Number Theory

Divisibility

- a "divides" b (a | b)

  if $ak=b$ for some $k \in \mathbb{Z}$

- Example: $5 \mid 15$ $k=3$

  $n \mid 0$

Perfect number

- Perfect number = sum of divisors other than self
- Examples:
  - 6=1+2+3
  - 28=1+2+4+7+14
- Any odd perfect numbers?

Divisibility facts

1. $a|b \Rightarrow a|bc$ for all c
2. $a|b$ and $b|c \Rightarrow a|c$
3. $a|b$ and $a|c \Rightarrow a|sb+tc$ for all s,t

   i.e., a divides every linear combination of b and c

4. for all $c \neq 0$, $a|b \Leftrightarrow ac|bc$

Proof of "$a|b$ and $b|c \Rightarrow a|c$"

$\exists\ k_1, k_2$ such that

$$ak_1 = b \quad \text{and} \quad bk_2 = c$$

so $ak_1 \cdot k_2 = c$

letting $k' = k_1 \cdot k_2$ gives $ak' = c$

$\Rightarrow a|c$

Division Theorem

For all n, for all $d>0$,

there is a unique q, r such that

$$n = qd + r$$

$(r = n \text{ rem } d)$

where $0 \leq r < d$

Example: $n = 6042 \quad d = 10$

$$6042 = 604 \cdot 10 + 2$$

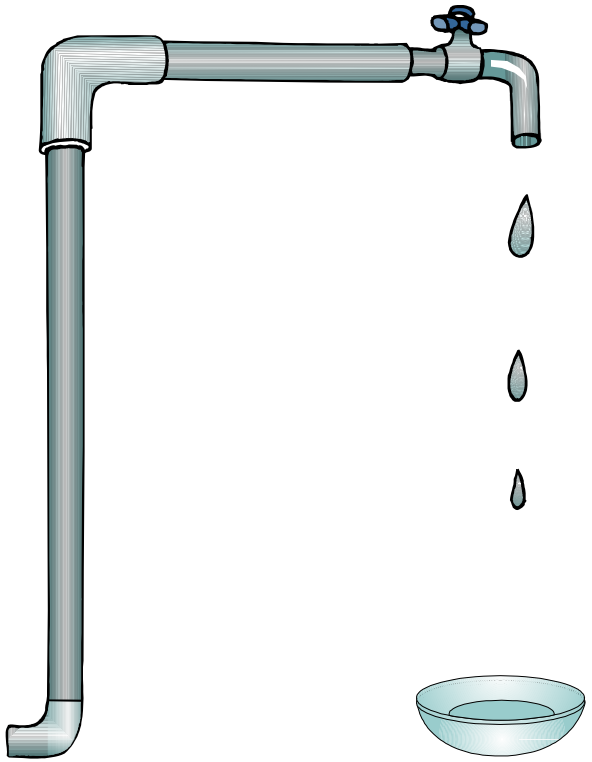
Die Hard 3

Die Hard 3

Picture source: http://movieweb.com/movie/diehard3/

Die Hard

Supplies:



Water

3 Gallon Jug

5 Gallon Jug

Die Hard

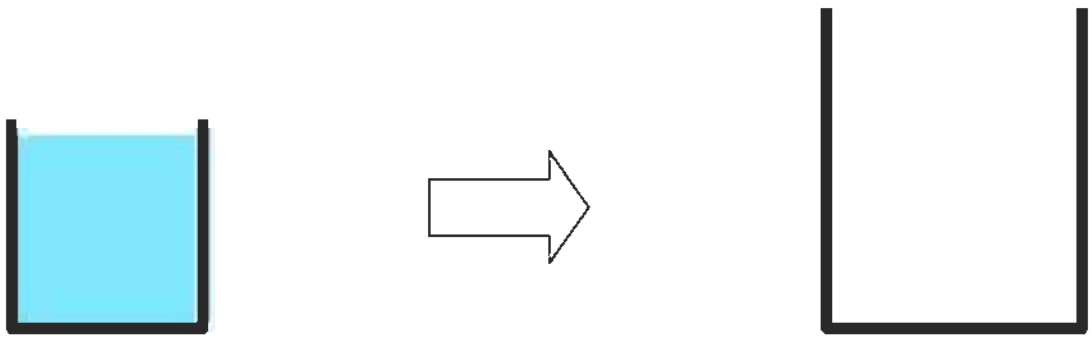
Psychopath's challenge:

Disarm bomb by putting 4 gallons of water on scale, or it will blow up.

