(e) (7 points)
In the new universe in which $X = 2$, we are asked to compute the conditional PDF of Y given the event $Y \geq 3$.

$$f_{Y|X,Y\geq3}(y \mid 2) = \frac{f_{Y|X}(y \mid 2)}{\mathbf{P}(Y \geq 3 \mid X = 2)}.$$

We first calculate the $\mathbf{P}(Y \geq 3 \mid X = 2)$.

$$\begin{aligned}
\mathbf{P}(Y \geq 3 \mid X = 2) &= \int_3^\infty f_{Y|X}(y \mid 2)dy \\
&= \int_3^\infty 2e^{-2y}dy \\
&= 1 - F_{Y|X}(3 \mid 2) \\
&= 1 - (1 - e^{-2 \cdot 3}) \\
&= e^{-6},
\end{aligned}$$

where $F_{Y|X(3|2)}$ is the CDF of an exponential random variable with $\lambda = 2$ evaluated at $y = 3$. Substituting the values of $f_{Y|X}(y \mid 2)$ and $\mathbf{P}(Y \geq 3 \mid X = 2)$ yields

$$f_{Y|X,Y\geq3}(y \mid 2) = \begin{cases} 2e^6 e^{-2y}, & y \geq 3 \\ 0, & \text{otherwise.} \end{cases}$$

Alternatively, $f_{Y|X}(y \mid 2)$ is an exponential random variable with $\lambda = 2$. To compute the conditional PMF $f_{Y|X,Y\geq3}(y \mid 2)$, we can apply the memorylessness property of an exponential variable. Therefore, this conditional PMF is also an exponential random variable with $\lambda = 2$, but it is shifted by 3.

(f) (7 points)
Let's define $Z = e^{2X}$. Since X is an exponential random variable that takes on non-negative values $(X \geq 0), Z \geq 1$. We find the PDF of Z by first computing its CDF.

$$\begin{aligned}
F_Z(z) &= \mathbf{P}(Z \leq z) \\
&= \mathbf{P}(e^{2X} \leq z) \\
&= \mathbf{P}(2X \leq \ln z) \\
&= \mathbf{P}(X \leq \frac{\ln z}{2}) \\
&= 1 - e^{-\frac{\ln z}{2}} \\
&= 1 - e^{\ln z^{-\frac{1}{2}}}
\end{aligned}$$

The CDF of Z is:

$$F_Z(z) = \begin{cases} 1 - z^{-\frac{1}{2}} & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Differentiating the CDF of Z yields the PDF

$$f_Z(z) = \begin{cases} \frac{1}{2} z^{-\frac{3}{2}} & z \geq 1 \\ 0, & z < 1 \end{cases}$$

Alternatively, you can apply the PDF formula for a strictly monotonic function of a continuous random variable. Recall if $z = g(x)$ and $x = h(z)$, then

$$f_Z(z) = f_X(h(z)) \left| \frac{dh}{dy}(z) \right|.$$

In this problem, $z = e^{2x}$ and $x = \frac{1}{2} \ln z$. Note that $f_Z(z)$ is nonzero for $z > 1$. Since X is an exponential random variable with $\lambda = 1$, $f_X(x) = e^x$. Thus,

$$\begin{aligned} f_Z(z) &= e^{-\frac{1}{2} \ln z} \left| \frac{1}{2z} \right| \\ &= e^{\ln z^{-\frac{1}{2}}} \frac{1}{2z} \\ &= \frac{1}{2} z^{-\frac{3}{2}} \quad z \geq 1, \end{aligned}$$

where the second equality holds since the expression inside the absolute value is always positive for $z \geq 1$.

Problem 3. (10 points)

(a) (5 points) The quantity $\mathbf{E}[X \mid Y]$ is always:

   (i) A number.

  (ii) A discrete random variable.

  (iii) A continuous random variable.

  (iv) $\boxed{\text{Not enough information to choose between (i)-(iii).}}$

If X and Y are not independent, then $\mathbf{E}[X \mid Y]$ is a function of Y and is therefore a continuous random variable. However if X and Y are independent, then $\mathbf{E}[X \mid Y] = \mathbf{E}[X]$ which is a number.

(b) (5 points) The quantity $\mathbf{E}[\mathbf{E}[X \mid Y, N] \mid N]$ is always:

   (i) A number.

  (ii) $\boxed{\text{A discrete random variable.}}$

  (iii) A continuous random variable.

  (iv) Not enough information to choose between (i)-(iii).

If X, Y and N are not independent, then the inner expectation $G(Y, N) = \mathbf{E}[X \mid Y, N]$ is a function of Y and N. Furthermore $\mathbf{E}[G(Y, N) \mid N]$ is a function of N, a discrete random variable. If X, Y and N are independent, then the inner expectation $\mathbf{E}[X \mid Y, N] = \mathbf{E}[X]$, which is a number. The expectation of a number given N is still a number, which is a special case of a discrete random variable.

