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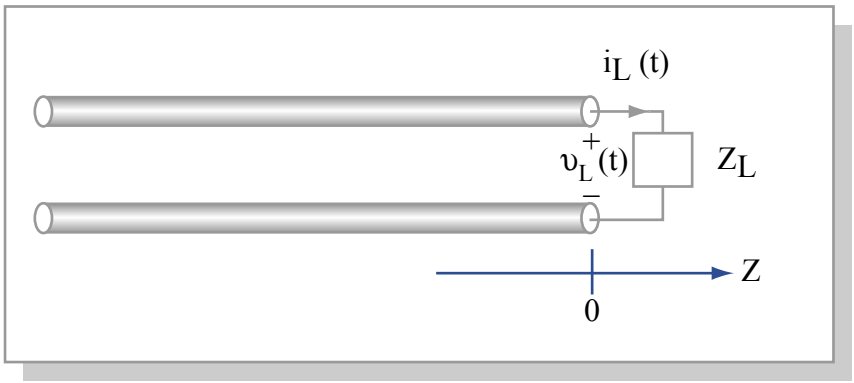
6.013 - Electromagnetics and Applications Fall 2005

Lecture 11 - Generalized Reflection Coefficient

Prof. Markus Zahn October 18, 2005

I. Arbitrary Impedance Terminations



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$$v(z=0,t) = v_L(t) = \text{Re}\left[\hat{V}_L e^{j\omega t}\right]$$

$$i(z=0,t) = i_L(t) = \text{Re}\left[\hat{I}_L e^{j\omega t}\right], \hat{I}_L = \frac{\hat{V}_L}{Z_L}$$

$$\hat{v}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz}$$

$$\hat{i}(z) = Y_0\left[\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz}\right]$$

$$\begin{aligned} \hat{v}(z=0) &= \hat{V}_L = \hat{V}_+ + \hat{V}_- \\ \hat{i}(z=0) &= \hat{I}_L = \frac{\hat{V}_L}{Z_L} = Y_0\left[\hat{V}_+ - \hat{V}_-\right] \end{aligned} \quad \Rightarrow \quad \begin{aligned} \hat{V}_+ + \hat{V}_- &= \hat{V}_L \\ \hat{V}_+ - \hat{V}_- &= \frac{\hat{V}_L}{Y_0 Z_L} \end{aligned}$$

$$\text{Add: } 2\hat{V}_+ = \hat{V}_L\left[1 + \frac{1}{Y_0 Z_L}\right] \Rightarrow \hat{V}_+ = \frac{\hat{V}_L}{2}\left[\frac{Y_0 Z_L + 1}{Y_0 Z_L}\right]$$

$$\text{Subtract: } 2\hat{V}_- = \hat{V}_L\left[1 - \frac{1}{Y_0 Z_L}\right] \Rightarrow \hat{V}_- = \frac{\hat{V}_L}{2}\left[\frac{Y_0 Z_L - 1}{Y_0 Z_L}\right]$$

$$\text{Load reflection coefficient: } \Gamma_L = \frac{\hat{V}_-}{\hat{V}_+} = \frac{Y_0 Z_L - 1}{Y_0 Z_L + 1}$$

$$= \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Generalized reflection coefficient:

$$\Gamma(z) = \frac{\hat{V}_- e^{+jkz}}{\hat{V}_+ e^{-jkz}} = \frac{\hat{V}_-}{\hat{V}_+} e^{2jkz} = \Gamma_L e^{2jkz}$$

$$\hat{v}(z) = \hat{V}_+ e^{-jkz} [1 + \Gamma(z)]$$
$$\hat{i}(z) = Y_0 \hat{V}_+ e^{-jkz} [1 - \Gamma(z)]$$
$$Z_n(z) = \frac{Z(z)}{Z_0} = \frac{\hat{v}(z)}{\hat{i}(z) Z_0} = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \qquad \text{Normalized impedance}$$
$$\Gamma(z) = \frac{Z_n(z) - 1}{Z_n(z) + 1}$$

Properties

A. $|\Gamma(z)| = |\Gamma_L| \leq 1$

B.

