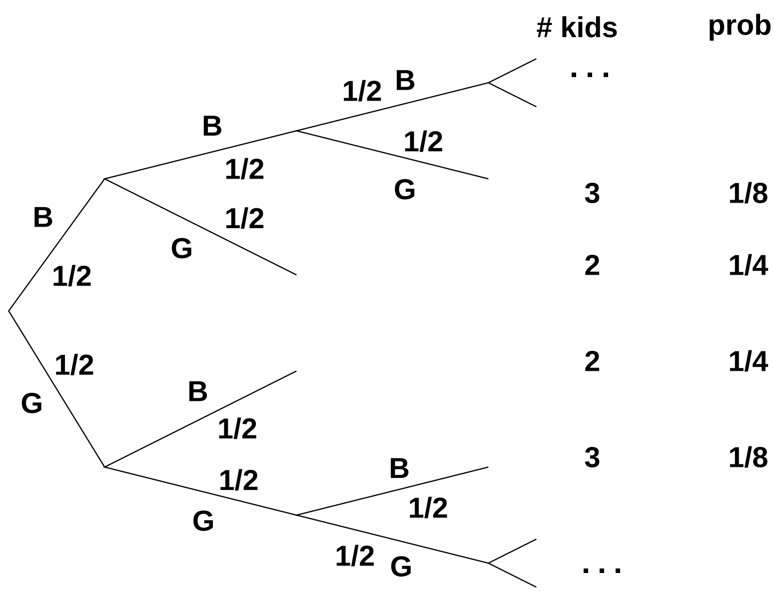
Solutions to In-Class Problems Week 13, Fri.

Problem 1. A couple decides to have children until they have both a boy and a girl. What is the expected number of children that they'll end up with? Assume that each child is equally likely to be a boy or a girl and genders are mutually independent.

Solution. There are many ways to solve this problem. We'll do it from first principles.

Suppose that a couple has children until they have both a boy and a girl. A tree diagram for this experiment is shown below.



Let the random variable R be the number of children the couple has. From the definition of expectation, we have:

$$
\begin{aligned}
\mathrm{E}[R] &= \sum_{w \in S} R(w) \cdot \Pr\{w\} \\
&= \left(2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots\right) + \left(2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots\right) \\
&= 2\left(2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots\right). \qquad (1)
\end{aligned}
$$

The only difficulty is evaluating the sum. We can use the general formula

$$1 + 2r + 3r^2 + 4r^3 + \ldots = \frac{1}{(1-r)^2}$$

which is obtained by differentiating the formula for the sum of an infinite geometric series. Setting $r = 1/2$ gives:

$$1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \ldots = 4$$

We have to tweak this a little to get the sum we're interested in. Subtracting 1 from each side and then dividing both sides by 2 does the trick:

$$2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots = \frac{4-1}{2} = \frac{3}{2}$$

So from (1) we have

$$\mathrm{E}\left[R\right] = 2\left(\frac{3}{2}\right) = 3.$$

A much simpler approach uses the fact that the "mean time to failure" is $1/p$ where p is the probability of failure in one step. If we consider having a child of opposite sex to the first a "failure" of that child, then the mean time to failure is the expected number of children after the first until the couple has both a boy and a girl. But the probability of a failure at the kth child after the first is 1/2 for all $k \geq 1$. So the expected number of children after the first is 1/(1/2) = 2, and the expected number of children including the first is 1+2 =3. ■

Problem 2. There is a nice formula for the expected value of a random variable R that takes on only nonnegative integer values:

$$\mathrm{E}\left[R\right] = \sum_{k=0}^{\infty} \Pr\left\{R > k\right\}$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{\infty} \Pr\left\{R > i\right\} &= \underbrace{\Pr\left\{R = 1\right\} + \Pr\left\{R = 2\right\} + \Pr\left\{R = 3\right\} + \cdots}_{\Pr\{R>0\}} \\
&\qquad\qquad + \underbrace{\Pr\left\{R = 2\right\} + \Pr\left\{R = 3\right\} + \cdots}_{\Pr\{R>1\}} \\
&\qquad\qquad\qquad\qquad + \underbrace{\Pr\left\{R = 3\right\} + \cdots}_{\Pr\{R>2\}} \\
&\qquad\qquad\qquad\qquad\qquad \vdots \\
&= \Pr\left\{R = 1\right\} + 2 \cdot \Pr\left\{R = 2\right\} + 3 \cdot \Pr\left\{R = 3\right\} + \cdots \\
&= \mathrm{E}\left[R\right].
\end{aligned}$$

□

Suppose we roll 6 fair, independent dice. Let R be the *largest* number that comes up. Use the formula above to compute $\mathrm{E}[R]$.

