6.1 Problem 6.1

GIVEN

Infinitely-long stripline for which $d = 1$ μm and the medium between the plates has $\mu = \mu_0$ and $\epsilon = 4\epsilon_0$.

6.1.1 FIND

Width, W, for which the characteristic impedance of the line is 50 Ω. Discuss whether ideal parallel plate model is valid for these dimensions.

WORK

For an ideal parallel plate transmission line, $Z_0 = \eta d/W$, where η is the characteristic impedance of the *medium* (for EM waves). As before, $\eta = \sqrt{\mu/\epsilon} = \sqrt{\mu_0/(4\epsilon_0)} = \eta_0/2 = 188.5\,[\Omega]$. Then the inverse problem is solved as $\boxed{W = d\eta/Z_0 = 1 \times 188.5/50 = 3.77\,[\mu\text{m}]}$. Since $W/d = 3.77$, it is acceptable to approximate the line as an ideal parallel plate configuration to first order, but we should not expect very accurate results.

6.1.2 FIND

For a 1 V DC step signal, what is the intensity, I, (time average Poynting vector magnitude [W]) of the TEM field propagating between the plates?

WORK

For the parallel plate waveguide, the electric field runs from top to bottom plate and has magnitude, $E = V/d = 10^6$ [V/m]. Then the time-average Poynting vector is found from $\langle S \rangle = |E^2/(\eta)| = 10^{12}/188.5 = 5.305 \times 10^9$ W/m^2. There is no factor of because the signal is a step and not a sinusoid. Multiplying by the cross-sectional area

$$\boxed{\begin{aligned} I &= \langle S \rangle A = \langle S \rangle W d = \langle S \rangle d^2 W/d = \langle S \rangle d^2 \eta/Z_0 \\ &= |E^2/(\eta)| d^2 \eta/Z_0 = (Ed)^2/(Z_0) = v_+^2/(Z_0) = 1/50 = 0.02\,[\text{W}] \end{aligned}}. \tag{6.1}$$

Even though we derived this for the parallel plate geometry, it is general for TEM lines (and will work for sinusoids after including the factor of).

Let's check the magnetic field. It can be found by drawing an Ampèrian loop, C encircling a single strip of the line, and noting that the field is much stronger between the plates than outside. Then $\oint_C \vec{H} \cdot \mathrm{d}\vec{l} = I_{enc} \approx HW \Longrightarrow H = I/W$. For a single pulse, v_+ and i_+ are related by $v_+ = Z_0 i_+$, so $i_+ = v_+/Z_0 = 1/50 = 0.02$ [A], and $H = I/W = 0.02/3.77 \times 10^{-6} \cong 5.305 \times 10^3$ [A/m]. But $\langle \vec{S} \rangle = \vec{E} \times \vec{H} = \hat{z}5.305 \times 10^9$ W/m^2, which is exactly what we got before.

6.1.3 FIND

Evaluate the time average electric and magnetic energy densities per meter, W'_e and W'_m [J/m].

WORK

The fields were determined in the previous section. The energy density stored in the electric field is $\epsilon E^2/2$; multiplying by the cross-sectional area gives the energy density per meter of the line,

$$\boxed{\begin{aligned} W'_e &= Wd\frac{\epsilon E^2}{2} = \epsilon\frac{\eta}{Z_0}\frac{d^2E^2}{2} = \frac{v_+^2}{2Z_0c} \\ &= 0.01/(1.5\times 10^8) = (2/3)\times 10^{-10} \approx 6.67\times 10^{-11}\ [\mathrm{J/m}] \end{aligned}}, \tag{6.2}$$

since $c = (L'C')^{-1/2} = (\mu\epsilon)^{-1/2} = c_0/2$.

Likewise, for the magnetic energy,

$$\boxed{\begin{aligned} W'_m &= Wd\mu H^2/2 = \mu\frac{Z_0}{\eta}\frac{W^2H^2}{2} = \frac{i_+^2 Z_0}{2c} = \frac{v_+^2}{2Z_0c} \\ &= (2/3)\times 10^{-10} \approx 6.67\times 10^{-11}\ [\mathrm{J/m}] \end{aligned}}, \tag{6.3}$$

the same as for the electric field, as expected.

6.1.4 FIND

Show that the average power on the line, $c(W'_e + W'_m)$, is equal to the intensity, I, found in part (b).

WORK

The quantity, $c(W'_e + W'_m)$, is

$$\boxed{c(W'_e + W'_m) = 2cW'_e = 2c\frac{v_+^2}{2Z_0c} = \frac{v_+^2}{Z_0}}. \tag{6.4}$$

This is exactly what we got before. Note that this implies the phase velocity, c, is also the group velocity (the line is non-dispersive). In general, transmission lines *are* dispersive.

6.1.5 FIND

Show

1. that for two arbitrary signals flowing in opposite directions, $f_+(t-z/c)$ and $f_-(t+z/c)$, with $v(z,t) = f_+ + f_-$, the total power flowing down the line in the $+\hat{z}$ direction at any (t,z) is the power flowing in $+\hat{z}$ less the power flowing in $-\hat{z}$;

2. whether or not this superposition holds for two waves traveling in the same direction.