$$\Gamma\left(z \pm \frac{\lambda}{2}\right) = \Gamma(z)$$
$$Z_n\left(z \pm \frac{\lambda}{2}\right) = Z_n(z)$$

C.

$$\Gamma\left(z \pm \frac{\lambda}{4}\right) = -\Gamma(z)$$
$$Z_n\left(z \pm \frac{\lambda}{4}\right) = \frac{1}{Z_n(z)} = Y_n(z) = \frac{Y(z)}{Y_0} = \frac{\hat{i}(z)}{Y_0 \hat{v}(z)}$$

D. If line is matched, $Z_L = Z_0, \Gamma_L = 0, Z_n(z) = 1$

II. Load Impedance Reflected Back to the Source

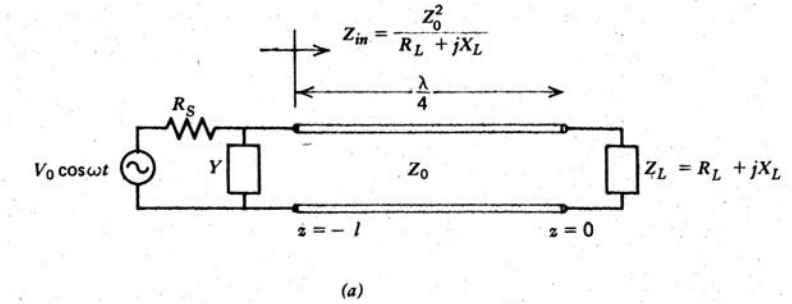
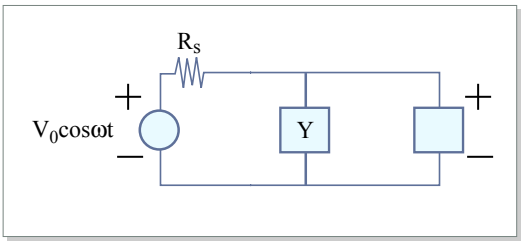


Figure 8-17 The normalized impedance reflected back through a quarter-wave-long line inverts. (*a*) The time-average power delivered to a complex load can be maximized if Y is adjusted to just cancel the reactive admittance of the load reflected back to the source with R_s equaling the resulting input resistance.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$Z_n(z = 0) = \frac{Z_L}{Z_0} = \frac{R_L + jX_L}{Z_0}$$
$$Z_n\left(z = -\frac{\lambda}{4}\right) = \frac{1}{Z_n(z = 0)} = \frac{Z_0}{R_L + jX_L} = \frac{Z\left(z = -\frac{\lambda}{4}\right)}{Z_0}$$



$$Z\left(z=-\frac{\lambda}{4}\right)=Z_0Z_n\left(z=-\frac{\lambda}{4}\right)=\frac{Z_0^2}{R_L+jX_L}$$

$$Y\left(z=-\frac{\lambda}{4}\right)=\frac{1}{Z\left(z=-\frac{\lambda}{4}\right)}=\frac{R_L+jX_L}{Z_0^2}$$

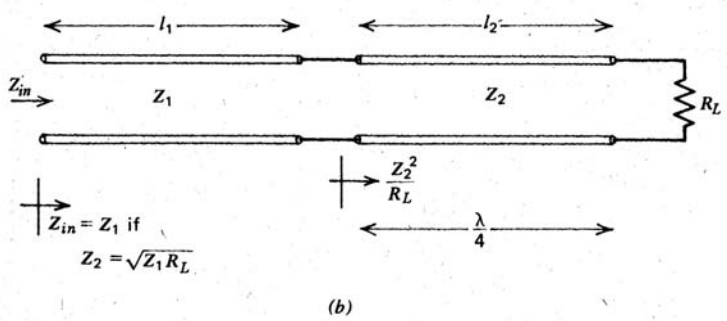
$$Y_T=Y+Y\left(z=-\frac{\lambda}{4}\right)=Y+\frac{R_L+jX_L}{Z_0^2}$$

Image by MIT OpenCourseWare.

To tune the line, choose $Y=-\frac{jX_L}{Z_0^2}\Rightarrow Y_T=\frac{R_L}{Z_0^2}$.

