (minimum) mean square error (MSE).

Work for Problem 2(c):

$$\begin{aligned}\hat{y}[5] &= \mu_y[5] + \rho\frac{\sigma_y[5]}{\sigma_y[4]}(y[4] - \mu_y[4]) \\ &= -\alpha y[4]\end{aligned}$$

$$\begin{aligned}MSE &= var(y[5])(1-\rho^2) \\ &= 9(1-\alpha^2)\end{aligned}$$

LMMSE estimator of $y[5]$ in terms of $y[4]$ is $\widehat{y}[5] = ay[4] + b = -\alpha y[4]$

Associated (minimum) MSE $= 9(1-\alpha^2)$

(2d) (8 points) Suppose $x[n] = y[n] + w[n]$, where $w[n]$ is a white process that is uncorrelated with $y[\cdot]$ and has power spectral density $S_{ww}(e^{j\Omega}) = 9\alpha^2$. Determine the power spectral density $S_{xx}(e^{j\Omega})$ and show that it can be written in the form

$$S_{xx}(e^{j\Omega}) = K(1 - \beta e^{-j\Omega})(1 - \beta e^{j\Omega})$$

for K and β that you should determine (expressed in terms of α, if necessary). Also determine the cross-power spectral density $S_{yx}(e^{j\Omega})$ in terms of α.

Begin work for Problem 2(d):

Since $y[\cdot]$ and $w[\cdot]$ are uncorrelated, and $E[w[n]] = 0$.

$$\begin{aligned} S_{xx}(e^{j\Omega}) &= S_{yy}(e^{j\Omega}) + S_{ww}(e^{j\Omega}) \\ &= 9(1 - \alpha e^{-j\Omega} - \alpha e^{j\Omega}) + 9\alpha^2 \\ &= K(1 + \beta^2 - \beta e^{-j\Omega} - \beta e^{j\Omega}) \end{aligned}$$

Yes, for $\beta = \alpha$, $K = 9$.

$R_{yx}[m] = R_{yy}[m]$, since y and w are uncorrelated and $E[w[n]] = 0$, so $S_{yx}(e^{j\Omega}) = S_{yy}(e^{j\Omega}) = 9(1 - 2\alpha\cos(\Omega))$.

$S_{xx}(e^{j\Omega}) = S_{yy}(e^{j\Omega}) + S_{ww}(e^{j\Omega}) = 9(1 - \alpha e^{-j\Omega} - \alpha e^{j\Omega}) + 9\alpha^2$

Also, $S_{xx}(e^{j\Omega}) = K(1 - \beta e^{-j\Omega})(1 - \beta e^{j\Omega})$ with

$K = 9$ $\beta = \alpha$

$S_{yx}(e^{j\Omega}) = S_{yy}(e^{j\Omega}) = 9(1 - 2\alpha\cos(\Omega))$

(2e) (5 points) Determine the frequency response $H(e^{j\Omega})$ of the noncausal Wiener filter that produces the LMMSE estimate $\widehat{y}[n]$ of $y[n]$ in terms of measurements of the entire process $x[\cdot]$.

Work for Problem 2(e):

$$\begin{aligned} H(e^{j\Omega}) &= \frac{S_{yx}(e^{j\Omega})}{S_{xx}(e^{j\Omega})} \\ &= \frac{S_{yy}(e^{j\Omega})}{S_{yy}(e^{j\Omega}) + S_{ww}(e^{j\Omega})} \\ &= \frac{9(1 - 2\alpha\cos\Omega)}{9(1 + \alpha^2 - 2\alpha\cos\Omega)} \end{aligned}$$

$H(e^{j\Omega}) = \frac{(1-2\alpha\cos\Omega)}{(1+\alpha^2-2\alpha\cos\Omega)}$

(2f) (6 points) Determine the frequency response $G(e^{j\Omega})$ of the *causal* Wiener filter that at time n uses measurements of $x[k]$ for all present and past times $k \leq n$ to produce an LMMSE prediction of the *measurement* at the next step, i.e., an LMMSE estimate $\widehat{x}[n+1]$ of $x[n+1]$. Also determine the associated MSE.

Begin work for Problem 2(f):

$$G(e^{j\Omega}) = \frac{1}{M_{xx}(e^{j\Omega})}\left[\frac{e^{j\Omega}S_{xx}(e^{j\Omega})}{M_{xx}(e^{-j\Omega})}\right]_+$$

where, $M_{xx}(e^{j\Omega}) = \sqrt{K}(1-\beta e^{-j\Omega}) = 3(1-\alpha e^{-j\Omega})$, stable and causal with a stable and causal inverse. So,

$$\begin{aligned} G(e^{j\Omega}) &= \frac{1}{3(1-\alpha e^{-j\Omega})}\left[e^{j\Omega}3(1-\alpha e^{-j\Omega})\right]_+ \\ &= \frac{-\alpha}{1-\alpha e^{-j\Omega}} \end{aligned}$$

$$\begin{aligned} \text{MSE} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|[e^{j\Omega}3(1-\alpha e^{-j\Omega})]_-\right|^2 d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|3e^{j\Omega}|^2 d\Omega \\ &= 9 \end{aligned}$$

$G(e^{j\Omega}) = \frac{-\alpha}{1-\alpha e^{-j\Omega}}$

Associated MSE $= 9$