Problem 4. (25 points)

(a)   (i) (5 points)
Using the Law of Iterated Expectations, we have

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X \mid Q]] = \mathbf{E}[Q] = \frac{1}{2}.$$

(ii) (5 points)
X is a Bernoulli random variable with a mean $p = \frac{1}{2}$ and its variance is $\text{var}(X) = p(1-p) = 1/4$.

(b) (7 points)
We know that $\text{cov}(X, Q) = \mathbf{E}[XQ] - \mathbf{E}[X]\mathbf{E}[Q]$, so first let's calculate $\mathbf{E}[XQ]$:

$$\mathbf{E}[XQ] = \mathbf{E}[\mathbf{E}[XQ \mid Q]] = \mathbf{E}[Q\mathbf{E}[X \mid Q]] = \mathbf{E}[Q^2] = \frac{1}{3}.$$

Therefore, we have

$$\text{cov}(X, Q) = \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

(c) (8 points)
Using Bayes' Rule, we have

$$f_{Q|X}(q \mid 1) = \frac{f_Q(q)p_{X|Q}(1 \mid q)}{p_X(1)} = \frac{f_Q(q)\mathbf{P}(X=1 \mid Q=q)}{\mathbf{P}(X=1)}, \quad 0 \le q \le 1.$$

Additionally, we know that
$$\mathbf{P}(X = 1 \mid Q = q) = q,$$

and that for Bernoulli random variables

$$\mathbf{P}(X = 1) = \mathbf{E}[X] = \frac{1}{2}.$$

Thus, the conditional PDF of Q given $X = 1$ is

$$f_{Q|X}(q \mid 1) = \frac{1 \cdot q}{1/2}$$
$$= \begin{cases} 2q, & 0 \le q \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 5. (21 points)

(a) (7 points)

$$\mathbf{P}(S \ge 1) = \mathbf{P}(\min\{X, Y\} \ge 1) = \mathbf{P}(X \ge 1 \text{ and } Y \ge 1) = \mathbf{P}(X \ge 1)\mathbf{P}(Y \ge 1)$$
$$= (1 - F_X(1))(1 - F_Y(1)) = (1 - \Phi(1))^2 \approx (1 - 0.8413)^2 \approx 0.0252.$$

(b) (7 points)

Recalling Problem 2 of Problem Set 6, we have

$$
\begin{aligned}
\mathbf{P}(s \leq S \text{ and } L \leq \ell) &= \mathbf{P}(s \leq \min\{X, Y\} \text{ and } \max\{X, Y\} \leq \ell) \\
&= \mathbf{P}(s \leq X \text{ and } s \leq Y \text{ and } X \leq \ell \text{ and } Y \leq \ell) \\
&= \mathbf{P}(s \leq X \leq \ell)\mathbf{P}(s \leq Y \leq \ell) \\
&= (F_X(\ell) - F_X(s))(F_Y(\ell)F_Y(s)).
\end{aligned}
$$

(c) (7 points)

Given that $s \leq s + \delta \leq \ell$, the event $\{s \leq S \leq s + \delta, \ell \leq L \leq \ell + \delta\}$ is made up of the union of two disjoint possible events:

$$
\{s \leq X \leq s + \delta, \ell \leq Y \leq \ell + \delta\} \cup \{s \leq Y \leq s + \delta, \ell \leq X \leq \ell + \delta\}.
$$

In other words, either $S = X$ and $L = Y$, or $S = Y$ and $L = X$. Because the two events are disjoint, the probability of their union is equal to the sum of their individual probabilities.

Using also the independence of X and Y, we have

$$
\begin{aligned}
\mathbf{P}(s \leq S \leq s + \delta, \ell \leq L \leq \ell + \delta) &= \mathbf{P}(s \leq X \leq s + \delta, \ell \leq Y \leq \ell + \delta) \\
&\quad + \mathbf{P}(s \leq y \leq s + \delta, \ell \leq X \leq \ell + \delta) \\
&= \mathbf{P}(s \leq X \leq s + \delta)\mathbf{P}(\ell \leq Y \leq \ell + \delta) \\
&\quad + \mathbf{P}(s \leq y \leq s + \delta)\mathbf{P}(\ell \leq X \leq \ell + \delta) \\
&= \int_s^{s+\delta} f_X(x)dx \int_\ell^{\ell+\delta} f_Y(y)dy \\
&\quad + \int_s^{s+\delta} f_Y(y)dy \int_\ell^{\ell+\delta} f_X(x)dx
\end{aligned}
$$

6.041SC Probabilistic Systems Analysis and Applied Probability
Fall 2013