Y is usually created by a variable length short circuited transmission line called a stub. There is maximum power into the line if $R_S=\frac{Z_0^2}{R_L}\Rightarrow\langle P\rangle_{\max}=\frac{1}{2}\frac{\left(\frac{1}{2}V_0\right)^2}{R_S}=\frac{1}{8}\frac{V_0^2}{R_S}=\frac{1}{8}\frac{V_0^2R_L}{Z_0^2}$.

III. Quarter Wavelength Matching



(*b*) If the length l_2 of the second transmission line shown is a quarter wave long or an odd integer multiple of λ/4 and its characteristic impedance is equal to the geometric average of Z_1 and R_L, the input impedance Z_{in} is matched to Z_1.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

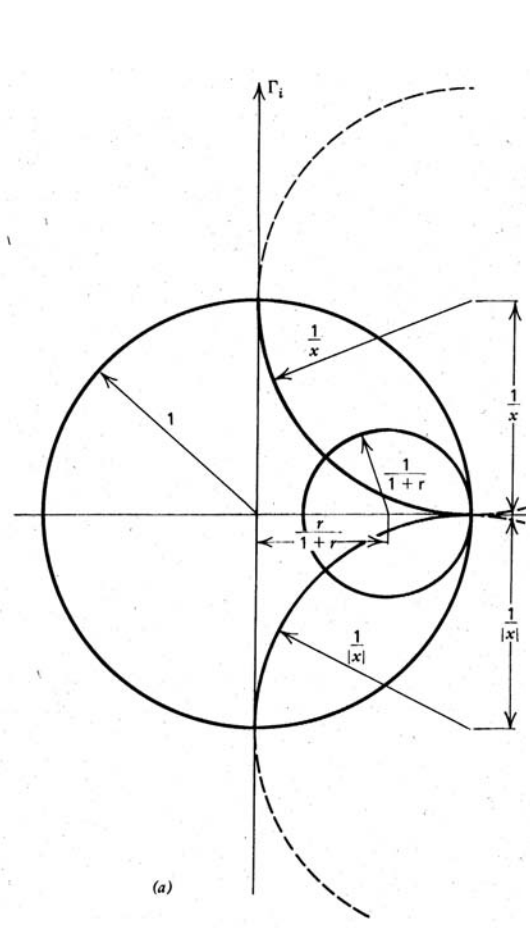
To match Z_1 to $R_L\Rightarrow Z_1=\frac{Z_2}{R_L/Z_2}=\frac{Z_2^2}{R_L}, Z_2=\sqrt{Z_1R_L}$.

IV. Smith Chart

$$\begin{array}{lcl} Z_n(z) = r + jx & & \Gamma(z) = \Gamma_r + j\Gamma_i \\ Z_n(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} & \Rightarrow & r + jx = \frac{1+\Gamma_r+j\Gamma_i}{1-\Gamma_r-j\Gamma_i} \\ r = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} & & x = \frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2} \end{array}$$

1. $\left(\Gamma_r - \frac{r}{1+r}\right)^2 + {\Gamma_i}^2 = \frac{1}{(1+r)^2}$ } Orthogonal
2. $(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \frac{1}{x^2}$ } Circles

1. Circle of radius $\frac{1}{1+r}$ Center at $\Gamma_i = 0, \Gamma_r = \frac{r}{1+r}$
2. Circle of radius $\frac{1}{|x|}$ Center at $\Gamma_i = \frac{1}{x}, \Gamma_r = 1$



