Frequency response: Passive Filters

Let's consider again the *RC* filter shown on Figure 1

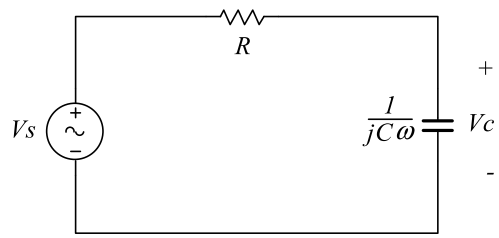


Figure 1

When the output is taken across the capacitor the magnitude of the transfer function is

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} \tag{1.1}$$

By letting $\omega_0 = \dfrac{1}{RC}$ the transfer function becomes

$$|H(\omega)| = \frac{1}{\sqrt{1+\left(\dfrac{\omega}{\omega_0}\right)^2}} \tag{1.2}$$

The overall characteristics of the transfer function may be determined by considering what happens at $\omega = 0$ and at $\omega \to \infty$.

$$\omega = 0, \qquad |H(\omega)| = 1$$

$$\omega \to \infty, \qquad |H(\omega)| \to 0$$

It is also interesting to look at the value of the transfer function at the frequency ω_0

For $\omega = \omega_0$ the magnitude of the transfer function becomes

$$|H(\omega)|_{\omega=\omega_0} = \frac{1}{\sqrt{2}} \tag{1.3}$$

The frequency ω_0 is called the corner, cutoff, or the ½ power frequency. Also, by considering the definition of the dB we have

$$\left|H(\omega)\right|_{dB} = 20\log\left(\left|H(\omega)\right|\right) \tag{1.4}$$

Which at $\omega = \omega_0$ gives

$$\left|H(\omega)\right|_{dB} = -3dB \tag{1.5}$$

And so the frequency ω_0 is also called the 3dB frequency.

For our example *RC* circuit with *R=10kΩ* and *C=47nF* the Bode plot of the transfer function is shown on Figure 2. In this case the corner frequency equals 2,127 rad/sec and it is indicated on Figure 2.

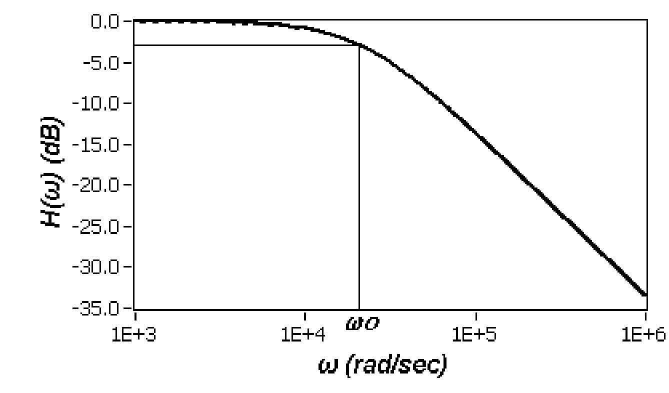


Figure 2

If the output is taken across the resistor, the magnitude of the transfer function becomes

$$\left|H(\omega)\right| = \frac{\dfrac{\omega}{\omega_0}}{\sqrt{1+\left(\dfrac{\omega}{\omega_0}\right)^2}} \tag{1.6}$$

