Problem Set 4: Solutions

1. (a) From the joint PMF, there are six (x, y) coordinate pairs with nonzero probabilities of occurring. These pairs are (1, 1), (1, 3), (2, 1), (2, 3), (4, 1), and (4, 3). The probability of a pair is proportional to the sum of the squares of the coordinates of the pair, $x^2 + y^2$. Because the probability of the entire sample space must equal 1, we have:

$$(1+1)c + (1+9)c + (4+1)c + (4+9)c + (16+1)c + (16+9)c = 1.$$

Solving for c, we get $c = \boxed{\frac{1}{72}}$.

(b) There are three sample points for which $y < x$:

$$\mathbf{P}(Y < X) = \mathbf{P}(\{(2,1)\}) + \mathbf{P}(\{(4,1)\}) + \mathbf{P}(\{(4,3)\}) = \frac{5}{72} + \frac{17}{72} + \frac{25}{72} = \boxed{\frac{47}{72}}.$$

(c) There are two sample points for which $y > x$:

$$\mathbf{P}(Y > X) = \mathbf{P}(\{(1,3)\}) + \mathbf{P}(\{(2,3)\}) = \frac{10}{72} + \frac{13}{72} = \boxed{\frac{23}{72}}.$$

(d) There is only one sample point for which $y = x$:

$$\mathbf{P}(Y = X) = \mathbf{P}(\{(1,1)\}) = \boxed{\frac{2}{72}}.$$

Notice that, using the above two parts,

$$\mathbf{P}(Y < X) + \mathbf{P}(Y > X) + \mathbf{P}(Y = X) = \frac{47}{72} + \frac{23}{72} + \frac{2}{72} = 1$$

as expected.

(e) There are three sample points for which $y = 3$:

$$\mathbf{P}(Y = 3) = \mathbf{P}(\{(1,3)\}) + \mathbf{P}(\{(2,3)\}) + \mathbf{P}(\{(4,3)\}) = \frac{10}{72} + \frac{13}{72} + \frac{25}{72} = \boxed{\frac{48}{72}}.$$

(f) In general, for two discrete random variable X and Y for which a joint PMF is defined, we have

$$p_X(x) = \sum_{y=-\infty}^{\infty} p_{X,Y}(x,y) \qquad \text{and} \qquad p_Y(y) = \sum_{x=-\infty}^{\infty} p_{X,Y}(x,y).$$

In this problem the ranges of X and Y are quite restricted so we can determine the marginal PMFs by enumeration. For example,

$$p_X(2) = \mathbf{P}(\{(2,1)\}) + \mathbf{P}(\{(2,3)\}) = \frac{18}{72}.$$

Overall, we get:

$$p_X(x) = \begin{cases} 12/72, & \text{if } x = 1, \\ 18/72, & \text{if } x = 2, \\ 42/72, & \text{if } x = 4, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad p_Y(y) = \begin{cases} 24/72, & \text{if } y = 1, \\ 48/72, & \text{if } y = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(g) In general, the expected value of any discrete random variable X equals

$$\mathbf{E}[X] = \sum_{x=-\infty}^{\infty} x p_X(x).$$

For this problem,

$$\mathbf{E}[X] = 1 \cdot \frac{12}{72} + 2 \cdot \frac{18}{72} + 4 \cdot \frac{42}{72} = \boxed{3}$$

and

$$\mathbf{E}[Y] = 1 \cdot \frac{24}{72} + 3 \cdot \frac{48}{72} = \boxed{\tfrac{7}{3}}\,.$$

To compute $\mathbf{E}[XY]$, note that $p_{X,Y}(x,y) \neq p_X(x)p_Y(y)$. Therefore, X and Y are not independent and we cannot assume $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$. Thus, we have

$$\begin{aligned}\mathbf{E}[XY] &= \sum_x \sum_y xy p_{X,Y}(x,y) \\ &= 1 \cdot \frac{2}{72} + 2 \cdot \frac{5}{72} + 4 \cdot \frac{17}{72} + 3 \cdot \frac{10}{72} + 6 \cdot \frac{13}{72} + 12 \cdot \frac{25}{72} = \boxed{\tfrac{61}{9}}\,.\end{aligned}$$

