MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.011 Introduction to Communication, Control and Signal Processing
Fall 2003

SECOND EVENING EXAM
Tuesday, November 18, 7:30 PM – 9:30 PM

- Put your name on **each** page of this booklet. Specify your recitation instructor and your recitation time.
- This is a closed book exam, but three $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides) are allowed. They can be as big as $8\frac{1}{2}'' \times 11''$ or as small as you'd like and you can write on one side or two sides of each, but only three sheets are allowed.
- Everything on the notes must be in your original handwriting (i.e., material cannot be xeroxed from solutions, tables, books, etc.).
- You have two hours for this exam.
- Calculators are NOT allowed.
- We will NOT provide a table of transforms.
- There are 4 problems on the exam with the percentage for each part and the total percentage for each problem as indicated. Note that the problems do not all have the same total percentage.
- Make sure you have seen all ? numbered sides of this answer booklet.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- We tried to provide ample space for you to write in. However, the space provided is not an indication of the length of the explanation required. Short, to the point, explanations are preferred to long ones that show no understanding.
- Please be neat—we cannot grade what we cannot decipher.

All work and answers must be in the space provided on the exam booklet. You are welcome to use scratch pages that we provide, but when you hand in the exam we will not accept any pages other than the exam booklet. There will be absolutely no exceptions.

Exam Grading

As in the first exam, in grading all of the 6.011 exams we will be focusing on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess, from your work, your level of understanding. On each part of an exam question we will also indicate the percentage of the total exam grade represented by that part, and your numerical score on the exam will then be calculated accordingly.

Our assessment of your level of understanding will be based upon what is given in your solution. A correct answer with no explanation will not receive full credit, and may not receive much—if any. An incorrect final answer having a solution and explanation that shows excellent understanding quite likely will receive full (or close to full) credit.

This page is intentionally left blank. Use it as scratch paper.
No work on this page will be evaluated.

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Full Name:____

	Points	Grader
1(a)		
1(b)		
2(a)		
2(b)		
2(c)		
2(d)		
3(a)		
3(b)		
3(c)		
3(d)		
4(a)		
4(b)		
4(c)		
Total		

Problem 1 (14%)

A pulse-amplitude modulation (PAM) transmitter sends

$$s(t) = \sum_{n=-\infty}^{\infty} a[n]p(t - nT)$$

to convey a message sequence $a[n]$. At the receiver, a sequence $b[n]$ is obtained by sampling $s(t)$ at integer multiples of T, i.e., the receiver's output sequence is $b[n] = s(nT)$.

Given below are two possible choices for $P(j\omega)$, the Fourier transform of the pulse shape $p(t)$. For each choice, specify whether there are values of T for which there will be no intersymbol interference (ISI), i.e., whether there are values of T for which $b[n] = ca[n]$ for all $a[n]$ and n, with c being a constant. If your answer for a particular $P(j\omega)$ is "yes", specify all possible values of T for which there is no ISI. Give a brief justification for each answer.

(a) (7%)

$$P_1(j\omega) = \frac{2\sin(\omega)}{\omega}$$

NO ISI POSSIBLE? **YES** **NO**

If yes, for which values of T?

Brief justification:

(b) (7%)

$$P_2(j\omega) = \begin{cases} e^{-j\omega/2} & \text{for } |\omega| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

NO ISI POSSIBLE? **YES** **NO**

If yes, for which values of T?

Brief justification:

Problem 2 (30%)

Consider the following state-space description of a causal, discrete-time system:

$$\begin{aligned} \mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \\ y[n] &= \mathbf{c}^T\mathbf{q}[n] \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{bmatrix}$$

The matrix $\mathbf{A}$ has the following eigenvalues and associated eigenvectors:

$$\lambda_1 = 2, \quad \mathbf{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = \frac{1}{2}, \quad \mathbf{v_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We want to describe the system in terms of a new set of variables, $\mathbf{r}[n] = \begin{pmatrix} r_1[n] \\ r_2[n] \end{pmatrix}$, where

$$\begin{aligned} r_1[n] &= w_{11}q_1[n] + w_{12}q_2[n] \\ r_2[n] &= w_{21}q_1[n] + w_{22}q_2[n]. \end{aligned}$$

In other words, we want to describe the system using the following equations

$$\begin{aligned} \mathbf{r}[n+1] &= \tilde{\mathbf{A}}\mathbf{r}[n] + \tilde{\mathbf{b}}x[n] \\ y[n] &= \tilde{\mathbf{c}}^T\mathbf{r}[n] \end{aligned}$$