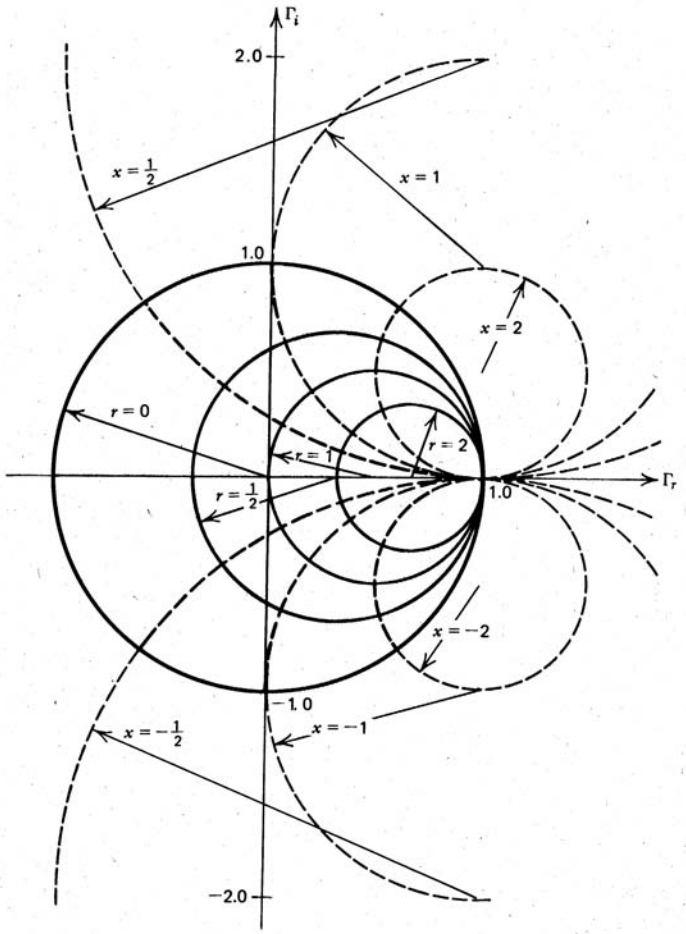


Figure 8-18 For passive loads the Smith chart is co
the complex Γ plane. (*a*) Circles of constant normal
are constructed with the centers and radii shown.
various values of *r* and *x*.