In this case the limits are

$$\omega = 0, \qquad \left|H(\omega)\right| = 0$$

$$\omega \to \infty, \qquad \left|H(\omega)\right| \to 1$$

The plot of this transfer function is shown on Figure 3

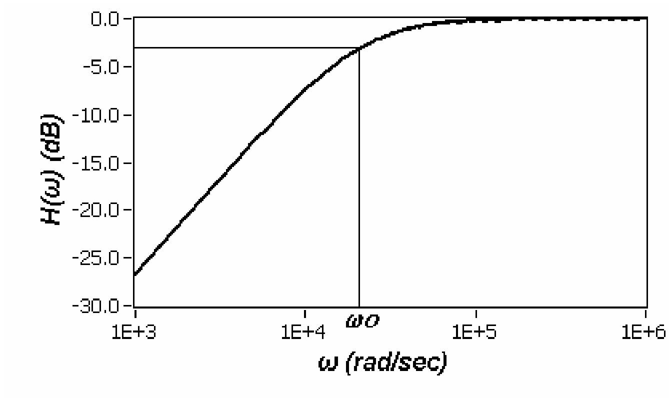


Figure 3

Filtering and Filters

By investigating Figure 2 and Figure 3 we see that the magnitude of the output signal is a very strong function of frequency.

The attenuation of the signal amplitude with frequency is also called filtering and the circuits that perform this operation are called filters. In general we say that filters are circuits which allow a specific range of frequencies to be passed (or rejected) as they are transmitted from an ac source to a load. Schematically the system is shown on Figure 4.

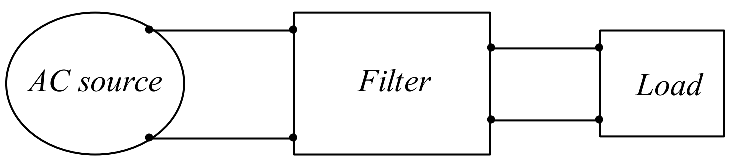
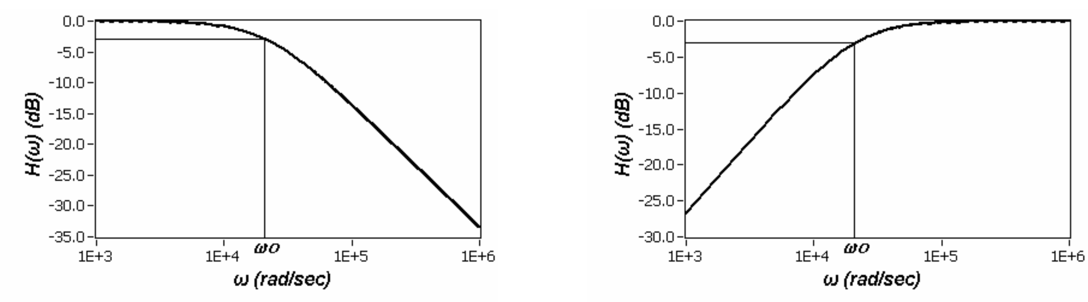


Figure 4

Filters in general fall into one of the following categories:

- Low Pass: passes low frequencies (that is signals with low frequencies) and attenuates high frequencies
- High Pass: passes high frequencies (that is signals with high frequencies) and attenuates low frequencies
- Band Pass: passes frequencies in a certain range and attenuates frequencies outside this range
- Band Stop: attenuates frequencies within a certain range and passes frequencies outside this range.

For the RC circuit, when the output is taken across the capacitor we obtain a Low Pass filter. By contrast when the output is taken across the resistor we have a High Pass filter. The corresponding plots are shown on Figure 5.



Low Pass Filter **High Pass Filter**

Figure 5

The transition frequency which indicates that range of frequencies that are allowed and those that are rejected is given by the cutoff frequency ω_0. In practical situations the design of a High pass or Low pass filter is guided by the value of the cutoff or corner frequency ω_0. For our example RC circuit, with *R=10kΩ* and *C=47nF,* the cutoff frequency is 338 Hz.

We may obtain a band pass filter by combining a low pas and a high pass filter. Consider the arrangement shown on Figure 6.