(a) (7%) Determine $w_{11}, w_{12}, w_{21}, w_{22}$ such that the state variables $r_1[n]$ and $r_2[n]$ obey the following evolution equations for the zero-input responses (ZIRs) for $n \geq 0$, given initial conditions $r_1[0]$ and $r_2[0]$:

$$\begin{aligned} r_1[n] &= \alpha_1\lambda_1^n \\ r_2[n] &= \alpha_2\lambda_2^n. \end{aligned}$$

Furthermore, determine α_1 and α_2 when $\mathbf{q}[0] = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$w_{11} =$ $w_{12} =$ $w_{21} =$ w_{22}

$\alpha_1 =$ $\alpha_2 =$

Work to be looked at:

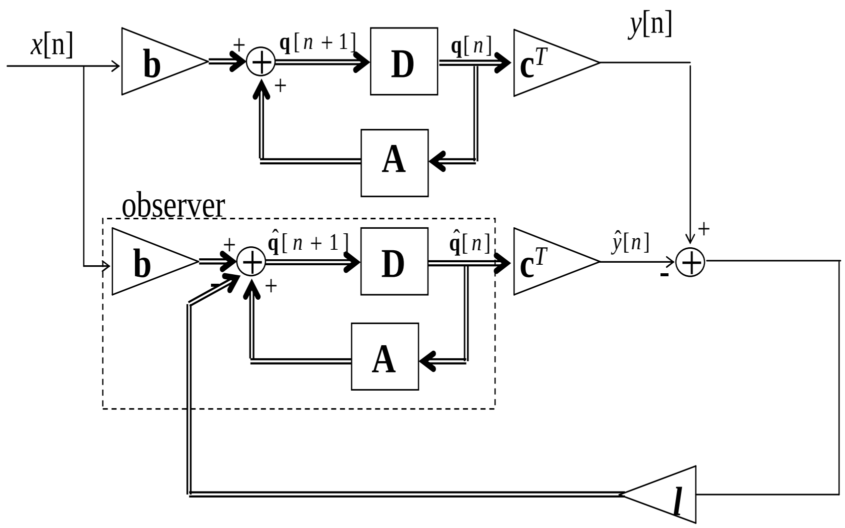
Work to be looked at (continued):

(b) (7%) Determine all possible values of $\tilde{\mathbf{b}}$ for which the system is bounded-input/bounded-output (BIBO) stable regardless of the value of $\tilde{\mathbf{c}}^T$.

$\tilde{\mathbf{b}} =$

Work to be looked at:

We would like to observe the evolution of the system using the following observer:



where $\hat{\mathbf{q}}$ is the state estimate and $\tilde{\mathbf{q}}[n] = \mathbf{q}[n] - \hat{\mathbf{q}}[n]$ is the error.
For parts (c) and (d), suppose that $\tilde{\mathbf{c}}^T$ is one of the three possibilities given below,

$$\tilde{\mathbf{c}}_1^T = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \tilde{\mathbf{c}}_2^T = \begin{pmatrix} 0 & 2 \end{pmatrix}, \quad \tilde{\mathbf{c}}_3^T = \begin{pmatrix} 1 & 2 \end{pmatrix}.$$

Answer the following questions and give brief justifications for your answers.

(c) (8%) For each of the above vectors $\tilde{\mathbf{c}}_i^T$, specify whether we can find an $\mathbf{l}$-vector such that $\tilde{q}_1[n]$ and $\tilde{q}_2[n]$ converge to 0 as $n \to \infty$? In other words, specify whether we can choose an observer for which the state-estimation error decays asymptotically to 0. Clearly state your reasoning.

For $\tilde{\mathbf{c}}_1^T$ can we find such an $\mathbf{l}$-vector ?	**YES**	**NO**
For $\tilde{\mathbf{c}}_2^T$ can we find such an $\mathbf{l}$-vector ?	**YES**	**NO**
For $\tilde{\mathbf{c}}_3^T$ can we find such an $\mathbf{l}$-vector ?	**YES**	**NO**

Reasoning:

Reasoning (continued):

(d) (8%) For each $\tilde{\mathbf{c}}_i^T$ for which the state-estimation error can be made to decay asymptotically to 0, can the decay rates for $\tilde{q}_1[n]$ and $\tilde{q}_2[n]$ be set to arbitrary values by choice of the observer gain vector l? Please circle your answer and again clearly state your reasoning.

Under $\tilde{\mathbf{c}}_1^T$ can the decay rate be set arbitrarily?

YES **NO** **NOT APPLICABLE**

Under $\tilde{\mathbf{c}}_2^T$ can the decay rate be set arbitrarily?

YES **NO** **NOT APPLICABLE**

Under $\tilde{\mathbf{c}}_3^T$ can the decay rate be set arbitrarily?