Figure 8-18

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

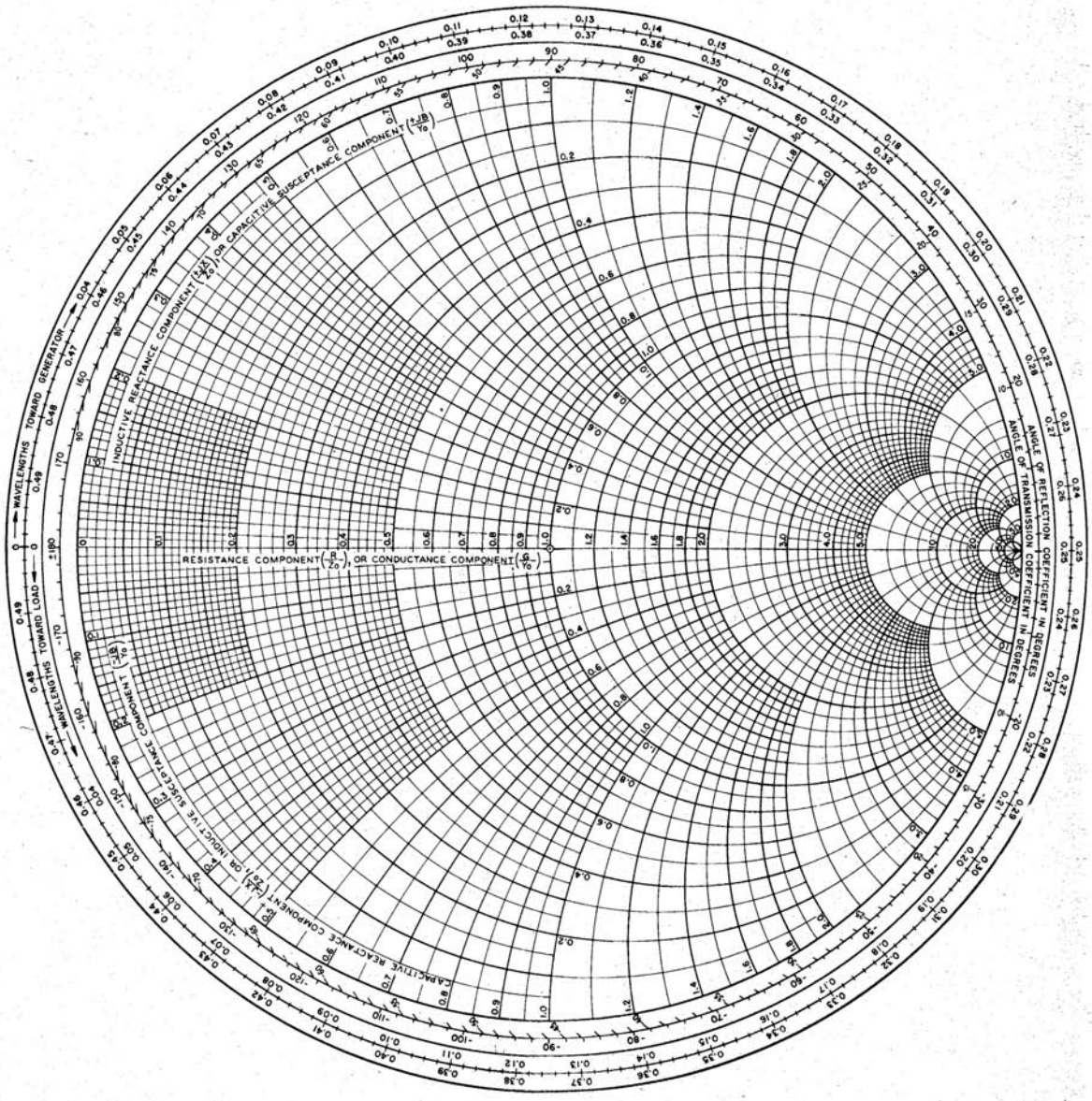
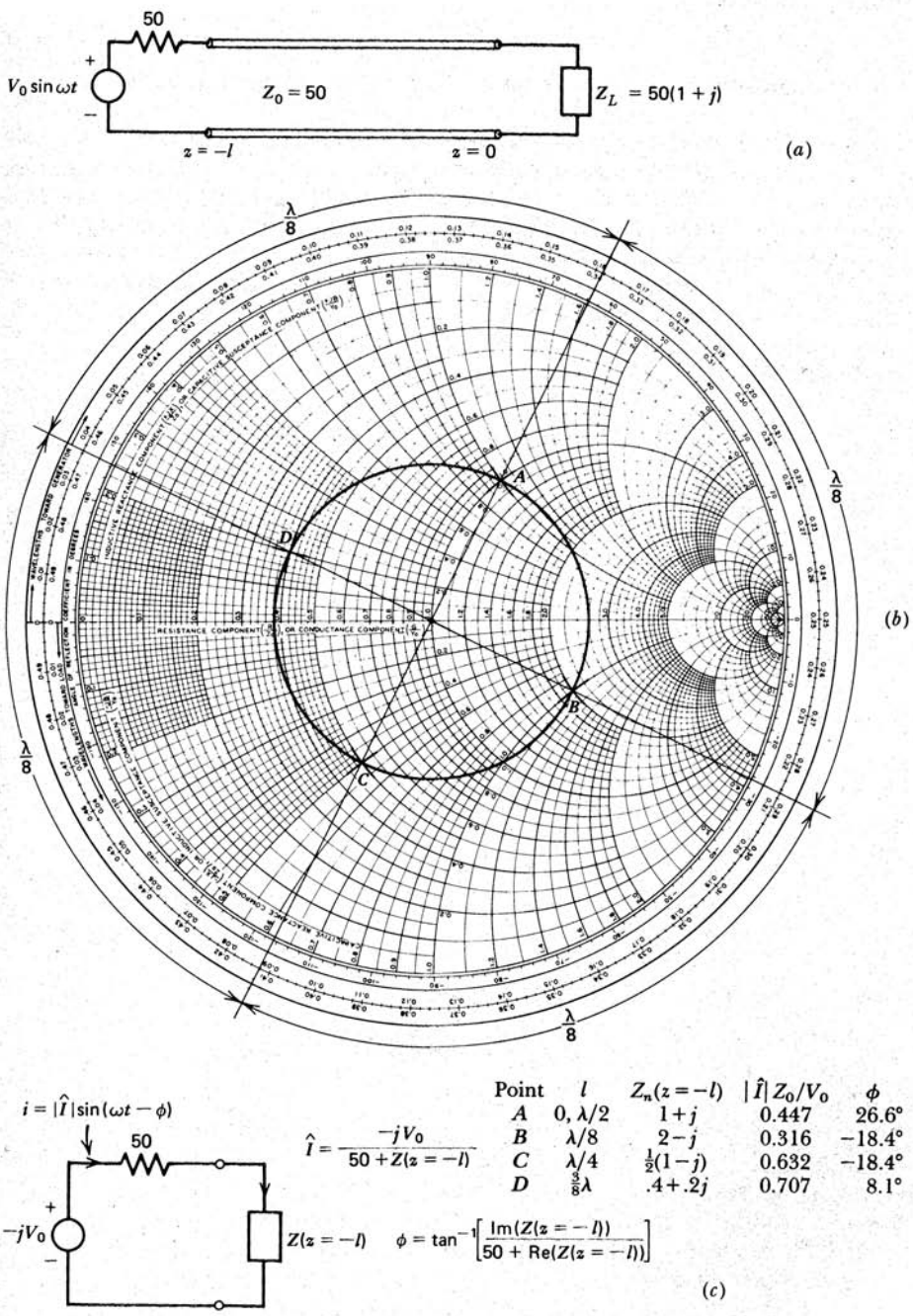