(h) The variance of a random variable X can be computed as $\mathbf{E}[X^2] - \mathbf{E}[X]^2$ or as $\mathbf{E}[(X - \mathbf{E}[X])^2]$. We use the second approach here because X and Y take on such limited ranges. We have

$$\text{var}(X) = (1-3)^2 \frac{12}{72} + (2-3)^2 \frac{18}{72} + (4-3)^2 \frac{42}{72} = \boxed{\tfrac{3}{2}}$$

and

$$\text{var}(Y) = (1 - \frac{7}{3})^2 \frac{24}{72} + (3 - \frac{7}{3})^2 \frac{48}{72} = \boxed{\tfrac{8}{9}}\,.$$

X and Y are not independent, so we cannot assume $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y)$. The variance of $X+Y$ will be computed using $\text{var}(X+Y) = \mathbf{E}[(X+Y)^2] - (\mathbf{E}[X+Y])^2$. Therefore, we have

$$\mathbf{E}[(X+Y)^2] = 4 \cdot \frac{2}{72} + 9 \cdot \frac{5}{72} + 25 \cdot \frac{17}{72} + 16 \cdot \frac{10}{72} + 25 \cdot \frac{13}{72} + 49 \cdot \frac{25}{72} = \frac{547}{18}\,.$$

$$(\mathbf{E}[X+Y])^2 = (\mathbf{E}[X] + \mathbf{E}[Y])^2 = \left(3 + \frac{7}{3}\right)^2 = \frac{256}{9}\,.$$

Therefore,

$$\text{var}(X+Y) = \frac{547}{18} - \frac{256}{9} = \boxed{\tfrac{35}{18}}\,.$$

(i) There are four $(x,\ y)$ coordinate pairs in A : (1,1), (2,1), (4,1), and (4,3). Therefore, $\mathbf{P}(A) = \frac{1}{72}(2 + 5 + 17 + 25) = \frac{49}{72}$. To find $\mathbf{E}[X \mid A]$ and $\text{var}(X \mid A)$, $p_{X|A}(x)$ must be calculated. We have

$$p_{X|A}(x) = \begin{cases} 2/49, & \text{if } x = 1, \\ 5/49, & \text{if } x = 2, \\ 42/49, & \text{if } x = 4, \\ 0, & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
\mathbf{E}[X \mid A] &= 1 \cdot \frac{2}{49} + 2 \cdot \frac{5}{49} + 4 \cdot \frac{42}{49} = \boxed{\frac{180}{49}}, \\
\mathbf{E}[X^2 \mid A] &= 1^2 \cdot \frac{2}{49} + 2^2 \cdot \frac{5}{49} + 4^2 \cdot \frac{42}{49} = \frac{694}{49}, \\
\mathrm{var}(X \mid A) &= \mathbf{E}[X^2 \mid A] - (\mathbf{E}[X \mid A])^2 = \frac{694}{49} - \left(\frac{180}{49}\right)^2 = \boxed{\frac{1606}{2401}},
\end{aligned}$$

2. Consider a sequence of six independent rolls of this die, and let X_i be the random variable corresponding to the ith roll.

   (a) What is the probability that exactly three of the rolls have result equal to 3? Each roll X_i can either be a 3 with probability 1/4 or not a 3 with probability 3/4. There are $\binom{6}{3}$ ways of placing the 3's in the sequence of six rolls. After we require that a 3 go in each of these spots, which has probability $(1/4)^3$, our only remaining condition is that either a 1 or a 2 go in the other three spots, which has probability $(3/4)^3$. So the probability of exactly three rolls of 3 in a sequence of six independent rolls is $\boxed{\binom{6}{3}(\frac{1}{4})^3(\frac{3}{4})^3}$.

   (b) What is the probability that the first roll is 1, given that exactly two of the six rolls have result of 1? The probability of obtaining a 1 on a single roll is 1/2, and the probability of obtaining a 2 or 3 on a single roll is also 1/2. For the purposes of solving this problem we treat obtaining a 2 or 3 as an equivalent result. We know that there are $\binom{6}{2}$ ways of rolling exactly two 1's. Of these $\binom{6}{2}$ ways, exactly $\binom{5}{1} = 5$ ways result in a 1 in the first roll, since we can place the remaining 1 in any of the five remaining rolls. The rest of the rolls must be either 2 or 3. Thus, the probability that the first roll is a 1 given exactly two rolls had an outcome of 1 is $\boxed{\frac{5}{\binom{6}{2}}}$.