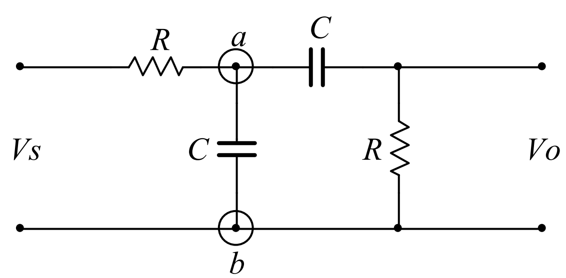


Figure 6

The transfer function may be calculated very easily if we first consider the equivalent circuit to the left of a-b as shown on Figure 7

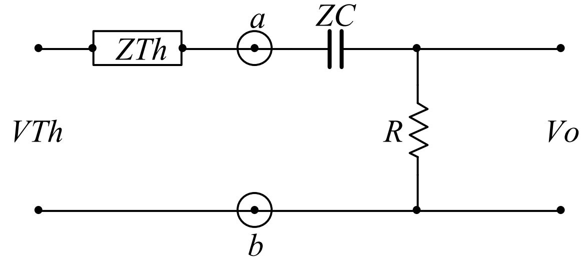


Figure 7

The voltage *VTh* is

$$VTh = Vs\frac{ZC}{ZC + R} \tag{1.7}$$

And

$$ZTh = \frac{R\ ZC}{R + ZC} \tag{1.8}$$

The transfer function now becomes

$$H(\omega) = \frac{Vo}{Vs} = \frac{ZC\ R}{(R + ZC)^2 + R\ ZC} \tag{1.9}$$

And upon simplification the magnitude becomes

$$\left|H(\omega)\right| = \frac{R}{\sqrt{(3R)^2 + \left(R^2\omega C - \frac{1}{\omega C}\right)^2}} \tag{1.10}$$

By looking at low and high values for ω we have

$$\omega = 0, \qquad \left|H(\omega)\right| = 0$$

$$\omega \to \infty, \qquad \left|H(\omega)\right| \to 0$$

Also we notice that for $\omega = \frac{1}{RC}$ the magnitude becomes $\left|H(\omega)\right| = \frac{1}{\sqrt{3}}$

The plot of Equation (1.10) is shown on Figure 8. This has the form of a band pass filter although the attenuation at the frequency $1/RC$ is not desirable. We will next look at ways to improve this type of filter by considering the RLC circuit.

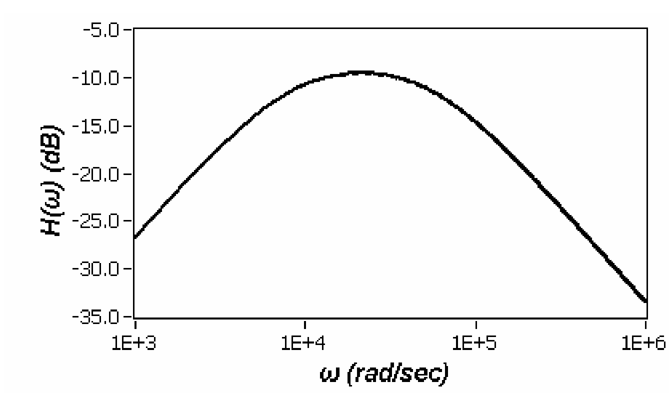
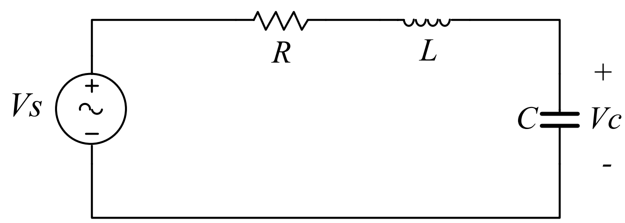


Figure 8

Now let's continue by exploring the frequency response of RLC circuits.