Figure 8-19 A complete Smith chart.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.



Point	l	$Z_n(z=-l)$	$\lvert\hat{I}\rvert Z_0/V_0$	ϕ
A	$0, \lambda/2$	$1+j$	0.447	26.6°
B	$\lambda/8$	$2-j$	0.316	−18.4°
C	$\lambda/4$	$\frac{1}{2}(1-j)$	0.632	−18.4°
D	$\frac{3}{8}\lambda$	$.4+.2j$	0.707	8.1°

Figure 8–20 (*a*) The load impedance at $z = 0$ reflected back to the source is found using the (*b*) Smith chart for various line lengths. Once this impedance is known the source current is found by solving the simple series circuit in (*c*).

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$i(t) = |\hat{I}| \sin(\omega t - \phi)$$

$$|\hat{I}| = \frac{V_0}{|50 + Z(z = -l)|}$$

$$\phi = \tan^{-1}\left[\frac{\text{Im}\,[Z(z = -l)]}{50 + \text{Re}\,[Z(z = -l)]}\right]$$

V. Standing Wave Parameters

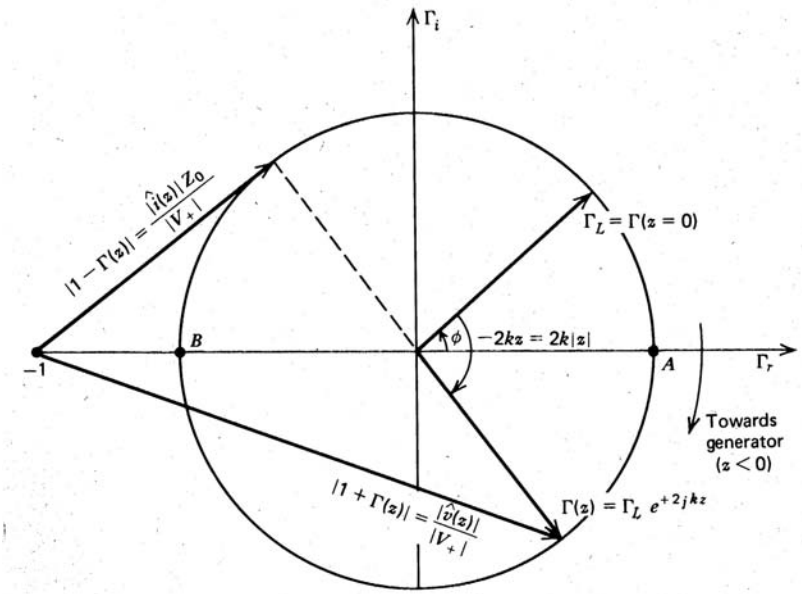


Figure 8-21 The voltage and current magnitudes along a transmission line are respectively proportional to the lengths of the vectors $|1+\Gamma(z)|$ and $|1-\Gamma(z)|$ in the complex Γ plane.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\hat{v}(z) = \hat{V}_+ e^{-jkz}\,[1+\Gamma(z)] \quad \Rightarrow \quad |\hat{v}(z)| = |\hat{V}_+||1+\Gamma(z)|$$
$$\hat{i}(z) = Y_0 \hat{V}_+ e^{-jkz}\,[1-\Gamma(z)] \quad \Rightarrow \quad |\hat{i}(z)| = Y_0|\hat{V}_+||1-\Gamma(z)|$$