Solution. The first task is to compute $\Pr\{R > k\}$; that is, the probability that *some* die is greater than k. Let's switch to computing the probability of the complementary event:

$$\Pr\{R > k\} = 1 - \Pr\{R \leq k\}$$

Now $\Pr\{R \leq k\}$ is the probability that all the dice show numbers in the set $\{1, \ldots k\}$. If $k \geq 6$, then this probability is 1. For smaller k, the probability that *one* die shows a value in this range is $k/6$. Since the dice are independent, the probability that *all 6* dice are in this range is $(k/6)^6$. Thus, we have:

$$\begin{aligned}
\mathrm{E}[R] &= \sum_{k=0}^{\infty} \Pr\{R > k\} \\
&= 1 + \left(1 - \left(\frac{1}{6}\right)^6\right) + \left(1 - \left(\frac{2}{6}\right)^6\right) + \ldots + \left(1 - \left(\frac{6}{6}\right)^6\right) \\
&= 7 - \frac{1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6}{6^6}
\end{aligned}$$

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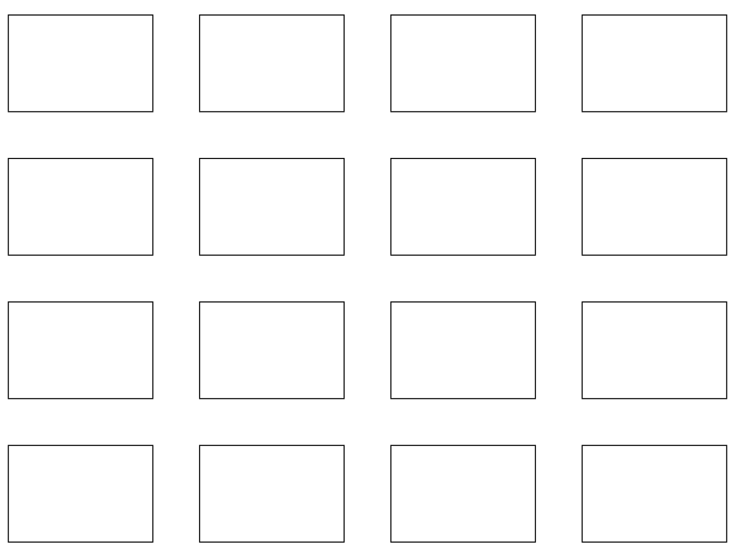
Problem 3. A classroom has sixteen desks arranged as shown below.



If there is a girl in front, behind, to the left, or to the right of a boy, then the two of them *flirt*. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied by boys and girls with equal probability and mutually independently. What is the expected number of flirting couples?

Solution. First, let's count the number of pairs of adjacent desks. There are three in each row and three in each column. Since there are four rows and four columns, there are $3 \cdot 4 + 3 \cdot 4 = 24$ pairs of adjacent desks.

Number these pairs of adjacent desks from 1 to 24. Let F_i be an indicator for the event that occupants of the desks in the i-th pair are flirting. The probability we want is then:

$$\begin{aligned} \mathrm{E}\left[\sum_{i=1}^{24} F_i\right] &= \sum_{i=1}^{24} \mathrm{E}\left[F_i\right] \\ &= \sum_{i=1}^{24} \Pr\left\{F_i = 1\right\} \end{aligned}$$

The first step uses linearity of expectation, and the second uses the fact that the expectation of an indicator is equal to the probability that it is 1.

The occupants of adjacent desks are flirting if the first holds a girl and the second a boy or vice versa. Each of these events happens with probability $1/2 \cdot 1/2 = 1/4$, and so the probability that the occupants flirt is

$$\Pr\left\{F_i = 1\right\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Plugging this into the previous expression gives:

$$\begin{aligned} \mathrm{E}\left[\sum_{i=1}^{24} F_i\right] &= \sum_{i=1}^{24} \Pr\left\{F_i = 1\right\} \\ &= 24 \cdot \frac{1}{2} \\ &= 12 \end{aligned}$$

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