The magnitude of the transfer function when the output is taken across the capacitor is

$$\left|\frac{Vc}{Vs}\right| = |H(\omega)| = \frac{1}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.11}$$

Here again let's look at the behavior of the transfer function, $|H(\omega)|$, for low and high frequencies.

$$\begin{aligned} &\omega = 0, \qquad |H(\omega)| = 1 \\ &\omega \to \infty, \qquad |H(\omega)| \to 0 \end{aligned} \tag{1.12}$$

There is another frequency that has a significant effect on the behavior of $|H(\omega)|$. This is the frequency ω_0 at which

$$1 = \omega_0^{\ 2} LC \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} \tag{1.13}$$

At this frequency the magnitude of the transfer function becomes

$$|H(\omega_0)| = \frac{\sqrt{LC}}{RC} \tag{1.14}$$

From the scaling given by Equation (1.12) we see that this circuit corresponds to a low pass filter. Indeed Figure 9 shows the plot for $|H(\omega)|$ for *R=2kΩ, L=47mH* and *C=47nF* (the values we also used in the laboratory).

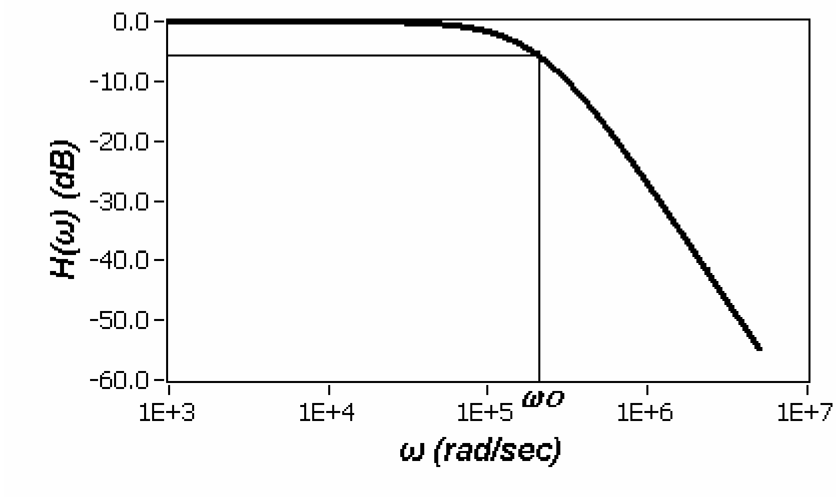


Figure 9

The cutoff frequency in this case is given by the frequency $\omega_0 = \dfrac{1}{\sqrt{LC}} = 21,276$ rad/sec.

This is also indicated on the plot of Figure 9.

The magnitude $|H(\omega)|$ at $\omega = \omega_0$ is inversely proportional to the resistor R. So let's now investigate the behavior of the transfer function with *R*. Figure 10 shows the *R* dependence of the transfer function. We have plotted

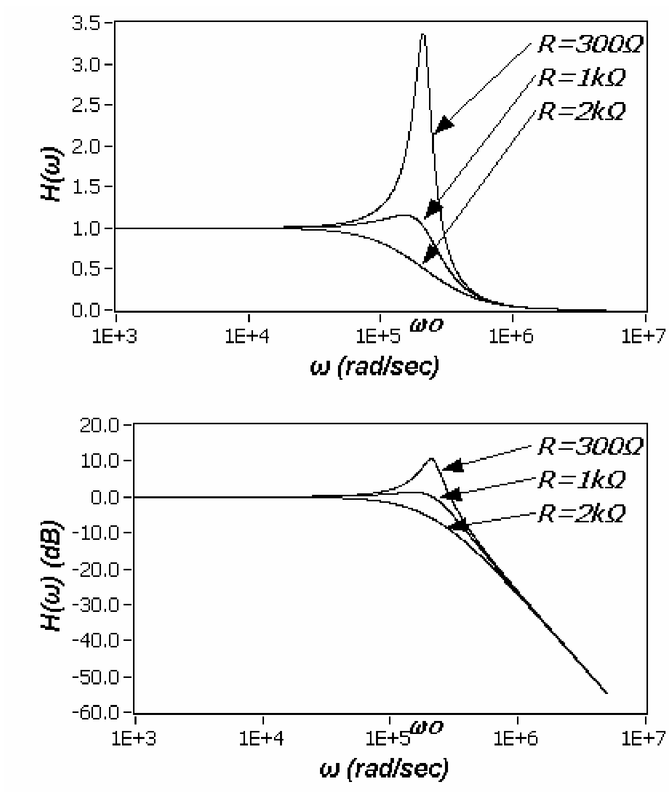


Figure 10

The peak observed at the frequency ω_0 is called the resonance peak and the frequency ω_0 is also referred to as the resonance frequency. A ω_0 the transfer function becomes