Question: How to do it?

Die Hard

Transferring water:

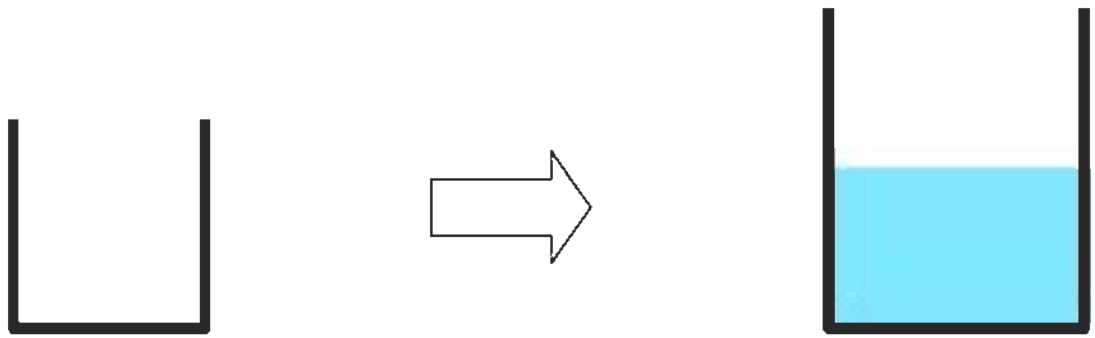


3 Gallon Jug

5 Gallon Jug

Die Hard

Transferring water:



3 Gallon Jug

5 Gallon Jug

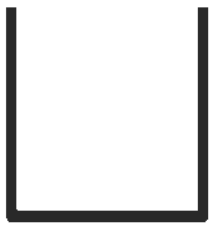
Die Hard

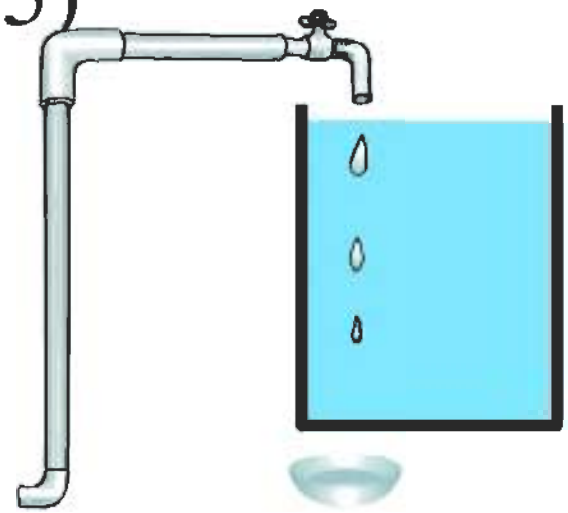
Work it out now!

How to do it

Start with empty jugs: (0,0)

Fill the big jug: (0,5)



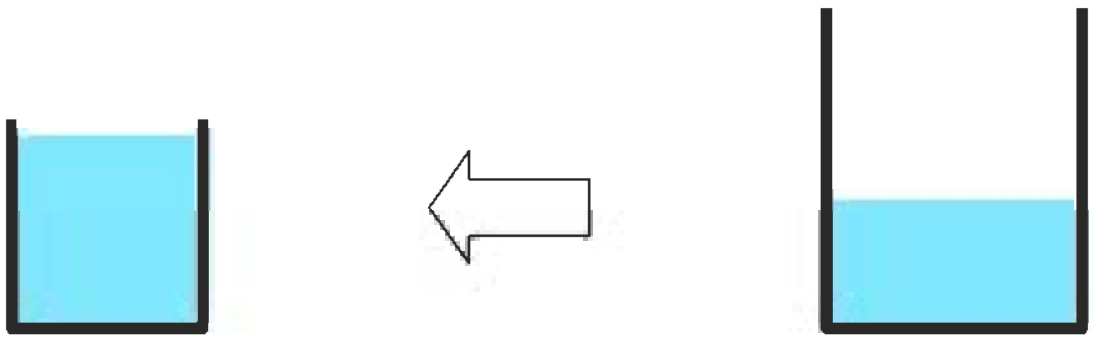


3 Gallon Jug

5 Gallon Jug

How to do it

Pour from big to little: (3,2)