YES **NO** **NOT APPLICABLE**

Reasoning:

Problem 3 (30%)

A signal $s[n]$ to be retrieved from storage is subject to errors due to faulty electronics. The retrieved signal $r[n]$ can be written as

$$r[n] = s[n] + e[n]$$

where $e[n]$ represents the error. Both $s[n]$ and $e[n]$ are independent identically distributed (i.i.d) random processes. The joint probablity density function (pdf) of $r[n]$ and $s[n]$ is shown in Figure 3-1.

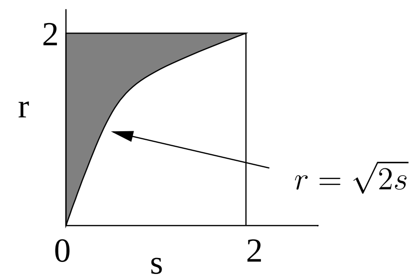


Figure 3-1: Joint pdf of R and S, $f_{R,S}(r,s)$

$$f_{R,S}(r,s) = \begin{cases} \frac{3}{4}, & \text{for } \sqrt{2s} \le r \le 2 \text{ and } 0 \le s \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

For the remainder of this problem, you may find some, none, or all of the following useful:

$f_R(r) = \frac{3}{8}r^2, \quad \text{for } 0 \le r \le 2.$

$f_S(s) = \frac{3}{4}[2 - \sqrt{2s}], \quad \text{for } 0 \le s \le 2.$

$E\{S \mid R = r\} = r^2/4.$

$E\{S\} = \frac{3}{5}.$

$E\{R\} = \frac{3}{2}.$

$E\{RS\} = 1.$

$E\{R^2\} = \frac{12}{5}.$

$E\{S^2\} = \frac{4}{7}.$

(a) (6%) Show that $f_R(r) = \frac{3}{8}r^2, \quad \text{for } 0 \le r \le 2.$

Work to be looked at:

(b) (6%) Determine $E\{R \mid S = s\}$.

$E\{R \mid S = s\} =$

Work to be looked at:

We would like to obtain an estimate $\hat{s}[n]$ of $s[n]$ from $r[n]$:

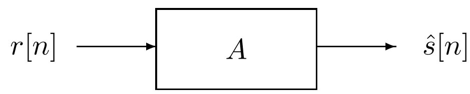


Figure 3-2:

The mean-squared error $\mathcal{E}$ of the estimate is defined as

$$\mathcal{E} = E\{(s[n] - \hat{s}[n])^2\}. \tag{1}$$

(c) (6%) Determine the memoryless system A that minimizes the mean-squared error, $\mathcal{E}$ from Equation (1).

Determine A:

Work to be looked at:

(d) (6%) In this part the output of system A is restricted to be of the form

$$\hat{s}[n] = a_0 + a_1 r[n]$$

where a_0 and a_1 are constants. Determine values a_0 and a_1 that minimize the mean-squared error, $\mathcal{E}$ specified in Equation (1).

$a_0 =$ $a_1 =$

Work to be looked at:

(e) (6%) In this part the system A is of the form

$$\hat{s}[n] = cr[n-1]$$

where c is a constant. Determine the value of c that minimizes the mean-squared error, $\mathcal{E}$ specified in Equation (1).

$c =$

Work to be looked at:

Problem 4 (26%)

A zero-mean, wide-sense stationary random process $s(t)$ is the input to an analog-to-digital (A/D) converter followed by a digital-to-analog (D/A) converter. As shown in Figure 1, the A/D-D/A cascade is modeled by an ideal continuous-to-discrete (C/D) converter, with additive quantization noise, followed by an ideal discrete-to-continuous converter.

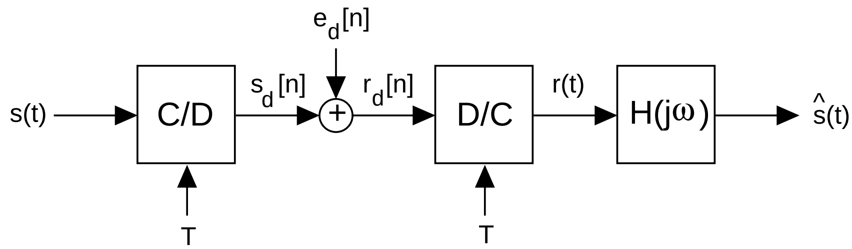


Figure 4-1: Model for A/D-D/A cascade with input $s(t)$ and output $r(t)$. The post-cascade filter $H(j\omega)$ is used to reduce the effect of quantization noise.