WORK

It is generally a good idea to apply superposition at the level of fields. In the transmission line problem, we get away with working in terms of voltage and current signals with the convention that $i_+ = Z_0 v_+$, but $i_- = Z_0 v_-$. We have seen that $I_+ = v_+^2/Z_0 = v_+ i_+$. The same would be true if the subscript were permuted to $-$. If there are two signals on the line, we may apply superposition at the level of the voltage and current in the following way:

$$\boxed{I_{+,net} = (v_+ + v_-)(i_+ - i_-) = (v_+ + v_-)(v_+ - v_-)/Z_0 = (v_+^2 - v_-^2)/Z_0 = I_+ - I_-}, \quad (6.5)$$

which is what we set out to demonstrate. However, for two signals propagating in the same direction (and at the same phase velocity),

$$\boxed{\begin{aligned} I_{+,net} &= (v_{+,1} + v_{+,2})(i_{+,1} + i_{+,2}) = (v_{+,1} + v_{+,2})(v_{+,1} + v_{+,2})/Z_0 \\ &= (v_{+,1}^2 + v_{+,2}^2 + 2v_{+,1}v_{+,2})/Z_0 = I_{+,1} + I_{+,2} + 2v_{+,1}v_{+,2}/Z_0 \end{aligned}}. \quad (6.6)$$

The cross-coupling term destroys the superposition of powers.

Superposition is one of the defining features of linear field and wave phenomena. It can be a subtle topic at times and a source of confusion, as this problem demonstrates. But you can see your way through by applying some physical reasoning and a few simple rules.

6.2 Problem 6.2

GIVEN

TEM line, $\ell = 30$ [cm], air-filled, $Z_0 = 100\,[\Omega]$, excited at one end by a matched voltage source, $V(t)$, where $V(t)$ is a step function $2u(t)$ volts.

6.2.1 FIND

Sketch and quantitatively dimension $V(z)$ and $I(z)$ on the line at $t_1 = 15 \times 10^{-10}$ [s] for the case where the load is a 300 $[\Omega]$ resistor.

WORK

Make the transformation, $z' = z - \ell$, so that at the load, $z' = 0$.

If the line is filled with air, then $c = c_0$, and the propagation time across the line is $\tau = \ell/c = 0.3/3 \times 10^8 = 10^{-9}$ [s]. This means that $t = 15 \times 10^{-10}$ [s] $= 3\ell/(2c)$. This means the line voltage is

$$v(z', t = t_1) = v_+(1 + \Gamma_L u((t - \tau - z'/c))) = v_+(1 + \Gamma_L u(\frac{\ell}{2} + z')). \tag{6.7}$$

At $t = 0$, the source sees a voltage divider across the terminals of the line, so $v_+ = V_s/2 = 1$ [V]. Γ_L is the reflection coefficient at the load, $\Gamma_L = (Z_L/Z_0 - 1)/(Z_L/Z_0 + 1) = 1/2$. As such, in terms of z,

$$\boxed{v(z, t = t_1) = v_+(1 + \Gamma_L u((t - \tau - z'/c))) = 1 + \frac{1}{2}u(z - \frac{\ell}{2})}. \tag{6.8}$$

Since the source is matched, there will be no further reflections. As expected, the forward and reflected steps sum to the steady-state voltage of 3/2 [V].

The current is found from

$$i(z', t = t_1) = \frac{v_+}{Z_0}(1 - \Gamma_L u((t - \tau - z'/c))) = \frac{v_+}{Z_0}(1 + \Gamma_L u(\frac{\ell}{2} + z')). \tag{6.9}$$

This is

$$\boxed{v(z, t = t_1) = v_+(1 + \Gamma_L u((t - \tau - z'/c))) = 1 + \frac{1}{2}u(z - \frac{\ell}{2})}. \tag{6.10}$$

Figure 6.1 shows the voltage and current distributions on the line.

6.2.2 FIND

Repeat for the case when the load is a capacitor with $C = 2 \times 10^{-12}$ [F].

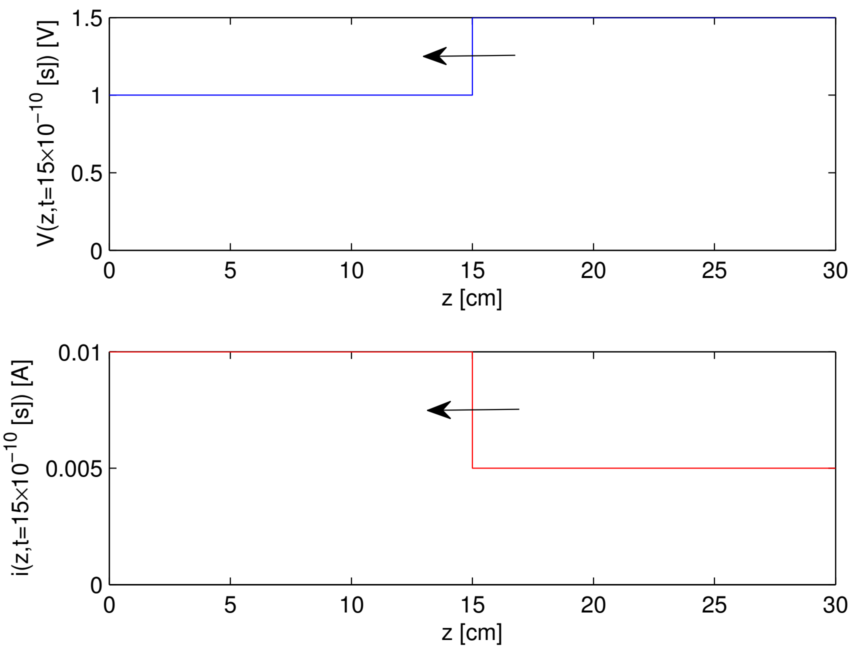

Figure 6.1: Current and voltage distribution at $3\ell/(2c)$ for 300 Ω load.

WORK

The approach to take here is to replace the line to the left of the load with its Thévenin equivalent circuit. This has a source of $v_{th} = 2v_+ u_0(t - \ell/c)$ and a source impedance of $Z_0 = 100\ \Omega$. The circuit equations lead to

$$\begin{aligned} i_R = \frac{v_{th} - v_c}{Z_0} = i_c = C\frac{\mathrm{d}v_c}{\mathrm{d}t} = C\frac{\mathrm{d}(v_c - v_{th})}{\mathrm{d}t} \\ = \frac{\mathrm{d}(v_c - v_{th})}{\mathrm{d}t} = -\frac{v_c - v_{th}}{Z_0 C} \end{aligned} \tag{6.11}$$

This is a first-order ordinary differential equation, and we write the solution by inspection,

$$v_c - v_{th} = V_0 e^{-t/(Z_0 C)} u(t - \ell/c) \Rightarrow v_c = 2v_+(1 - e^{-t/(Z_0 C)})u(t - \ell/c), \tag{6.12}$$

where we were able to factor out $v_{th} = 2v_+$ because of the initial condition that the capacitor is a short.

The next step is to determine how this load voltage translates into waves on the line. The load voltage is the sum of the forward and backward waves, $v_c = v_+ + v_- = 2v_+(1 - e^{-t/(Z_0 C)})u(t - \ell/c)$. Let $t' \equiv t - \ell/c$ and solve this relation for v_-:

$$v_-(z' = 0, t) = v_+(1 - 2e^{-t'/(Z_0 C)})u(t') \tag{6.13}$$

Lastly, we must make this wave travel in the $-\hat{z}$-direction:

$$v_-(z', t) = v_+(1 - 2\exp\left(-\frac{t' + z'/c}{Z_0 C}\right))u(t' + z'/c) \tag{6.14}$$

Let us make the transformation to the problem coordinates, $z' = z - \ell$ and $t' = t - \ell/c$, and add in v_+:

$$\boxed{v_t(z,t) = v_+ + v_= v_+ + v_+ u(t + z/c - 2\ell/c)(1 - 2\exp\left(-\frac{t + (z - 2\ell)/c}{Z_0 C}\right))}. \tag{6.15}$$

The current problem is solved similarly:

$$i_-(z', t) = \frac{v_+}{Z_0}(1 - 2\exp\left(-\frac{t' + z'/c}{Z_0 C}\right))u(t' + z'/c), \tag{6.16}$$

so

$$\boxed{i_t(z,t) = i_+ - i_- = \frac{v_+}{Z_0} + \frac{v_+}{Z_0}u(t + \frac{z - 2\ell}{c})(1 - 2\exp\left(-\frac{t + (z - 2\ell)/c}{Z_0 C}\right))}. \tag{6.17}$$

and note that $\boxed{v_+ = 1\,\mathrm{V}}$, as before.

Figure 6.2 illustrates the solution.

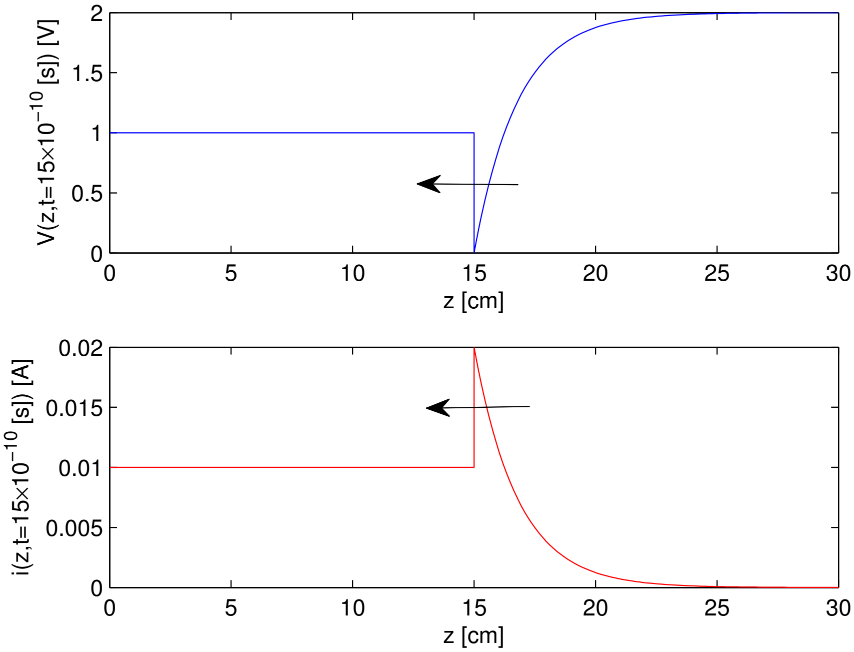

Figure 6.2: Current and voltage distribution at $3\ell/(2c)$ for capacitive load with $C = 2 \times 10^{-12}$ F.

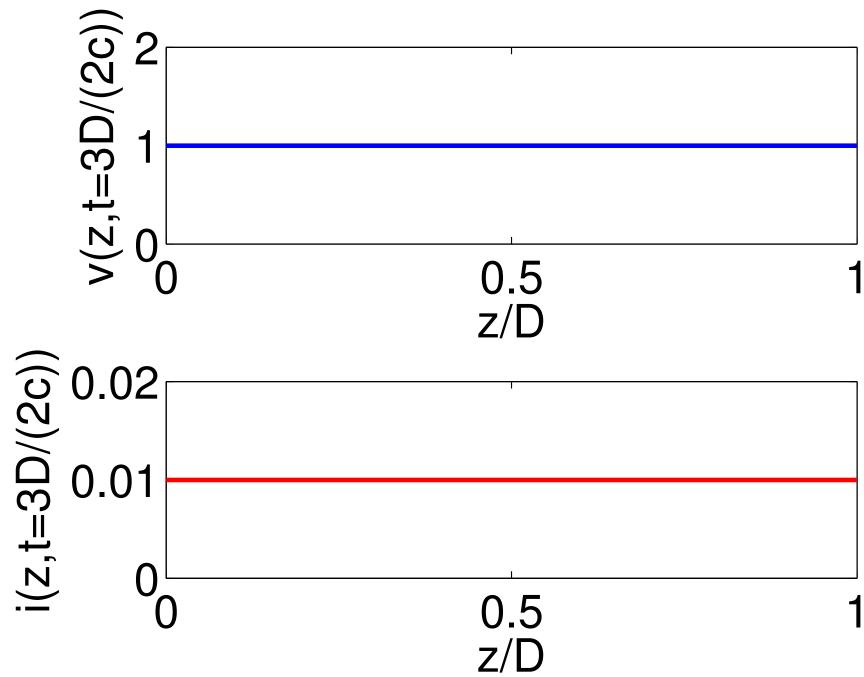

Figure 6.3: Current and voltage distribution at $3\ell/(2c)$.

6.2.3 FIND

Repeat for the case when the load is a diode triggering at 1 V (i.e. an ideal diode back-biased by a 1 V battery).

WORK

This problem is amenable to either the "bounce" approach we used in part (a) or the Thévenin equivalent circuit used in part (b). We could also work backwards. In the steady-state, the voltage on the line will be unity if the diode is shorted, while the current will be (2-1) V/100 Ω = 0.01 A (since there is no voltage differential on the line). And we cannot have reflections at the source. As such, we can be sure that the diode will reach its threshold on the first bounce, and indeed, it does: because $v_+ = 1$ V, the diode will appear as a short.

Now, since we short voltage sources, $\Gamma_L = -1$, and $v_- = -v_+$. So how do we arrive at the steady state solution? The answer is that another wave will be launched when the diode becomes a short, again of height, 1 V. This precisely cancels out the effect of the reflected wave, so that the system actually reaches steady-state in minimum time.

At $t = 3\ell/(2c)$, $\boxed{v(z, t_1) = 1\,[\text{V}]}$ and $\boxed{i(z, t) = 0.01\,[\text{A}]}$.

Figure 6.3 illustrates the solution.

6.3 Problem 3

GIVEN

A line driver at one end of a 2 cm-long, $Z_0 = 200\ \Omega$ TEM transmission line triggers a flip-flop at the other end with a step function, as illustrated. The dielectric in the line has $\epsilon = 4\epsilon_0$ and $\mu = \mu_0$. The input to the flip-flop can be treated as a 50 Ω load; it triggers (changes the *output* state of the flip-flop) at 4 V.

6.3.1 FIND

Sketch and dimension $v(t, z)$ on the line at $t = 0.1$ ns (10^{-10} s).

WORK

The time required for a signal to traverse the length of the line is $\tau = \ell/c$. $\ell = 0.02$ [m] and $c = 1/\sqrt{\mu\epsilon} = 1/\sqrt{\mu_0 4\epsilon_0} = c_0/2$. As such, $\tau = 2 \times 10^{-2}/(3 \times 10^8/2) = (4/3) \times 10^{-10}$ s. If $t = t_1 = 10^{-10} = (3/3) \times 10^{-10}$ s, the signal will have reached 3/4 of the way down the line. As such, the first reflection, v_-, will not have been generated yet.

v_+ is found from a voltage divider at the source: $v_+ = V_s 200/(50 + 200) = 4V_s/5 = 4 \times 10/5 = 8$ [V].

As such, the voltage distribution will be

$$\boxed{v(z) = \begin{cases} v_+ = 8\,[\mathrm{V}] & \text{for } 0 \le z < \frac{3\ell}{4} \\ 0 & \text{for } \frac{3\ell}{4} \le z \le \ell, \end{cases}} \tag{6.18}$$

with $\ell = 2$ [cm].

Figure 6.4 illustrates the voltage distribution for parts (a) and (b).

6.3.2 FIND

Repeat (a) for t=0.2 ns.

WORK:

Now, with $t = t_2 = 2 \times 10^{-10}$ [s], the signal will have travelled another $3\ell/4$. This means it reaches the load after travelling a distance, $\ell/4$, and then a reflection travels $\ell/2$ back toward the source.

Let's examine the reflection. The *reflection coefficient* at the load is

$$\Gamma_L \equiv \frac{v_-}{v_+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_{L,n} - 1}{Z_{L,n} + 1}, \tag{6.19}$$

where $Z_{L,n}$ is the load impedance *normalized* by the characteristic impedance of the line, Z_0. This ratio is $\Gamma_L = (0.25 - 1)/(0.25 + 1) = (-3/4)/(5/4) = -3/5$.

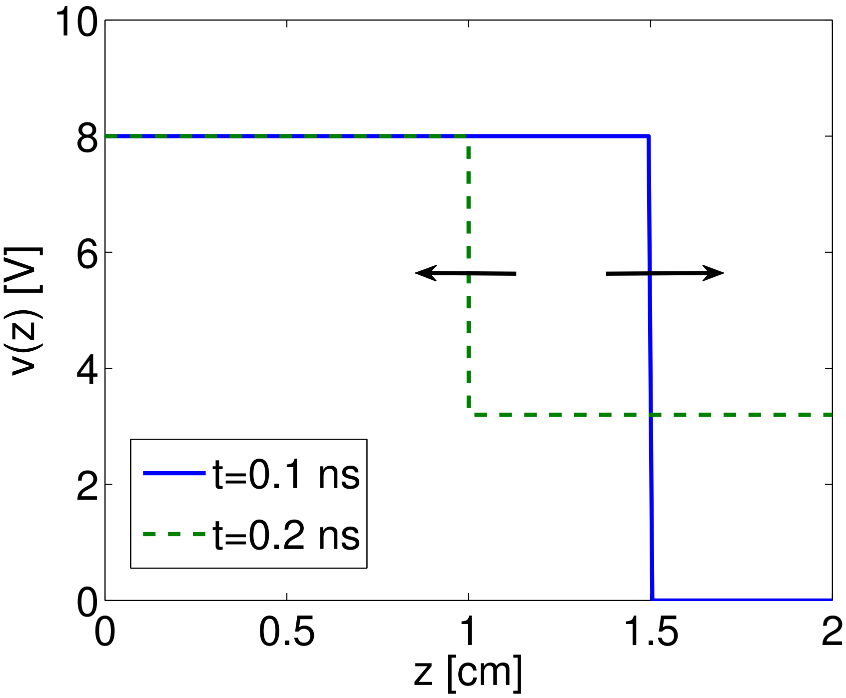

Figure 6.4: Voltage distribution for parts (a) and (b) of problem 6.3.

Since $v_+ = 8$ V, $v_- = \Gamma_L v_+ = -3 \times 8/5 = -4.8$ V. The total voltage is the sum of the forward and backward waves. This results in

$$\boxed{v(z) = \begin{cases} v_+ = 8\,[\mathrm{V}] & \text{for } 0 \le z < \ell/2 \\ v_+ + v_- = 8 - 4.8 = 3.2\,[\mathrm{V}] & \text{for } \ell/2 \le z \le \ell, \end{cases}} \tag{6.20}$$

with $\ell = 2$ [cm].

Figure 6.4 illustrates the voltage distribution for parts (a) and (b).

6.3.3 FIND

Sketch quantitatively the load voltage, $v_L(t)$, until the flip-flop is triggered; its trigger voltage is 4 V. Note that triggering is excessively delayed.

WORK

The load will see step increments every $2\tau = 2\ell/c$ time interval, starting at $t = \ell/c$. The first step is from zero (the assumed initial condition on the line) to $v_{L,1} = v_{+,1} + v_{-,1}$, which we already saw yielded 3.2 V. The second step will include the second forward and backward reflections, $v_{L,2} = v_{L,1} + v_{+,2} + v_{-,2}$. $v_{L,1}$ and all other signal persist because the source is a step. You can think of the problem also as applying to infintesimal pulses from the source; each one traverses the line, is reflected at the load, goes back to the source and is reflected again, and so forth. The source puts out a continuous supply of these infinitesimal pulses because it is a step.

The next two reflections are found as before. Define the source reflection coefficient analogously with that at the load, $\Gamma_S = (Z_s - Z_0)/(Z_s + Z_0)$. For this problem, since $Z_s = Z_L$, $\Gamma_S = \Gamma_L = -3/5$. Then $v_{+,2} = \Gamma_S v_{-,1} = \Gamma_S \Gamma_L v_{+,1} = 9 \times 8/25 = 2.88$ [V], while $v_{-,2} = \Gamma_L v_{+,2} = \Gamma_L \Gamma_S \Gamma_L v_{+,1} = -27 \times 8/125 = -1.728$ [V]. Then $v_{L,2} = v_{+,1} + v_{-,1} + v_{+,2} + v_{-,2} = 8 - 4.8 + 2.88 - 1.728 = 4.352$ [V]. This is above the trigger voltage, so the flip-flop triggers at $3\tau = 4 \times 10^{-10}$ [s].

In general, the n^{th}-step load voltage occurs at time $t = (2n-1)\tau$, and has step height,

$$\boxed{\begin{aligned} v_L\big|_{(2n-1)\tau \le t < 2n\tau} &= \sum_{i=1}^{n} v_{+,i} + v_{-,i} = \sum_{i=1}^{n} v_{+,i}(1+\Gamma_L) \\ &= (1+\Gamma_L)\sum_{i=1}^{n} (\Gamma_L\Gamma_S)^{i-1} v_{+,1} = v_{+,1}(1+\Gamma_L)\sum_{i=0}^{n-1}(\Gamma_L\Gamma_S)^i \\ &= v_{+,1}(1+\Gamma_L)\frac{1-(\Gamma_L\Gamma_S)^n}{1-\Gamma_L\Gamma_S} \end{aligned}}, \tag{6.21}$$

where we have summed the geometric series. For $n \to \infty$, this geometric series results in $v_{L,t\to\infty} = v_{+,1}(1+\Gamma_L)\frac{1}{/}(1-\Gamma_L\Gamma_S) = V_S \frac{Z_0}{Z_S+Z_0}(1+\Gamma_L)/(1-\Gamma_L\Gamma_S)$. Algebra will show that this is $v_{L,t\to\infty} = V_S \frac{Z_L}{Z_S+Z_L} = v_{L,ss}$, as expected. In this case, with $V_S = 10$ V and $Z_S = Z_L = 50$ Ω, $v_{L,ss} = 5$ [V], above the trigger voltage. However, after interval, τ, at which time the load sees the first pulse, the load voltage is 3.2 V, and the flip-flop will not trigger, but will suffer at least another two bounce intervals.

Figure 6.3.3 shows the voltage trace for the load.

6.3.4 FIND

Load voltage as $t \to \infty$.

WORK

We already determined this limit in the last section. It was $\boxed{v_L(t \to \infty) = 5\,[\mathrm{V}]}$. This corresponds to a voltage divider between the load and source impedances. It underscores the lessons that transmission lines are wires, and in the steady state, after the transient has decayed, they may be treated like circuit nodes, just as we have always treated them prior to learning about transmission lines in 6.013.

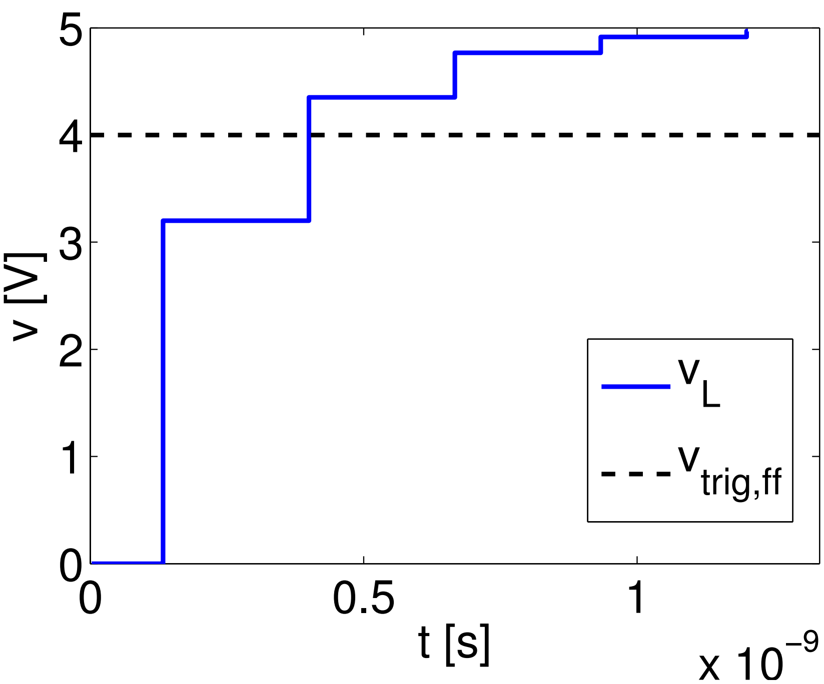

Figure 6.5: Load voltage trace. Triggering occurs when load voltage exceeds flip-flop trigger voltage (here, 4×10^{-10} [s]).

FIND

If the line impedance is matched at 50 Ω, would there still be excessive delay?

WORK

Matching the line impedance to the source and load eliminates reflections. We achieve the steady-state result with the first traversal of the line, so that $v_L = 0$, V, $t < \tau$, and $5V$, $t \geq \tau$. Since the load triggers at 4 V, the system achieves the minimum possible delay between firing the signal at the source and getting a result at the (flip-flop) load.

This is one of the many reasons why understanding transients on transmission lines is important. A flip-flop is a bit in digital memory, the source, an attempt to change that bit's state. Matching impedances increased the speed of this memory circuit by a factor of three.

6.3.5 FIND

Write a simple equation for $v(z,t)$ valid for $0 < t < 0.1$ ns, then extend it to 0.2 ns.

WORK

$$\boxed{v(z,t) = v_{+,1}u(t - z/c) + v_{-,1}u(t - \tau + \frac{z-\ell}{c}) = v_{+,1}u(t - z/c) + v_{-,1}u(t + \frac{z-2\ell}{c})} \tag{6.22}$$

where $\boxed{v_{+,1} = V_S Z_0/(Z_S + Z_0) = 8\,[\mathrm{V}]}$, $\boxed{v_{-,1} = \Gamma_L v_{+,1} = -4.8\,[\mathrm{V}]}$ and $u(x)$ is the unit step function (also called "Heaviside" step function, after Oliver Heaviside, a pioneer of the application of Maxwell's equations to transmission lines).

This summation can be continued, resulting in a similar expression as obtained for the load voltage trace earlier. The differences are that now, a phase contribution from position will also occur, and also, there will be two steps per each 2τ interval instead of one, corresponding to the appearance of the forward and backward edge at a particular location.

6.4 Problem 4

GIVEN

Unit-step current source drives given circuit (transmission line of length, D, with Z_0 and c, terminated in two more transmission lines in parallel, each of infinite length and same Z_0, c).

6.4.1 FIND

Sketch and dimension the *voltage* on all lines at time,

1. $t = t_1 = D/(2c)$,
2. $t = t_2 = 3D/(2c)$, and
3. $t = t_3 = 5D/(2c)$.

WORK

For this problem, it is more convenient to begin by studying the current problem. In the steady state, we expect 1 A to flow down the main line, and A to flow down each of the parallel branches.

At $z = D$, there will be a reflection because of an impedance mismatch. The look-in impedance for the parallel branches is simply the parallel combination of their characteristic impedances, $Z_D = Z_0||Z_0 = Z_0/2$. The voltage reflection coefficient is then

$$\Gamma_D = \frac{Z_{D,n} - 1}{Z_{D,n} + 1} = \frac{1/2 - 1}{1/2 + 1} = -\frac{1}{3}. \tag{6.23}$$

The voltage reflection coefficient at the source is trickier. But remember that when we superimpose the contribution from different sources (including incoming signals), we short all other voltage sources and *open* all other current sources. Then the reflected signal sees the current source as an open circuit, which means that the source voltage reflection coefficient is $\Gamma_s = 1$.

Now, the first current signal edge, $i_{+,1}$, will have height, 1 A, since the source is a unit step. The reflection will be $i_1 = -\Gamma_D i_{+,1}$, where the negative sign appears because $i_+ = v_+/Z_0$ but $i_- = -v_-/Z_0$, and $v_- = \Gamma_D v_+$. After $t = 2D/c$, there will be another reflection at the current source, with $i_{+,2} = -\Gamma_s i_{-,1} = \Gamma_s \Gamma_D i_{+,1}$.

Putting these pieces together gives the following line current for $0 \leq t < (6D/(2c) = 3D/c)$:

$$i(z, 0 \leq t < 3D/c) = i_{+,1}\left(u(t - z/c) - \Gamma_D u(t + \frac{z - 2D}{c}) + \Gamma_s\Gamma_D u(t - \frac{z + 2D}{c})\right). \tag{6.24}$$

Noting again that $v_+ = i_+ Z_0$ but $v_- = -i_- Z_0$ allows for the transformation to voltage:

$$v(z, 0 \le t < 3D/c) = Z_0 i_{+,1} \left(u(t - z/c) + \Gamma_D u(t + \frac{z - 2D}{c}) + \Gamma_s \Gamma_D u(t - \frac{z + 2D}{c}) \right). \tag{6.25}$$

We can now simply evaluate this expression at the three given time snapshots:

$$\boxed{v(z) = \begin{cases} Z_0 i_{+,1} u((D/2) - z) & t = t_1 = D/(2c) \\ Z_0 i_{+,1} \left(1 + \Gamma_D u(z + (3D/2) - 2D)\right) = Z_0 i_{+,1} \left(1 + \Gamma_D u(z - (D/2))\right) & t = t_2 = 3D/(2c) \\ Z_0 i_{+,1} \left(1 + \Gamma_D + \Gamma_s \Gamma_D u((5D/2) - z - 2D)\right) & \\ \quad = Z_0 i_{+,1} \left(1 + \Gamma_D + \Gamma_s \Gamma_D u((D/2) - z)\right) & t = t_3 = 5D/(2c) \end{cases}} \tag{6.26}$$

where $\boxed{i_{+,1} = 1\,[\mathrm{A}]}$, $\boxed{\Gamma_D = -1/3}$, and $\boxed{\Gamma_s = 1}$.

The formula derived in the last section for the load voltage may be used again after replacing Γ_L with Γ_D; then the voltage at $z = D$ as $t \to \infty$ is $v_D(t \to \infty) = Z_0 i_{+,1}(1 + \Gamma_D)\frac{1}{1 - \Gamma_S \Gamma_D} = Z_0 i_{+,1}(2/3)\frac{1}{4/3} = v_{+,1}/2$

Again, it is useful to interpret this steady-state result in terms of the current. In the steady state, we expect the current on each branch to be half of the current coming out of the source, A. A steady-state voltage at D, and everywhere else on the primary line from $0 \le z < D$, of $v_{+,1}/2 = Z_0 i_{+,1}/2$ produces exactly this branch current.

This is only the voltage on the primary line, $0 \le z < D$. The voltage on the branches is identical, since they are in parallel. Here, at time $t = D/c$, an identical voltage step will be launched down each of these lines of magnitude, $v_L = v_{+,1}(1 + \Gamma_D)$, so that the voltage on these branches for $0 \le t < 3D/c$ is

$$v_b ranches(z', 0 \le t < 3D/c) = Z_0 i_{+,1} u(t - D/c - z'/c) \tag{6.27}$$

, where $z' = 0$ at $z = D$, and z' measures the length along either of the branches.

Evaluating at the three snapshots gives:

$$\boxed{v_{branches}(z', t) = \begin{cases} 0 & 0 \le t = t_1 = D/(2c) \\ Z_0 i_{+,1} u(3D/(2c) - D/c - z'/c) = Z_0 i_{+,1} u(D/2 - z') & 0 \le t = t_2 = 3D/(2c) \\ Z_0 i_{+,1} u(5D/(2c) - D/c - z'/c) = Z_0 i_{+,1} u(3D/2 - z') & 0 \le t = t_3 = 5D/(2c) \end{cases}}. \tag{6.28}$$

Figure 6.6 illustrates the solution at the three snapshots for both the primary and branch lines.

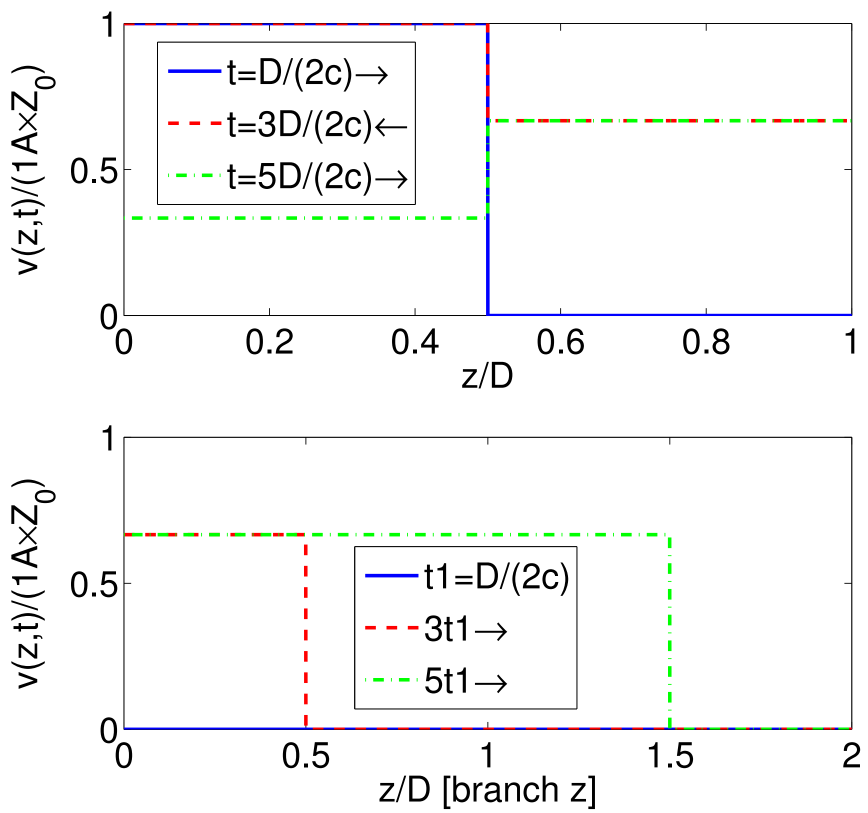

Figure 6.6: Voltage distribution on the primary (top) and branch (bottom) lines.

6.5 Problem 5

GIVEN

A current source delivering I_0 drives a delicate transistor that has an input impedance of $4Z_0$ through a TEM line of impedance, Z_0. The system has reached a steady state.

6.5.1 FIND

At $t = 0$, the switch at $z = D/2$ opens for a time interval, $D/10c$, and then recloses. Sketch the voltage, $v(z)$, on the line at $t = D/(5c)$.

WORK

The initial current distribution on the line is $i(z, t < 0) = I_0$. This requires the voltage distribution to be $V_0 \equiv v(z, t < 0) = I_0 Z_L = 4I_0 Z_0$.

An intuitive way to think about this problem is that the switch sends a wave proclaiming that it has changed from a short to an open circuit it ("darn it"). At time, $D/(10c)$, the switch sounds a counter-wave saying that it is now, again, a short ("darn it"). In the language of matching boundary conditions, this information is embodied by the statement that the current through an open circuit is zero. Since the current on the line is initially I_0, the switch sends a pulse, $i_- = -I_0$ travelling to the left of the open circuit, and another pulse, $i_+ = -I_0$, traveling to the right. Each pulse lasts for a time, $D/(10c)$, so that its length is $D/10$. Behind the pulse, the distributions return to their steady-state values.

Then, with the transformations, $v_+ = Z_0 i_+$ and $v_- = -Z_0 i_-$, we have $v_+ = -Z_0 I_0$ and $v_- = -(-Z_0 I_0) = Z_0 I_0$. This means that the total voltage on the left-moving pulse is $v_{left} = V_0 + v_- = 5Z_0 I_0$, while the total voltage on the right-moving pulse is $v_{right} = V_0 + v_+ = 3Z_0 I_0$.

As such, we can write the total voltage everywhere on the line at time, $t = D/(5c)$, as

$$\boxed{v(z, t = D/(5c)) = V_0 + v_- \left[u(z - 3D/10) - u(z - 4D/10)\right] + v_+ \left[u(z - 6D/10) - u(z - 7D/10)\right].} \tag{6.29}$$

Figure 6.7 illustrates this voltage distribution.

6.5.2 FIND

Will $v_L(t)$ ever exceed the transistor's breakdown limit of $7Z_0 I_0$? Explain.

WORK

At the load, the voltage reflection coefficient is $\Gamma_L = (Z_{L,n} - 1)/(Z_{L,n} + 1) = 3/5$. At the source, remembering that for superposition problems, we short voltage sources and open current sources, we know that the current source appears as an open circuit, so that $\Gamma_S = 1$.

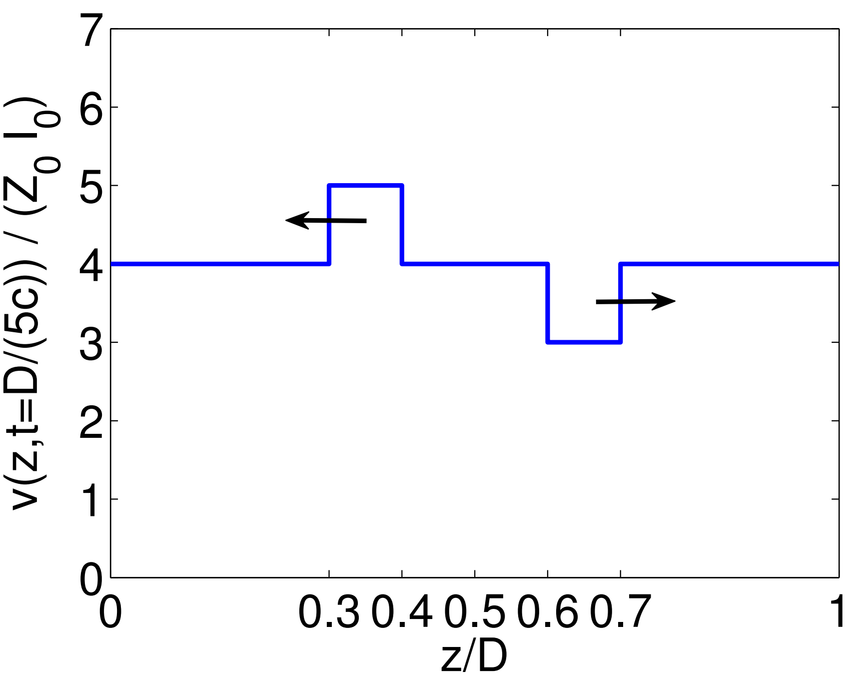

Figure 6.7: Voltage distribution on the line at time, $t = D/(5c)$, after the switch was initially opened.

The load voltage will be *smallest* after the rightward-moving pulse, v_+, hits the load for the first time, since this pulse is beneath the initial voltage and $\Gamma_L = 3/5 > 0$. It will be *largest* after the leftward-moving pulse, v_- reaches the load. This will happen after it has reflected off of the source, where it suffers no loss because $\Gamma_S = 1$, and then traverses the entire length of the line before reflecting at the load. All subsequent bounces for either pulse will have smaller pulses, as the pulse heights/depths will diminish by a factor of $\Gamma_L = 3/5$ with each round trip. As such, we need only concern ourselves with a maximum load voltage of $\boxed{v_{L,max} = V_0 + v_-(1+\Gamma_L) = Z_0 I_0 \left(4 + 1 \times (1+3/5)\right) = 5.6 Z_0 I_0 < 7 Z_0 I_0}$. This is beneath the breakdown voltage of the transistor. Note that the period for which the load voltage reaches this height is $D/(10c)$, the duration of the pulse.

The minimum load voltage is $\boxed{v_{L,min} = V_0 + v_+(1+\Gamma_L) = Z_0 I_0 \left(4 - 1 \times (1+3/5)\right) = 2.4 Z_0 I_0}$, which also is (presumably) safe for the transistor. Again, this minimum voltage lasts for a duration of $D/(10c)$.

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