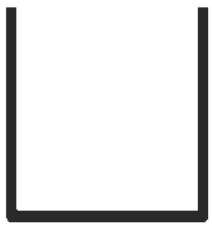


3 Gallon Jug

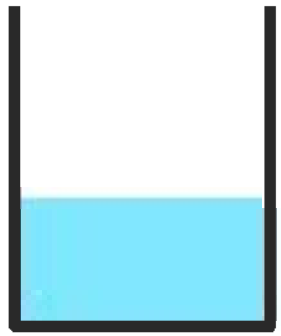
5 Gallon Jug

How to do it

Empty the little: (0,2)



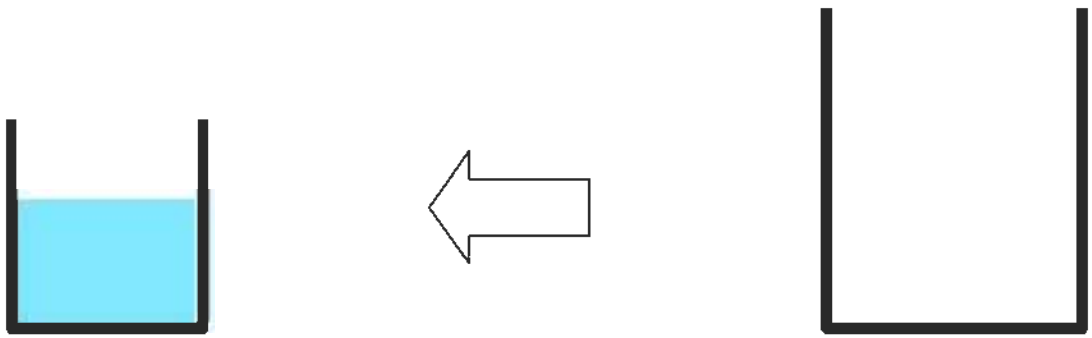
3 Gallon Jug



5 Gallon Jug

How to do it

Pour from big to little: (2,0)



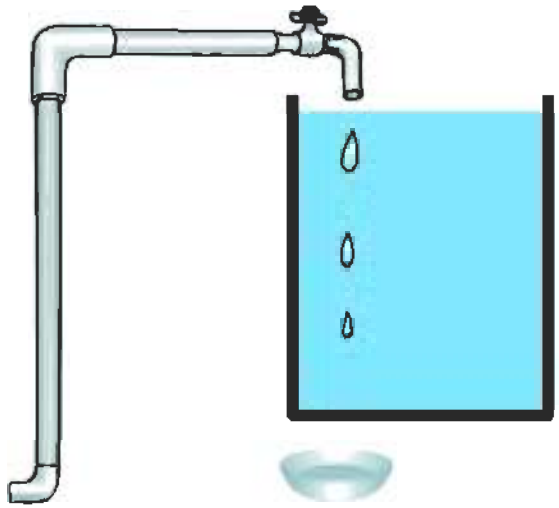
3 Gallon Jug

5 Gallon Jug

How to do it

Fill the big jug: (2,5)

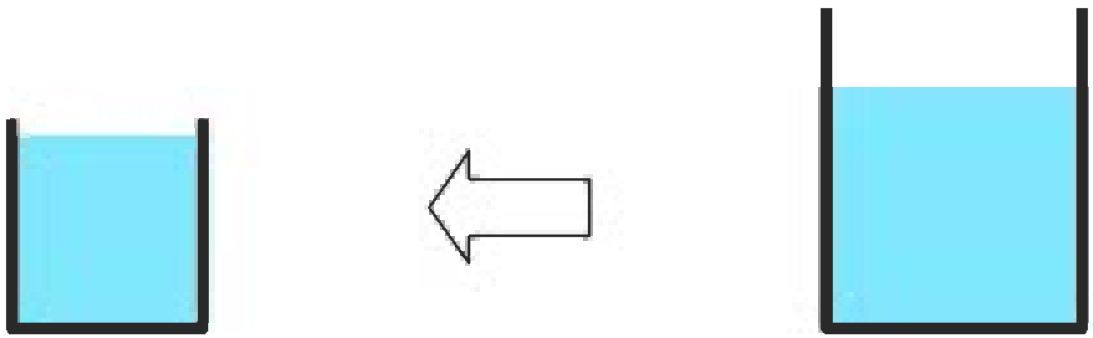
3 Gallon Jug



5 Gallon Jug

How to do it

Pour from big to little: (3,4)

