Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

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Problem Set 3[1]

Problem 3.1

Find out which of the functions $V : \mathbf{R}^2 \to \mathbf{R}$,

(a) $V(x_1, x_2) = x_1^2 + x_2^2$;

(b) $V(x_1, x_2) = |x_1| + |x_2|$;

(c) $V(x_1, x_2) = \max |x_1|, |x_2|$;

are valid Lyapunov functions for the systems

(1) $\dot{x}_1 = -x_1 + (x_1 + x_2)^3$, $\dot{x}_2 = -x_2 - (x_1 + x_2)^3$;

(2) $\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2)$, $\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$;

(3) $\dot{x}_1 = x_2|x_1|$, $\dot{x}_2 = -x_1|x_2|$.

Problem 3.2

Show that the following statement is not true. Formulate and prove a correct version: *if $V : \mathbf{R}^n \mapsto \mathbf{R}$ is a continuously differentiable functional and $a : \mathbf{R}^n \mapsto \mathbf{R}^n$ is a continuous function such that*

$$\nabla V(\bar{x})a(\bar{x}) \leq 0 \quad \forall \ \bar{x} : \ V(\bar{x}) = 1, \tag{3.1}$$

then $V(x(t)) \leq 1$ for every solution $x : [0, \infty) \to \mathbf{R}^n$ of

$$\dot{x}(t) = a(x(t)) \tag{3.2}$$

with $V(x(0)) \leq 1$.

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Problem 3.3

The optimal minimal-time controller for the double integrator system with bounded control

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \end{cases} \qquad |u(t)| \leq 1$$

has the form

$$u(t) = \operatorname{sgn}(x_1(t) + 0.5x_2(t)^2 \operatorname{sgn}(x_2(t))).$$

(Do you know why ?)

(a) Find a Lyapunov function $V : \mathbf{R}^2 \mapsto \mathbf{R}^2$ for the closed loop system, such that $V(x(t))$ is strictly decreasing along all solutions of system equations except the equilibrium solution $x(t) \equiv 0$.

(b) Find out whether the equilibrium remains asymptotically stable when the same controller is used for the perturbed system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\epsilon x_1(t) + u(t), \end{cases} \qquad |u(t)| \leq 1,$$

where $\epsilon > 0$ is small.