$$\frac{|\hat{v}(z)|_{\max}}{|\hat{v}(z)|_{\min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \text{VSWR (Voltage Standing Wave Ratio)}$$
$$|\Gamma_L| = \frac{\text{VSWR}-1}{\text{VSWR}+1}$$
$$\Gamma_L = |\Gamma_L| e^{j\phi}$$
$$2kd_{\min} = \phi + \pi \Rightarrow \phi = \pi\left(\frac{4d_{\min}}{\lambda} - 1\right)$$

$d_{\min}$ is the shortest distance from load to first voltage minimum (at B in Figure 8-21 above)

Special Cases:

A. Matched Line: $\Gamma_L = 0, \text{VSWR} = 1$

B. Short or open circuited line: $|\Gamma_L| = 1, \text{VSWR} = \infty$

C. $\left|\frac{\hat{v}(z)}{\hat{V}_+}\right|_{\text{peak}} = 1 + |\Gamma_L|; \left|\frac{\hat{v}(z)}{\hat{V}_+}\right|_{\min} = 1 - |\Gamma_L|$

D. $\left|\frac{\hat{v}(z=0)}{\hat{V}_+}\right| = |1+\Gamma_L|; \left|\frac{\hat{i}(z=0)}{Y_0\hat{V}_+}\right| = |1-\Gamma_L|$

E. If $Z_L = R_L$ (real), then Γ_L real. If $Z_L > Z_0, \text{VSWR} = \frac{Z_L}{Z_0}$. If $Z_L < Z_0, \text{VSWR} = \frac{Z_0}{Z_L}$