$$H(\omega) = \frac{Vc}{Vs} = \frac{1}{j\frac{RC}{\sqrt{LC}}} = \frac{1}{\sqrt{\frac{(RC)^2}{LC}}} e^{-\frac{\pi}{2}} \tag{1.15}$$

Which shows that there is 90 degree phase difference between *Vc* and *Vs*.
The current flowing through the capacitor is

$$I = j\omega C Vc \tag{1.16}$$

And thus the phase difference between the current *I* and the source voltage *Vs* is zero.

Resonance if defined as the condition at which the voltage and the current at the input of a circuit is in phase.

If we take the output across the inductor the magnitude of the transfer function is

$$\left|\frac{VL}{Vs}\right| = \left|H(\omega)\right| = \frac{\omega^2 LC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.17}$$

In this case, consideration of the frequency limits gives

$$\begin{aligned} &\omega = 0, \qquad \left|H(\omega)\right| = 0 \\ &\omega \to \infty, \qquad \left|H(\omega)\right| \to 1 \end{aligned} \tag{1.18}$$

And it corresponds to a high pass filter.

Figure 11 shows the plot of $\left|H(\omega)\right|$ for various values of *R*. Here again we observe the resonance phenomenon at $\omega_0 = \frac{1}{\sqrt{LC}}$.

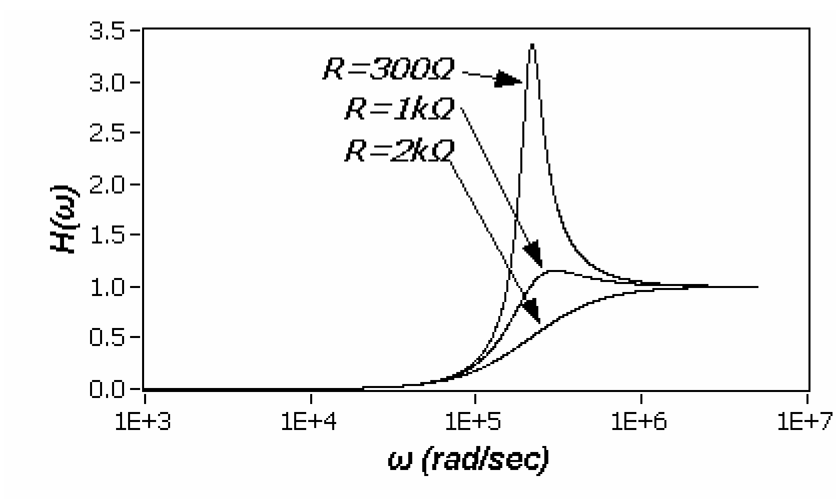


Figure 11

If we take the voltage across the resistor the transfer function becomes

$$\left|\frac{VR}{Vs}\right| = \left|H(\omega)\right| = \frac{\omega RC}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.19}$$

In this case, consideration of the frequency limits gives

$$\begin{aligned} \omega = 0, \qquad & \left|H(\omega)\right| = 0 \\ \omega \to \infty, \qquad & \left|H(\omega)\right| \to 0 \end{aligned} \tag{1.20}$$

And it corresponds to a band pass filter.

Figure 12 shows the plot $\left|H(\omega)\right|$ for various values of R. Here again we observe the peak at $\omega_0 = \frac{1}{\sqrt{LC}}$. In this case however, the magnitude of the transfer function does not exceed 1.

