MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Fall 2008 6.436J/15.085J

Midterm exam, 7-9pm (120 mins/100 pts) 10/21/08

Problem 1: (15 points)
Let $\{X_n\}$ be a sequence of random variables (i.e., measurable functions) defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show, from first principles (i.e., only using the definition of σ-fields and of measurability) that the event $\{\sup_n X_n < c\}$ is measurable.

Solution: Observe that the event

$$\sup_n X_n \leq c,$$

is measurable for any c because

$$\{\sup_n X_n \leq c\} = \bigcap_n \{X_n \leq c\},$$

and now we can just write

$$\{\sup_n X_n < c\} = \bigcup_{n=1}^{\infty} \{\sup_n X_n \leq c - 1/n\},$$

so the event in question is a countable union of measurable sets.

Problem 2: (15 points)
Let $\{X_n\}$ and $\{Y_n\}$ be two nondecreasing sequences of random variables, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (That is, $X_n(\omega) \leq X_{n+1}(\omega)$, for all ω, and similarly for Y_n.) Let $X = \lim_{n\to\infty} X_n$ and $Y = \lim_{n\to\infty} Y_n$. Assume that for every n, the random variables X_n and Y_n are independent. Show (using only the definitions and basic facts about measures) that X and Y are independent. *Hint:* Use the definition of independence in terms of CDFs.

Solution: Observe that

$$\{X \leq x, Y \leq y\} = \bigcap_{i=1}^{\infty} \{X_i \leq x, Y_i \leq y\}.$$

Indeed, if ω is in the set on the left-hand side, then it must be in each set onn the right-hand side since $X_i(\omega) \leq X(\omega)$ and $Y_i(\omega) \leq Y(\omega)$ for all i. Conversely, if

ω belongs to each of the sets on the right hand side, then $X_i(\omega) \leq x, Y_i(\omega) \leq y$ for all i, and passing to the limit we get $X(\omega) \leq x, Y(\omega) \leq y$, so ω must belong to the set on the left-hand side.

The monotonicity of X_i, Y_i implies that $\{X_i \leq x, Y_i \leq y\}$ is a decreasing sequence. So, using the continuity property of probability measures

$$\mathbb{P}(X \leq x, Y \leq y) = \lim_n \mathbb{P}(X_n \leq x, Y_n \leq y),$$

and now we use the independence of X_n, Y_n to get

$$\mathbb{P}(X \leq x, Y \leq y) = \lim_n \mathbb{P}(X_n \leq x)\mathbb{P}(Y_n \leq y) = \lim_n \mathbb{P}(X_n \leq x) \lim_n \mathbb{P}(Y_n \leq y),$$

and now using continuity of probabilities again,

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y),$$

which is what we needed to show.

Problem 3: (15 points)
Let F_1 and F_2 be two CDFs, and suppose that $F_1(t) < F_2(t)$, for all t. Assume that F_1 and F_2 are continuous and strictly increasing. Show that there exist random variables X_1 and X_2, with CDFs F_1 and F_2, defined on the same probability space such that $X_1 > X_2$.

Solution: Observe that since $F_1(t), F_2(t)$ are continuous and strictly increasing, the functions $F_1^{-1}(t), F_2^{-1}(t)$ are well-defined when $t \in (0, 1)$. Let U be a uniform random variable in $(0, 1)$ and let

$$\begin{aligned} X_1 &= F_1^{-1}(U) \\ X_2 &= F_2^{-1}(U) \end{aligned}$$

Then the CDF of X_1 is F_1 since

$$\mathbb{P}(X_1 \leq t) = \mathbb{P}(F_1^{-1}(U) \leq t) = \mathbb{P}(U \leq F_1(t)) = F_1(t),$$

and similarly the DF of X_2 is F_2. Since $F_2 > F_1$ it follows that $F_2^{-1} < F_1^{-1}$ and so $X_1 > X_2$.

Problem 4: (20 points)
Let $\{X_n\}$ be a sequence of independent random variables. Show that $\sup_n X_n < \infty$, a.s., if and only if $\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) < \infty$ for some c.

Remark: The problem indicates that X_i are random variables (as opposed to extended-valued random variables), which means that their range is the real numbers. On the other hand, we can't say the same about $\sup_i X_i$, since it can take the value of $+\infty$.

Solution: Suppose $\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) = \infty$ for all c. For any c, the probability of $\sup_i X_i < c$ must be 0 since the event $X_i > c$ occurs infinitely often with probability 1 by the Borel-Cantelli lemma. It follows that

$$\mathbb{P}(\sup_i X_i \text{ finite }) = \mathbb{P}(\cup_{n=0}^{\infty} \sup_i X_i < n) \le \sum_n \mathbb{P}(\sup_i X_i < n) = \sum_n 0 = 0,$$

so $\sup_i X_i$ must equal $+\infty$ with probability 1.

Suppose that for some c, $\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) < \infty$. By the Borel-Cantelli lemma, this means that with probability 1, the number of times $X_i > c$ is finite. If $X_i > c$ occurs finitely many times, then $\sup_i X_i$ is finite, and it follows that $\sup_i X_i$ is finite with probability 1.

Problem 5: (20 points)
Let X be a continuous nonnegative random variable with PDF f. Let $N = \lfloor X \rfloor$ be the integer part of X (i.e., N is the largest integer that satisfies $N \le X$). Let $U = X - N$.

(a) Write down an expression (in terms of f) for the conditional PDF $f_{U|N}(u \mid n)$ of U, given N.

(b) Write down an expression for the PDF of U.

(c) Write down an expression for the conditional PMF of N given U.

Solution:

(a)

$$f_{U|N}(u \mid n) = \frac{f_{U,N}(u,n)}{f_N(n)} = \frac{f(u+n)}{\int_n^{n+1} f(x)dx},$$

where $u \in [0,1)$, and n is a nonnegative integer.

(b)

$$f_U(u) = \sum_{n=0}^{\infty} f_{U|N}(u|n) f_N(n) = \sum_{n=0}^{\infty} \frac{f(u+n)}{\int_n^{n+1} f(x)dx} \int_n^{n+1} f(x)dx = \sum_{n=0}^{\infty} f(u+n),$$

for $u \in [0,1)$.

(c)

$$f_{N|U}(n \mid u) = \frac{f_{N,U}(n,u)}{f_U(u)} = \frac{f(n+u)}{\sum_{k=0}^{\infty} f(k+u)},$$

for n integer, $u \in [0,1)$.

Problem 6: (15 points)
Let X and Y be positive continuous random variables with known joint PDF $f_{X,Y}$. Find an expression for the joint PDF of X and XY.

Solution: If $U = X, V = XY$, then $X = U, Y = V/U$, and the determinant of the Jacobian of the inverse transformation is

$$\left| \begin{bmatrix} 1 & 0 \\ -v/u^2 & 1/u \end{bmatrix} \right| = \frac{1}{u},$$

so that

$$f_{U,V}(u,v) = \frac{1}{u} f_{X,Y}(u, v/u),$$

for $u, v > 0$.

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