   (c) We are now told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. What is the probability of the sequence 121212? We want to find

$$\mathbf{P}(121212 \,|\, \text{exactly three 1's and three 2's}) = \frac{\mathbf{P}(121212)}{\mathbf{P}(\text{exactly 3 ones and 3 twos})}.$$

   Any particular sequence of three 1's and three 2's will have the same probability: $(1/2)^3(1/4)^3$. There are $\binom{6}{3}$ possible rolls with exactly three 1's and three 2's. Therefore,

$$\mathbf{P}(121212 \,|\, \text{exactly three 1's and three 2's}) = \boxed{\frac{1}{\binom{6}{3}}}.$$

   (d) Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's. Let A be the event that at least one roll results in a 3. Then

$$\mathbf{P}(A) = 1 - \mathbf{P}(\text{no rolls resulted in 3}) = 1 - \left(\frac{3}{4}\right)^6.$$

   Now let K be the random variable representing the number of 3's in the 6 rolls. The (unconditional) PMF $p_K(k)$ for K is given by

$$p_K(k) = \binom{6}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{6-k}.$$

We find the conditional PMF $p_{k|A}(k \mid A)$ using the definition of conditional probability:

$$p_{K|A}(k \mid A) = \frac{\mathbf{P}(\{K = k\} \cap A)}{\mathbf{P}(A)}.$$

Thus we obtain

$$p_{K|A}(k \mid A) = \begin{cases} \frac{1}{1-(3/4)^6}\binom{6}{k}\left(\frac{1}{4}\right)^k\left(\frac{3}{4}\right)^{6-k} & \text{if } k = 1, 2, \ldots, 6, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $p_{K|A}(0 \mid A) = 0$ because the event $\{K = 0\}$ and the event A are mutually exclusive. Thus the probability of their intersection, which appears in the numerator in the definition of the conditional PMF, is zero.

3. By the definition of conditional probability,

$$\mathbf{P}(X = i \mid X + Y = n) = \frac{\mathbf{P}(\{X = i\} \cap \{X + Y = n\})}{\mathbf{P}(X + Y = n)}.$$

The event $\{X = i\} \cap \{X + Y = n\}$ in the numerator is equivalent to $\{X = i\} \cap \{Y = n - i\}$. Combining this with the independence of X and Y,

$$\mathbf{P}(\{X = i\} \cap \{X + Y = n\}) = \mathbf{P}(\{X = i\} \cap \{Y = n - i\}) = \mathbf{P}(X = i)\mathbf{P}(Y = n - i).$$

In the denominator, $\mathbf{P}(X + Y = n)$ can be expanded using the total probability theorem and the independence of X and Y:

$$\begin{aligned} \mathbf{P}(X + Y = n) &= \sum_{i=1}^{n-1} \mathbf{P}(X = i)\mathbf{P}(X + Y = n \mid X = i) \\ &= \sum_{i=1}^{n-1} \mathbf{P}(X = i)\mathbf{P}(i + Y = n \mid X = i) \\ &= \sum_{i=1}^{n-1} \mathbf{P}(X = i)\mathbf{P}(Y = n - i \mid X = i) \\ &= \sum_{i=1}^{n-1} \mathbf{P}(X = i)\mathbf{P}(Y = n - i) \end{aligned}$$

Note that we only get non-zero probability for $i = 1, \ldots, n - 1$ since X and Y are geometric random variables.

The desired result is obtained by combining the computations above and using the geometric

PMF explicitly:

$$\begin{aligned}
\mathbf{P}(X = i \mid X + Y = n) &= \frac{\mathbf{P}(X = i)\mathbf{P}(Y = n - i)}{\sum_{i=1}^{n-1} \mathbf{P}(X = i)\mathbf{P}(Y = n - i)} \\
&= \frac{(1-p)^{i-1}p(1-p)^{n-i-1}p}{\sum_{i=1}^{n-1}(1-p)^{i-1}p(1-p)^{n-i-1}p} \\
&= \frac{(1-p)^n}{\sum_{i=1}^{n-1}(1-p)^n} \\
&= \frac{(1-p)^n}{(1-p)^n \sum_{i=1}^{n-1} 1} \\
&= \frac{1}{n-1}, \quad i = 1, \dots, n-1.
\end{aligned}$$

4. (a) Since $\mathbf{P}(A) > 0$, we can show independence through $\mathbf{P}(B) = \mathbf{P}(B \mid A)$:

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(B \cap A)}{\mathbf{P}(A)} = \frac{\binom{8}{6}p^6(1-p)^2 p}{\binom{8}{6}p^6(1-p)^2} = p = \mathbf{P}(B).$$

Therefore, A and B are independent.