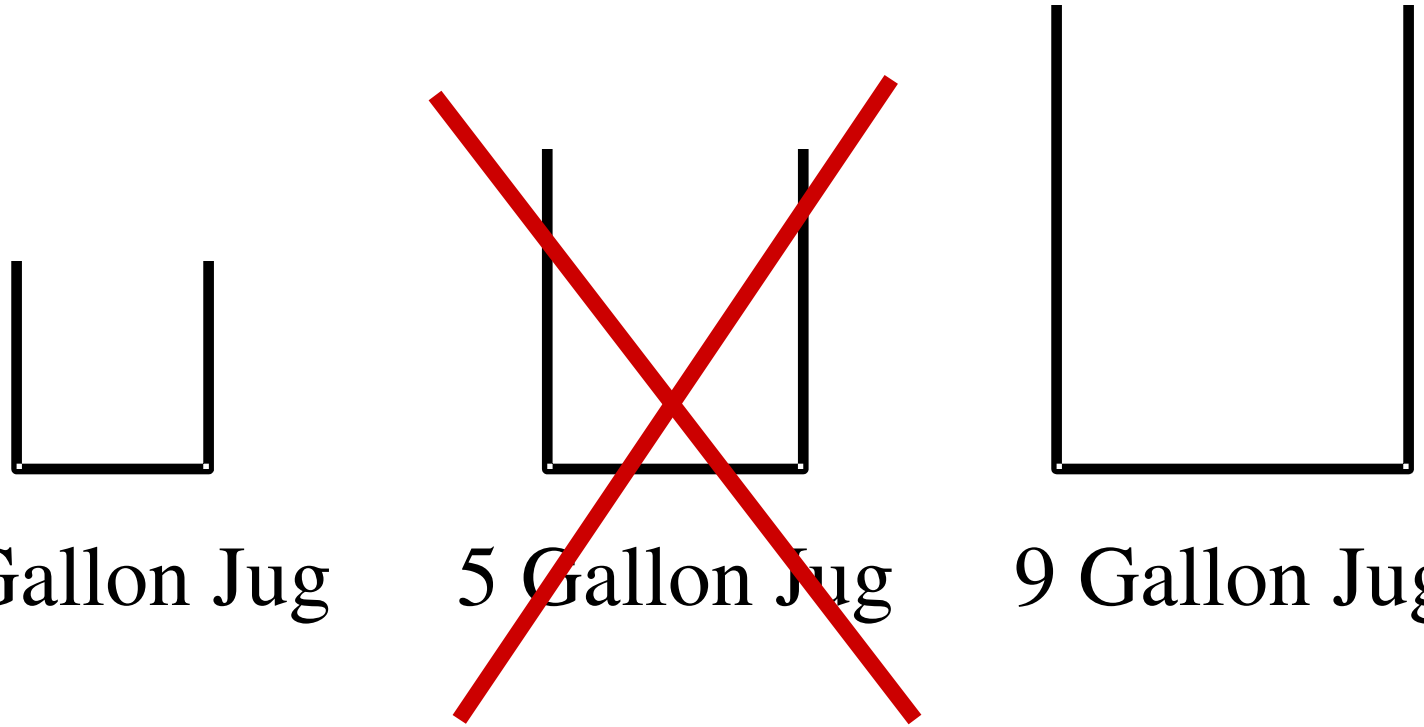


3 Gallon Jug 5 Gallon Jug

Done!!

Die Hard Once and For All

What if you have a 9 gallon jug instead?



Can you do it? Can you prove it?

- Allowed operations:
  - Fill a jug with water
  - Empty a jug onto the sidewalk
  - Transfer water from one jug to another until first jug is empty or the other jug is full

Class Problems 1 and 2