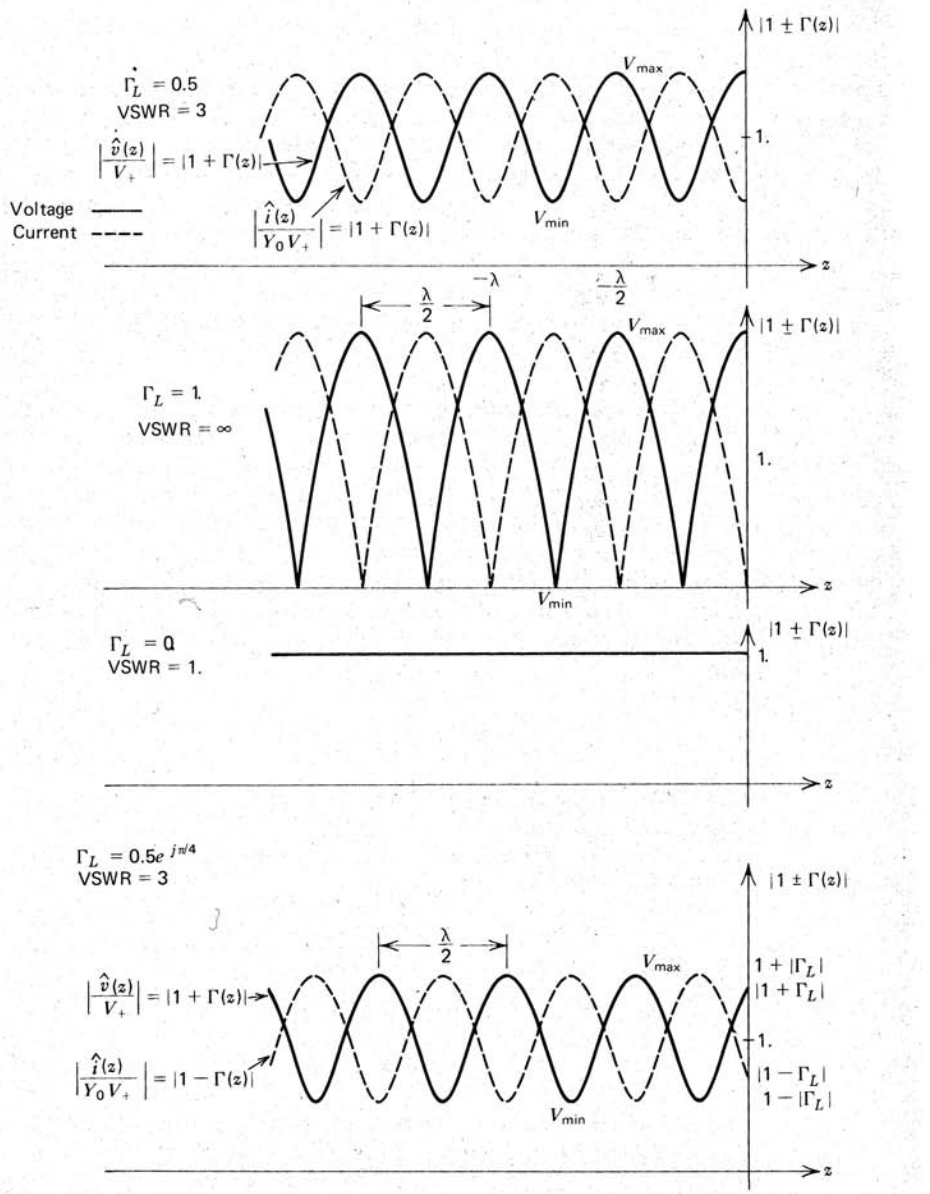


Figure 8-22 Voltage and current standing wave patterns plotted for various values of the VSWR.

From *Electromagnetic Field Theory: A Problem Solving Approach*, by Markus Zahn, 1987. Used with permission.

$$\text{Load Impedance: } Z_L = Z_0 \frac{1 + |\Gamma_L| e^{j\phi}}{1 - |\Gamma_L| e^{j\phi}}$$

$$= Z_0 \left[\frac{\text{VSWR} + 1 + (\text{VSWR} - 1) e^{j\phi}}{\text{VSWR} + 1 - (\text{VSWR} - 1) e^{j\phi}} \right]$$

$$= Z_0 \frac{\left[\text{VSWR} - j \tan\left(\frac{\phi}{2}\right) \right]}{1 - j \text{VSWR} \tan\left(\frac{\phi}{2}\right)}$$

$$= Z_0 \frac{[1 - j \text{VSWR} \tan(k d_{\min})]}{[\text{VSWR} - j \tan(k d_{\min})]}$$

Example: $Z_0 = 50\,\Omega, \text{VSWR} = 2$

$d =$ distance between successive voltage minima

$$
\begin{aligned}
&= 40 \text{ cm} = \frac{\lambda}{2} \\
d_{\min} &= \text{distance from load to first minimum} \\
&= 10 \text{ cm} \\
\lambda = 2d = 80 \text{ cm} &\Rightarrow k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.8} = 2.5\pi \\
d_{\min} = 10 \text{ cm} = 0.1 \text{ m} &\Rightarrow kd_{\min} = 2.5\pi(0.1) = \frac{\pi}{4} \\
|\Gamma_L| &= \frac{\text{VSWR} - 1}{\text{VSWR} + 1} = \frac{1}{3} \\
\frac{\phi}{\pi} &= \frac{4d_{\min}}{\lambda} - 1 = \frac{4(0.1)}{0.8} - 1 = -\frac{1}{2} \Rightarrow \phi = -\frac{\pi}{2} \\
\Gamma_L &= |\Gamma_L| e^{j\phi} = \frac{1}{3} e^{-j\pi/2} = -\frac{j}{3} \\
Z_L &= Z_0 \frac{[1 - j\text{VSWR}\tan(kd_{\min})]}{\text{VSWM} - j\tan(kd_{\min})} \\
&= \frac{50\left[1 - j(2)\tan\left(+\frac{\pi}{4}\right)\right]}{2 - j\tan\left(+\frac{\pi}{4}\right)} \\
&= \frac{50(1-2j)}{2-j} = 40 - 30j \text{ ohms}
\end{aligned}
$$