

6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: Uniform Probabilities on a Square

In this problem, we will be helping Romeo and Juliet meet up for a date. And in the process, also we'll review some concepts in basic probability theory, including sample spaces and probability laws.

This problem, the basic setup is that Romeo and Juliet are trying to meet up for a date. And let's say they're trying to meet up for lunch tomorrow at noon. But they're not necessarily punctual. So they may arrive on time with a delay of 0, or they may actually be up to 1 hour late and arrive at 1:00 PM.

So the other thing that we assume in this problem is that all pairs of arrival times-- so the time that Romeo arrives paired with the time they Juliet arrives-- all of these pairs are equally likely. And I've put this in quotes, because we haven't really specify exactly what this means. And we'll come back to that in a little bit. The last important thing is that each person will wait for 15 minutes for the other person to arrive. If within that 15-minute window the other person doesn't arrive, then they'll give up and they'll end up not meeting up for lunch.

So to solve this problem, let's first try to set up a sample space and come up with a probability law to describe this scenario. And let's actually start with a simpler version of this problem. And instead of assuming that they can arrive at any delay between 0 and 1 hour, let's pretend instead that Romeo and Juliet can only arrive in 15-minute increments. So Romeo can arrive on time with a delay 0, or be 15 minutes late, 30 minutes late, 45 minutes late, or one hour late. But none of the other times are possible. And the same thing for Juliet.

Let's start out with just the simple case first, because it helps us get the intuition for the problem, and it's an easier case to analyze. So it's actually easy to visualize this. It's a nice visual tool to group this sample space into a grid. So the horizontal axis here represents the arrival time of Romeo, and the vertical axis represents the arrival time of Juliet. And so, for example, this point here would represent Romeo arriving 15 minutes late and Juliet arriving 30 minutes late.

So this is our sample space now. This is our Ω . And now let's try to assign a probability law. And we'll continue to assume that all pairs of arrival times are equally likely. And now we can actually specifically specify what this term means.

And in particular, we'll be invoking the discrete uniform law, which basically says that all of these points, which are just outcomes in our probabilistic experiment-- all of these outcomes are equally likely. And so since there are 25 of them, each one of these outcomes has a probability of 1 over 25.

So now we've specified our sample space and our probability law. So now let's try to answer the question, what is the probability that Romeo and Juliet will meet up for their date? So all that amounts to now is just identifying which of these 25 outcomes results in Romeo and Juliet arriving within 15 minutes of each other.

So let's start with this one that I've picked out. If Romeo arrives 15 minutes late and Juliet arrives 30 minutes late, then they will arrive within 15 minutes of each other. So this outcome does result in the two of them meeting. And so we can actually highlight all of these. And it turns out that these outcomes that I'm highlighting result in the two of them arriving within 15 minutes of each other.

So because each one has a probability of $\frac{1}{25}$, all we really need to do now is just count how many outcomes there are. So there's 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. So the probability in the end is for the discrete case.

The discrete case-- I'm referring to the case where we simplified it and considered only arrival times with increments of 15 minutes. In this case, the probability is $\frac{13}{25}$. So now we have an idea of how to solve this problem. It amounts to basically coming up with a sample space, a probability law, and then identifying the events of interest in calculating the probability of that event.

So now let's actually solve the problem that we really are interested in, which is that instead of confining Romeo and Juliet to arrive in only 15-minute increments, really, time is continuous, and Romeo and Juliet can arrive at any time. So they don't necessarily have to arrive 15 minutes late. Romeo could arrive 15 minutes and 37 seconds late if he wanted to.

So now our new sample space is actually just, instead of only these 25 points in the grid, it's this entire square. So any point within the square could be a possible pair of meeting times between Romeo and Juliet. So that is our new sample space, our new Ω .

And now let's assign a new probability law. And now, instead of being in the discrete world, we're in the continuous world. And the analogy here is to consider probabilities as areas. So the area of this entire square is one. And that also corresponds to the probability of Ω , the sample space. And imagine just spreading probability evenly across this square so that the probability of any event-- which in this case would just be any shape within this square-- is exactly equal to the area of that shape. So now that is our new sample space and our new probability law.

So what we have to do now is just to identify the event of interest, which is still the event that Romeo and Juliet arrive within 15 minutes of each other. So let's do that. If Romeo and Juliet arrive both on time, then obviously they'll meet. And if Romeo's on time and Juliet is 15 minutes late, then they will still meet.

And in fact, any pairs of meeting times between these would still work, because now Romeo can be on time, and Juliet can arrive at any time between 0 and 15 minutes late. But you notice that if Juliet is even a tiny bit later than 15 minutes, then they won't end up meeting. So this segment here is part of the event of interest.

And similarly, this segment here is also part of the event. And if you take this exercise and extend it, you can actually verify that the event of interest is this strip shape in the middle of the square. Which, if you think about it, makes sense, because you want the arrival times between

Romeo and Juliet to be close to each other, so you would expect it to be somewhere close to a diagonal in this square.

So now we have our event of interest. We have our sample space and our probability law. So all we have to do now is just calculate what this probability is. And we've already said that the probability in this probability law is just areas. So now it actually just boils down to not a probability problem, but a problem in geometry.

So to calculate this area, you can do it in lots of ways. One way is to calculate the area of the square, which is 1, and subtract the areas of these two triangles. So let's do that.

So in the continuous case, the probability of meeting is going to be 1 minus the area of this triangle. The base here is $\frac{3}{4}$ and $\frac{3}{4}$, so it's $\frac{1}{2}$ times $\frac{3}{4}$ times $\frac{3}{4}$. That's the area of one of these triangles. There's two of them, so we'll multiply by two. And we end up with 1 minus $\frac{9}{16}$, or $\frac{7}{16}$ as our final answer.

So in this problem, we've reviewed some basic concepts of probability, and that's also helped us solve this problem of helping Romeo and Juliet meet up for a date. And if you wanted to, you could even extend this problem even further and turn it on its head. And instead of calculating given that they arrive within 15 minutes of each other, what is the probability that they'll meet, let's say that Romeo really wants to meet up with Juliet, and he wants to assure himself a least, say, a 90% chance of meeting Juliet. Then you can ask, if he wants to have at least a 90% chance of meeting her, how long should he be willing to wait? And so that's the flip side of the problem.

And you can see that with just some basic concepts of probability, you can answer some already pretty interesting problems. So I hope this problem was interesting, and we'll see you next time.

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Fall 2013

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