Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Problem Set 6
Due: April 5, 2006

1. Suppose that
$$M_X(s) = \frac{1}{3} \cdot \frac{1}{1-s} + \frac{2}{3} \cdot \frac{3}{3-s}.$$
What is the PDF of X?

2. Find the transform of the random variable X with density function:
$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x} + (1-p)\mu e^{-\mu x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
where p is a constant with $0 \leq p \leq 1$.

3. Consider random variable Z with transform
$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}.$$
   (a) Find the numerical value for the parameter a.
   (b) Find $\mathbf{P}(Z \geq 0.5)$.
   (c) Find $\mathbf{E}[Z]$ by using the probability distribution of Z.
   (d) Find $\mathbf{E}[Z]$ by using the transform of Z and without explicity using the probability distribution of Z.
   (e) Find $\text{var}(Z)$ by using the probability distribution of Z.
   (f) Find $\text{var}(Z)$ by using the transform of Z and without explicity using the probability distribution of Z.

4. A coin is tossed repeatedly, heads appearing with probability q on each toss. Let random variable T denote the number of tosses when a run of n consecutive heads has appeared for the first time.
   (a) Show that the PMF for T can be expressed as
$$p_T(k) = \begin{cases} 0 & , \quad k < n \\ q^n & , \quad k = n \\ \left(\sum_{i=k-n}^{\infty} p_T(i) \right) (1-q)q^n & , \quad k \geq n+1 \end{cases}.$$
   (b) Determine the transform $M_T(s)$ associated with random variable T.
   (c) Compute $\mathbf{E}[T]$, the expectation of random variable T.

5. This problem is based on an example covered in Monday's lecture (lecture 11). Let X and N be two independent normal random variables. Say $X \sim N(0, \sigma_x^2)$ and $N \sim N(0, \sigma_n^2)$. Let $Y = X + N$. In lecture we saw that Y is also normal. The lecture slides also prove that the conditional PDF $f_{Y|X}(y|x)$ is normal.

   Prove that for every value of y, the conditional density $f_{X|Y}(x|y)$ is normal.

6. Four fair 6-sided dice are rolled independently of each other. Let X_1 be the sum of the numbers on the first and second dice, and X_2 be the sum of the numbers on the third and fourth dice. Convolve the PMFs of the random variables X_1 and X_2 to find the probability that the outcomes of the four dice rolls sum to 8.

7. Consider two independent random variables X and Y. Let $f_X(x) = 1 - x/2$ for $x \in [0, 2]$ and 0 otherwise. Let $f_Y(y) = 2 - 2y$ for $y \in [0, 1]$ and 0 otherwise. Give the PDF of $W = X + Y$.

G1†. Let $X_1, X_2, \ldots, X_n$ be drawn *i.i.d* from the uniform distribution on $[0, 1]$. Let Y be the minimum of the X_i, and let Z be the maximum of the X_i. Let $W = Y + Z$. Compute $f_W(w)$, and prove that for all $\epsilon > 0$, $\lim_{n\to\infty} \mathbf{P}(|W - 1| > \epsilon) = 0$. Thus, for large n, with very high probability W is close to 1.

†Required for 6.431; optional for 6.041