6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013 Transcript – Recitation: Probability that Three Pieces Form a Triangle

In this problem, we're going to look at the probability that when you take a stick and break it into three pieces randomly that these three pieces can actually be used to form a triangle. All right, so we start out with a stick of unit length, so-- length 1. And we'll choose a point along the stick to break. And we'll choose that point uniformly at random.

So let's say that we chose it here, that was the point where we'll break it. And then independently of this first choice we'll again choose a second point to break it. Again, uniformly at random along the entire stick. So let's say the second point we chose was here. So what we have now is, we'll break it here, here, and so we'll have three pieces-- the first one, the left one and the right one.

And we want to know, what's the probability that when you take these three pieces you could form a triangle? So the first thing we should ask ourselves is, what do you need-- what conditions must be satisfied in order to actually be able to form a triangle with three pieces? So you could think about, what would stop you from being able to do that?

Well, one possibility is that you have pieces that look like this. So in that case you would try to form something that looks like this. But you can't get a triangle because these two pieces are too short and they can't touch each other. So actually the condition that must be satisfied is that when you take any two of the three pieces, their combined length has to be greater than the length of the remaining third piece. And that has to be true for any two pieces.

And really that's just so that any two pieces, they can touch and still form a triangle. So let's try to add some probability to this. So we have a unit length stick. So let's actually give a coordinate system. The stick goes from 0 to 1. And let's say that we break it at these two points.

So the first point where we choose, we'll call that x. So that's the first point that we choose to break it. And then the second point we choose, we'll call that y. Now note that I've drawn it so that x is to the left of y. But it could actually be the case that the first point I chose is here and the second point that I chose is to the left.

But for now, let's first assume that this scenario holds. That the first point is to the left of the second point. So under this assumption, we can see that-- from the definition of these random variables-- we can actually see that the lengths are given by these three lengths.

So the lengths are x, the left most piece has length x. The second, middle piece has length y minus x. And the last piece has length 1 minus y. And now let's recall our three conditions.

So the conditions were that any two of these, the sum of any two lengths, has to be at least-- has to be greater than the length of the third piece. So let's do these together. So x plus y minus x has to be greater than 1 minus y. So with these two pieces you can cover this third piece.

We also need that with the first and third pieces, we can cover the middle piece. And we need with the second and third pieces, we can cover the first piece. Now this looks kind of messy, but in fact we can actually simplify this.

So this actually simplifies. x minus x, that disappears. And so this actually simplifies to 2y has to be at least 1. Or even more simply, y has to be greater than 1/2.

What about this one? This one, we can rearrange things again. x we can move over. y we can move over here. And we get that 2x plus 1 has to be greater than 2y. Or put in other words, y is less than x plus 1/2.

And for the last one, again we can simplify. The y's cancel each other out. And we're left with 2x is less than 1. Or x is less than 1/2. So these are our three conditions that need to be satisfied.

So now we just have to figure out what's the probability that this is actually satisfied? Now let's go back to original definition and see what are the actual distributions for these random variables, x and y. Remember, we defined them to be x is the location of the first break and y is the location of the second break. And as we said in the problem, these are chosen uniformly at random and they're independent.

And so we can actually draw out their joint PDF. So x and y, you can cover any point in the square. And moreover, it's actually uniform within the square. Because each one is chosen uniformly at random and they're independent. So it's anywhere in here.

And so what do we need to do? We just need to identify, what is the probability that these three conditions hold? Rewrite this, line these up. So these are our three conditions that we need. And now remember, we're still working under the assumption that the first point that we chose is actually to the left of the second point.

So what does that mean? That means that we are actually in this top triangle, top half-- x is less than y. All right, so what do we need? We need y to be at least 1/2, so here's 1/2. So we need y to be above this line.

We need x to be less than 1/2. So we need x to be to the left of here. So now so far we're stuck in this upper square. And the last thing we need is y to be less than x plus 1/2.

What is y? The line y equals x and a 1/2, x plus 1/2, is this one. So y has to be less than that, so it would have to be in this triangle here. So these three conditions tell us that in order for us to have a triangle we need to for x and y to fall jointly in this small triangle here.

Now because the joint distribution is uniform, we know that the density is just 1, right? Because the area here is just 1. So the height is just 1 as well. And so the density, or the probability of falling within this small triangle, is just going to be also the area of this triangle.

And what is the area of this triangle? Well, you can fit 8 of these triangles in here, or you could think of it as 1/2 times 1/2 times 1/2. So the area is just 1/8. So assuming that x is less than y, then the probability of forming a triangle is 1/8.

Now, that's only half this story, though. Because it's possible that when you chose these two break points that we actually had the opposite result. That x, the point that you chose first, falls to the right of the point that you chose second.

In which case everything kind of flips. Now we assume that y is less than x, which means that now we're in this lower triangle in the square. Now we can go through this whole exercise again.

But really, what we can see is that all we've really done is just swap the names. Instead of having x and y we now call x-- we call x y and we call y x. And so if we just swap names, we can see that-- let's just fast forward through all these steps and see that we could just swap names here, too, as well, in the three conditions.

So instead of needing y to be greater than 1/2, we just need x to be greater than 1/2. Instead of having x less than 1/2, we need y less than 1/2. We also swap this.

So we need x to be less than y plus 1/2 or y is greater than x minus 1/2. All right, now let's figure out what this corresponds to. We need x to be greater than 1/2, so it needs to be to the right of here. We need y to be less than 1/2, so we need it to be below this line. And we need y to be greater than x minus 1/2.

What is the line y equals x minus 1/2? That is this line here. And we need y to be greater than that, so it needs to be above this line. And so we get that this is the triangle, the small triangle that we need in this case. And notice that it's exactly the same area as this one, right? And so we get another contribution of 1/8 here.

So the final answer is 1/8 plus 1/8 is 1/4. So the probability of forming a triangle using this three pieces is exactly 1/4. And so notice that we've done is, you've set things up very methodically in the beginning by assigning these random variables. And you consider different cases.

Because you don't actually know the order in which x and y might fall, let's just assume that one particular order and work from there. And then do the other case, as well. And it just so happened that because of the symmetry of the problem the second case was actually very simple. We could just see that it is actually symmetric and so we get the same answer.

So this is kind of an interesting problem because it's actually a practical application of something that you might actually do. And you can see that just by applying these probability concepts you can actually--

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