(b) Let C be the event "3 heads in the first 4 tosses" and let D be the event "2 heads in the last 3 tosses". Since there are no overlap in tosses in C and D, they are independent:

$$\begin{aligned}
\mathbf{P}(C \cap D) &= \mathbf{P}(C)\mathbf{P}(D) \\
&= \binom{4}{3}p^3(1-p) \cdot \binom{3}{2}p^2(1-p) \\
&= 12p^5(1-p)^2.
\end{aligned}$$

(c) Let E be the event "4 heads in the first 7 tosses" and let F be the event "2nd head occurred during 4th trial". We are asked to find $\mathbf{P}(F \mid E) = \mathbf{P}(F \cap E)/\mathbf{P}(E)$. The event $F \cap E$ occurs if there is 1 head in the first 3 trials, 1 head on the 4th trial, and 2 heads in the last 3 trials. Thus, we have

$$\begin{aligned}
\mathbf{P}(F \mid E) = \frac{\mathbf{P}(F \cap E)}{\mathbf{P}(E)} &= \frac{\binom{3}{1}p(1-p)^2 \cdot p \cdot \binom{3}{2}p^2(1-p)}{\binom{7}{4}p^4(1-p)^3} \\
&= \frac{\binom{3}{1} \cdot 1 \cdot \binom{3}{2}}{\binom{7}{4}} = \frac{9}{35}.
\end{aligned}$$

Alternatively, we can solve this by counting. We are given that 4 heads occurred in the first 7 tosses. Each sequence of 7 trials with 4 heads is equally probable, the discrete uniform probability law can be used here. There are $\binom{7}{4}$ outcomes in E. For the event $E \cap F$, there are $\binom{3}{1}$ ways to arrange 1 head in the first 3 trials, 1 way to arrange the 2nd head in the 4th trial and $\binom{3}{2}$ ways to arrange 2 heads in the first 3 trials. Therefore,

$$\mathbf{P}(F \mid E) = \frac{\binom{3}{1} \cdot 1 \cdot \binom{3}{2}}{\binom{7}{4}} = \frac{9}{35}.$$

(d) Let G be the event "5 heads in the first 8 tosses" and let H be the event "3 heads in the last 5 tosses". These two events are not independent as there is some overlap in the tosses (the 6th, 7th, and 8th tosses). To compute the probability of interest, we carefully count all the disjoint, possible outcomes in the set $G \cap H$ by conditioning on the number of heads in the 6th, 7th, and the 8th tosses. We have

$$\begin{aligned}\mathbf{P}(G \cap H) = {} & \mathbf{P}(G \cap H \mid 1 \text{ head in tosses 6–8})\mathbf{P}(1 \text{ head in tosses 6–8}) \\ & + \mathbf{P}(G \cap H \mid 2 \text{ heads in tosses 6–8})\mathbf{P}(2 \text{ heads in tosses 6–8}) \\ & + \mathbf{P}(G \cap H \mid 3 \text{ heads in tosses 6–8})\mathbf{P}(3 \text{ heads in tosses 6–8})\end{aligned}$$

$$\begin{aligned} = {} & \binom{5}{4} p^4(1-p) \cdot p^2 \cdot \binom{3}{1} p(1-p)^2 \\ & + \binom{5}{3} p^3(1-p)^2 \cdot \binom{2}{1} p(1-p) \cdot \binom{3}{2} p^2(1-p) \\ & + \binom{5}{2} p^2(1-p)^3 \cdot (1-p)^2 \cdot p^3.\end{aligned}$$

$$= 15p^7(1-p)^3 + 60p^6(1-p)^4 + 10p^5(1-p)^5.$$

5. Let I_k be the reward paid at time k. We have

$$\mathbf{E}[I_k] = \mathbf{P}(I_k = 1) = \mathbf{P}(\text{T at time } k \text{ and H at time } k-1) = p(1-p).$$