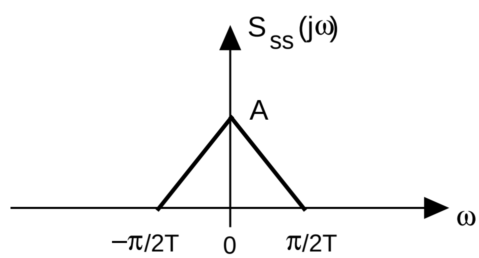


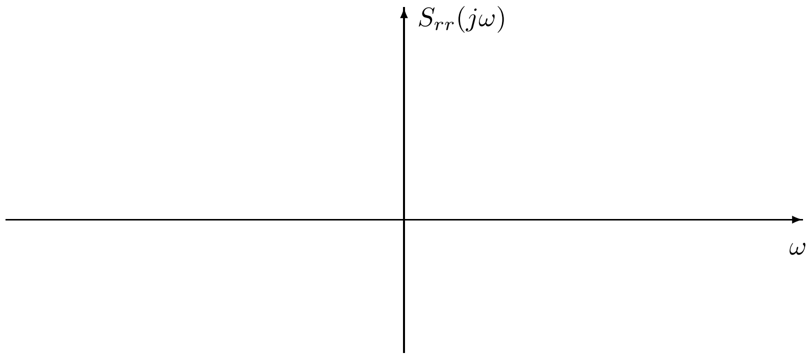
Figure 4-2: Power spectral density of $s(t)$.

In particular:

- The input process $s(t)$ has the bandlimited power spectral density shown in Figure 2.
- The C/D output is $s_d[n] = s(nT)$.
- The quantization noise $e_d[n]$ is a zero-mean, wide-sense stationary, white noise process with power spectral density $S_{e_d e_d}(e^{j\Omega}) = \sigma_e^2$.
- The process $e_d[n]$ is statistically independent of the process $s(t)$.
- The A/D output is $r_d[n] = s_d[n] + e_d[n]$.
- The D/A output is

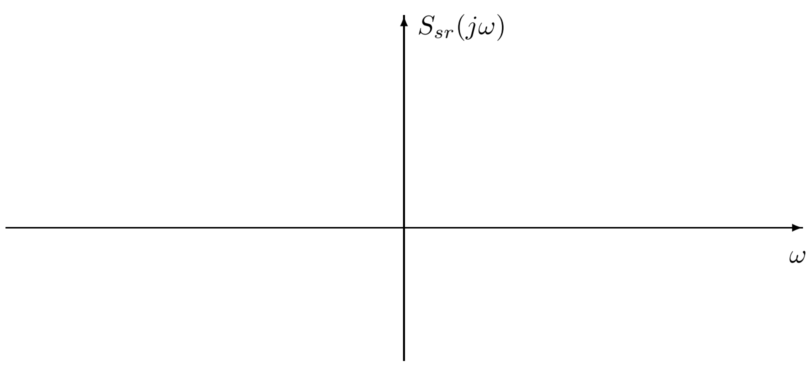
$$r(t) = \sum_{n=-\infty}^{\infty} r_d[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}.$$

(a) (7%) Determine and make a labeled sketch of $S_{rr}(j\omega)$, the power spectral density of $r(t)$.



Work to be looked at:

(b) (7%) Determine and make a labeled sketch of $S_{sr}(j\omega)$, the cross-power spectral density of $s(t)$ and $r(t)$.



Work to be looked at:

To reduce the effect of the quantization noise, we want to pass $r(t)$ through a linear time-invariant filter with frequency response $H(j\omega)$ to obtain an estimate $\hat{s}(t)$ of $s(t)$. The error measure that we will want to minimize is the mean-squared error,

$$\mathcal{E} = E\{[s(t) - \hat{s}(t)]^2\}. \quad (1)$$

(c) (12%) In this part we restrict the filter $H(j\omega)$ to be an ideal low-pass filter with unity gain and cutoff frequency ω_c, as shown in Figure 3. Determine the ω_c value, in terms of A, T, and σ_e^2, that minimizes the mean-squared error, $\mathcal{E}$, defined in Eq (1).

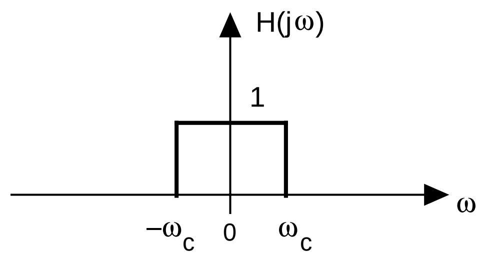


Figure 4-3: Frequency response of the ideal low-pass filter.

$w_c =$

Work to be looked at: