MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2006)

Problem Set 3

Due: March 1, 2006

1. Mary and Tom park their cars in an empty parking lot that consists of N parking spaces in a row. Assume that each possible pair of parking locations is equally likely. Calculate the probability that the parking spaces they select are adjacent.

2. Two fair, three-sided dice[1] are rolled simultaneously.

   (a) Let X be the sum of the two rolls. Calculate the PMF, the expected value, and the variance of X.

   (b) As a gambling game, you pay a dollars in advance and get paid $5X$, with X defined as in part (a). What value of a makes it a fair game, i.e., one in which you break even on average?

   (c) Repeat parts (a) and (b) for the case where X is the square of the sum of the two rolls.

3. Consider another game played with dice. Each of two players rolls a fair, four-sided die. Player A scores the maximum of the two dice minus 1, which is denoted X. Player B scores the minimum of the two dice, which is denoted Y.

   (a) Find the expectations of X, Y, and $X - Y$.

   (b) Find the variances of X, Y, and $X - Y$.

4. Suppose wish to estimate a random variable X by some constant $\hat{x}$. There are many ways to measure how good of an estimate $\hat{x}$ is. Here you will derive an important property of *minimum mean-squared error estimation*

   Define the *mean-squared estimation error* by

$$e(\hat{x}) = \mathbf{E}\left[(X - \hat{x})^2\right].$$

   (This is a deterministic function of the real variable $\hat{x}$.) Show that $e(\hat{x})$ is minimized by $\hat{x} = \mathbf{E}[X]$.

5. Random variables X and Y have the joint PMF

$$p_{X,Y}(x, y) = \begin{cases} cxy, & x \in \{1, 2, 4\} \quad \text{and} \quad y \in \{1, 3\} \\ 0, & \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?

   (b) What is $\mathbf{P}(Y < X)$?

   (c) What is $\mathbf{P}(Y > X)$?

   (d) What is $\mathbf{P}(Y = X)$?

   (e) What is $\mathbf{P}(Y = 3)$?

[1] With coins and dice, "fair" means that all outcomes are equally likely. Unless otherwise indicated, an n-sided die has faces labeled 1, 2, $\ldots$, nOne can't really build a three-sided die, but it is nevertheless a well-defined probabilistic model.

(f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.

(g) Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[Y]$.

(h) Find the variances $\text{var}(X)$ and $\text{var}(Y)$.

G1†. The *Cauchy-Schwarz* inequality tells us that for two vectors v and w in an inner product space,

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$$

with equality if and only if one vector is a constant multiple of the other.

Prove the analogue of the Cauchy-Schwartz inequality for random variables:

$$|\mathbf{E}[XY]| \leq \sqrt{\mathbf{E}[X^2]}\ \sqrt{\mathbf{E}[Y^2]}.$$

This is consistent with the fact that one can define vector spaces of random variables.

Hint: Use the fact that $\mathbf{E}[(\alpha X + Y)^2]$ must be nonnegative for all real constants α.

†Required for 6.431; optional for 6.041