6.055J / 2.038J The Art of Approximation in Science and Engineering
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The volume of the pyramid is $V \sim hb^2$, and the missing constant must make volume 4/3. Since $hb^2 = 4$ for these pyramids, the missing constant is 1/3. Voilà:

$$V = \frac{1}{3}hb^2 = \frac{4}{3}.$$

8.2 Mechanics

8.2.1 Atwood machine

The next problem illustrates dimensional analysis and special cases in a physical problem. Many of the ideas and methods from the geometry example transfer to this problem, and it introduces more methods and ways of reasoning.

The problem is a staple of first-year physics: Two masses, m_1 and m_2, are connected and, thanks to a pulley, are free to move up and down. What is the acceleration of the masses and the tension in the string? You can solve this problem with standard methods from first-year physics, which means that you can can check the solution that we derive using dimensional analysis, educated guessing, and a feel for functions.
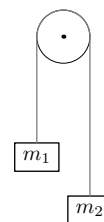
The first problem is to find the acceleration of, say, m_1. Since m_1 and m_2 are connected by a rope, the acceleration of m_2 is, depending on your sign convention, either equal to m_1 or equal to $-m_1$. Let's call the acceleration a and use dimensional analysis to guess its form. The first step is to decide what variables are relevant. The acceleration depends on gravity, so g should be on the list. The masses affect the acceleration, so m_1 and m_2 are on the list. And that's it. You might wonder what happened to the tension: Doesn't it affect the acceleration? It does, but it is itself a consequence of m_1, m_2, and g. So adding tension to the list does not add information; it would instead make the dimensional analysis difficult.

These variables fall into two pairs where the variables in each pair have the same dimensions. So there are two dimensionless groups here ripe for picking: $G_1 = m_1/m_2$ and $G_2 = a/g$. You can make any dimensionless group using these two obvious groups, as experimentation will convince you. Then, following the usual pattern,

Var	Dim	What
a	LT^{-2}	accel. of m_1
g	LT^{-2}	gravity
m_1	M	block mass
m_2	M	block mass

$$\frac{a}{g} = f\left(\frac{m_1}{m_2}\right),$$

where f is a dimensionless function.

Pause a moment. The more thinking that you do to choose a clean representation, the less algebra you do later. So rather than find f using m_1/m_2 as the dimensionless group, first choose a better group. The ratio m_1/m_2 does not respect the symmetry of the problem in that only the sign of the acceleration changes when you interchange the labels m_1 and m_2. Whereas m_1/m_2 turns into its reciprocal. So the function f will have to do lots of work to turn the unsymmetric ratio m_1/m_2 into a symmetric acceleration.

Back to the drawing board for how to fix G_1. Another option is to use $m_1 - m_2$. Wait, the difference is not dimensionless! I fix that problem in a moment. For now observe the virtue of $m_1 - m_2$. It shows a physically reasonable symmetry under mass interchange: $G_1 \rightarrow -G_1$. To make it dimensionless, divide it by another mass. One candidate is m_1:

$$G_1 = \frac{m_1 + m_2}{m_1}.$$

That choice, like dividing by m_2, abandons the beloved symmetry. But dividing by $m_1 + m_2$ solves all the problems:

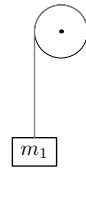$$G_1 = \frac{m_1 - m_2}{m_1 + m_2}.$$

This group is dimensionless and it respects the symmetry of the problem.

Using this G_1, the solution becomes

$$\frac{a}{g} = f\left(\frac{m_1 - m_2}{m_1 + m_2}\right),$$

where f is another dimensionless function.

To guess $f(x)$, where $x = G_1$, try special cases. First imagine that m_1 becomes huge. A quantity with mass cannot be huge on its own, however. Here huge means *huge relative to m_2*, whereupon $x \approx 1$. In this thought experiment, m_1 falls as if there were no m_2 so $a = -g$. Here we've chosen a sign convention with positive acceleration being upward. If m_2 is huge relative to m_1, which means $x = -1$, then m_2 falls like a stone pulling m_1 upward with acceleration $a = g$. A third limiting case is $m_1 = m_2$ or $x = 0$, whereupon the masses are in equilibrium so $a = 0$.

Here is a plot of our knowledge of f:
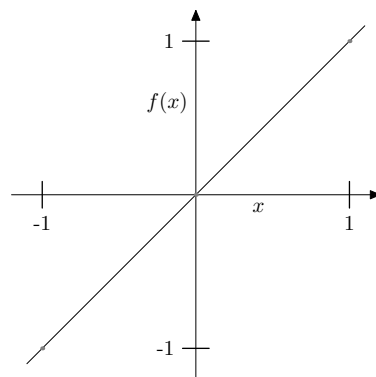
The simplest conjecture – an educated guess – is that $f(x) = x$. Then we have our result:

$$\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}.$$

Look how simple the result is when derived in a symmetric, dimensionless form using special cases!