Computing $\mathbf{E}[R]$ is immediate because

$$\mathbf{E}[R] = \mathbf{E}\left[\sum_{k=1}^{n} I_k\right] = \sum_{k=1}^{n} \mathbf{E}[I_k] = np(1-p).$$

The variance calculation is not as easy because the I_ks are not all independent:

$$\begin{aligned}\mathbf{E}[I_k^2] &= p(1-p) \\ \mathbf{E}[I_k I_{k+1}] &= 0 \quad \text{because rewards at times } k \text{ and } k+1 \text{ are inconsistent} \\ \mathbf{E}[I_k I_{k+\ell}] &= \mathbf{E}[I_k]\mathbf{E}[I_{k+\ell}] = p^2(1-p)^2 \quad \text{for } \ell \geq 2\end{aligned}$$

$$\begin{aligned}
\mathbf{E}[R^2] &= \mathbf{E}[(\sum_{k=1}^{n} I_k)(\sum_{m=1}^{n} I_m)] = \sum_{k=1}^{n}\sum_{m=1}^{n} \mathbf{E}[I_k I_m] \\
&= \underbrace{np(1-p)}_{n \text{ terms with } k=m} + \underbrace{0}_{2(n-1) \text{ terms with } |k-m|=1} + \underbrace{(n^2-3n+2)p^2(1-p)^2}_{n^2-3n+2 \text{ terms with } |k-m|>1}
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}(R) &= \mathbf{E}[R^2] - (\mathbf{E}[R])^2 \\
&= np(1-p) + (n^2-3n+2)p^2(1-p)^2 - n^2p^2(1-p)^2 \\
&= np(1-p) - (3n-2)p^2(1-p)^2.
\end{aligned}$$

G1[†]. (a) We know that I_A is a random variable that maps a 1 to the real number line if ω occurs within an event A and maps a 0 to the real number line if ω occurs outside of event A. A similar argument holds for event B. Thus we have,

$$I_A(\omega) = \begin{cases} 1, & \text{with probability } \mathbf{P}(A) \\ 0, & \text{with probability } 1 - \mathbf{P}(A) \end{cases}$$

$$I_B(\omega) = \begin{cases} 1, & \text{with probability } \mathbf{P}(B) \\ 0, & \text{with probability } 1 - \mathbf{P}(B) \end{cases}$$

If the random variables, A and B, are independent, we have $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$. The indicator random variables, I_A and I_B, are independent if, $\mathbf{P}_{I_A,I_B}(x,y) = \mathbf{P}_{I_A}(x)\mathbf{P}_{I_B}(y)$ We know that the intersection of A and B yields.

$$\begin{aligned}
\mathbf{P}_{I_A,I_B}(1,1) &= \mathbf{P}_{I_A}(1)\mathbf{P}_{I_B}(1) \\
&= \mathbf{P}(A)\mathbf{P}(B) \\
&= \mathbf{P}(A \cap B)
\end{aligned}$$

We also have,

$$\begin{aligned}
\mathbf{P}_{I_A,I_B}(1,1) &= \mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B) = \mathbf{P}_{I_A}(1)\mathbf{P}_{I_B}(1) \\
\mathbf{P}_{I_A,I_B}(0,1) &= \mathbf{P}(A^c \cap B) = \mathbf{P}(A^c)\mathbf{P}(B) = \mathbf{P}_{I_A}(0)\mathbf{P}_{I_B}(1) \\
\mathbf{P}_{I_A,I_B}(1,0) &= \mathbf{P}(A \cap B^c) = \mathbf{P}(A)\mathbf{P}(B^c) = \mathbf{P}_{I_A}(1)\mathbf{P}_{I_B}(0) \\
\mathbf{P}_{I_A,I_B}(0,0) &= \mathbf{P}(A^c \cap B^c) = \mathbf{P}(A^c)\mathbf{P}(B^c) = \mathbf{P}_{I_A}(0)\mathbf{P}_{I_B}(0)
\end{aligned}$$

(b) If $X = I_A$, we know that

$$\mathbf{E}[X] = \mathbf{E}[I_A] = 1 \cdot \mathbf{P}(A) + 0 \cdot (1 - \mathbf{P}(A)) = \mathbf{P}(A)$$

[†]Required for 6.431; optional for 6.041