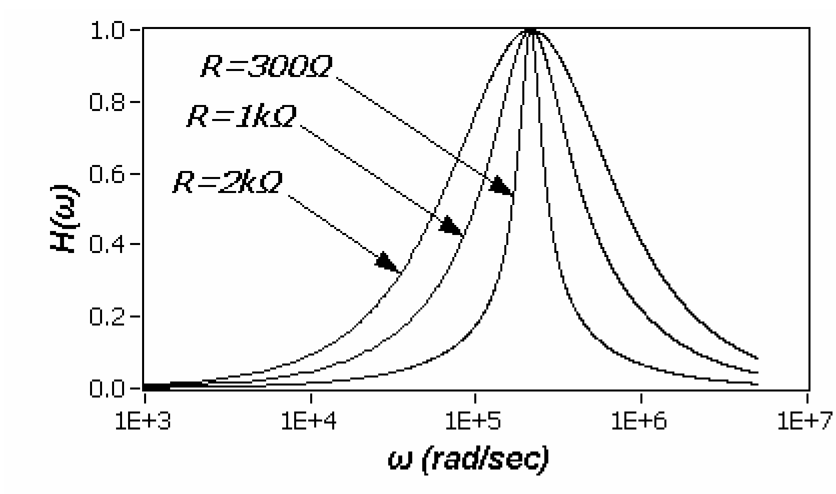


Figure 12

Figure 12 also shows that as the resistance increases, the peak becomes broader. This is a direct consequence of the increases damping provided by the resistor. We will analyze this phenomenon in detail next class.

Now we consider the voltage across the capacitor and inductor combination. In this case the magnitude of the transfer function is

$$\left|\frac{VR}{Vs}\right| = \left|H(\omega)\right| = \frac{\left|1-\omega^2 LC\right|}{\sqrt{\left(1-\omega^2 LC\right)^2 + \left(\omega RC\right)^2}} \tag{1.21}$$

In this case, consideration of the frequency limits gives

$$\begin{aligned} &\omega = 0, \qquad \left|H(\omega)\right| = 1 \\ &\omega \to \infty, \qquad \left|H(\omega)\right| \to 1 \end{aligned} \tag{1.22}$$

And it corresponds to a band reject filter.

Figure 13 shows the plot $\left|H(\omega)\right|$ for various values of R. Here again we observe the peak at $\omega_0 = \frac{1}{\sqrt{LC}}$. Here again we see the effect that R has on the details of the filter. We will investigate this phenomenon next class.

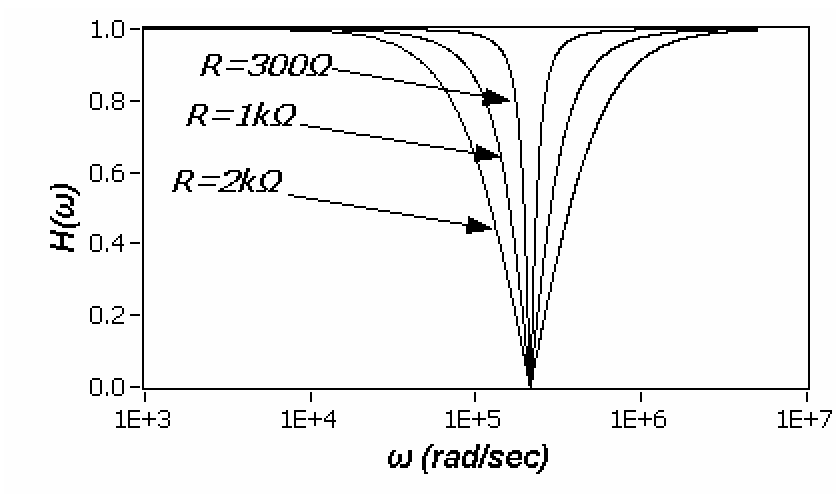


Figure 13