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6.641 Electromagnetic Fields, Forces, and Motion
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Problem Set 5 - Solutions

Problem 5.1

$$m \frac{d\vec{v}}{dt} = (q\vec{v}) \times (\mu_0\vec{H}); \vec{H} = H_0\hat{i}_z$$

$$m \frac{d\vec{v}}{dt} = q\mu_0 \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ v_x & v_y & v_z \\ 0 & 0 & H_0 \end{vmatrix}$$

$$\frac{d\vec{v}}{dt} = \frac{q\mu_0}{m} (v_y H_0 \hat{i}_x - v_x H_0 \hat{i}_y)$$

(i)

$$\frac{dv_x}{dt} = \frac{q\mu_0 v_y}{m} H_0$$

(ii)

$$\frac{dv_y}{dt} = -\frac{q\mu_0 v_x}{m} H_0$$

$$(i) \Rightarrow \frac{d^2 v_x}{dt^2} = \frac{q\mu_0 H_0}{m} \frac{dv_y}{dt}$$

Substitute this into (ii):

$$\frac{m}{q\mu_0 H_0} \frac{d^2 v_x}{dt^2} = -\frac{q\mu_0 v_x}{m} H_0$$

$$\frac{d^2 v_x}{dt^2} = -\frac{q^2 \mu_0^2 H_0^2}{m^2} v_x$$

$$v_x = A \cos\left(\frac{q\mu_0 H_0}{m} t\right) + B \sin\left(\frac{q\mu_0 H_0}{m} t\right)$$

$$v_y = C \cos\left(\frac{q\mu_0 H_0}{m} t\right) + D \sin\left(\frac{q\mu_0 H_0}{m} t\right)$$

$$v_x(t=0) = v_{x_0} = A$$

$$v_y(t=0) = v_{y_0} = C$$

Need two more initial conditions:

I. Acceleration in x direction at $t = 0$

$$\hat{i}_x \left. \frac{dv_x}{dt} \right|_{t=0} = \frac{(qv_{y_0} \hat{i}_y) \times (H_0 \hat{i}_z) \mu_0}{m}$$

$$B \frac{q\mu_0 H_0}{m} = \frac{q\mu_0 H_0}{m} v_{y_0}$$

$$B = v_{y_0}$$

II. Acceleration in y direction at $t = 0$

$$\hat{i}_y \left. \frac{dv_y}{dt} \right|_{t=0} = \frac{(qv_{x_0}\hat{i}_x) \times (\mu_0 H_0 \hat{i}_z)}{m}$$

$$D = -v_{x_0}$$

$$v_x(t) = v_{x_0} \cos\left(\frac{q\mu_0 H_0}{m}t\right) + v_{y_0} \sin\left(\frac{q\mu_0 H_0}{m}t\right)$$

$$v_y(t) = v_{y_0} \cos\left(\frac{q\mu_0 H_0}{m}t\right) - v_{x_0} \sin\left(\frac{q\mu_0 H_0}{m}t\right)$$

$$v_z(t) = v_{z_0}$$

Note: $\sqrt{v_x^2 + v_y^2 + v_z^2} = \text{constant in time (for this case) (easy to check and verify)}$

$$= \sqrt{v_{x_0}^2 + v_{y_0}^2 + v_{z_0}^2}$$

$$\Rightarrow v_{xy} = \sqrt{v_{x_0}^2 + v_{y_0}^2} = \text{velocity on } xy \text{ plane}$$

From 8.01, centripetal acceleration, a , is

$$a = \frac{v_{xy}^2}{r}$$

where r is radius of circle. So:

$$\underbrace{\left| m \frac{v_{xy}^2}{r} \right|}_{F=ma} = \underbrace{|q\vec{v} \times \mu_0 \vec{H}|}_{\text{Force due to } \vec{B} \text{ field}}$$

$$m \frac{v_{xy}^2}{r} = qv_{xy}\mu_0 H_0$$

$$r = \frac{mv_{xy}}{q\mu_0 H_0} = \frac{m\sqrt{v_{x_0}^2 + v_{y_0}^2}}{q\mu_0 H_0}$$

A

i

$$\vec{f} = 0 = q\vec{E} + q\vec{v} \times \mu_0 \vec{H}$$

$$0 = q\vec{E} + q(v_0\hat{i}_y) \times (\mu_0 H_0 \hat{i}_z)$$

$$\vec{E} = -v_0\mu_0 H_0 \hat{i}_x$$

$$V = -v_0\mu_0 H_0 s$$

ii

$$d = 2r = \frac{2mv_0}{q\mu_0 H_0}; v_0 = \frac{V}{-\mu_0 H_0 s}$$

$$d = \frac{2m|v|}{qB_0^2 s} = 0.50\text{cm for } 12\text{Mg}^{24}$$

$$0.52\text{cm for } 12\text{Mg}^{25}$$

$$0.54\text{cm for } 12\text{Mg}^{26}$$

B

i

$$m \frac{d\vec{v}}{dt} = q\vec{E} + q\vec{v} \times (\mu_0\vec{H}) = -e \left(-\frac{V_0}{s}\hat{i}_x + \vec{v} \times (\mu_0 H_0 \hat{i}_z) \right)$$

$$\vec{v} \times (\mu_0 H_0 \hat{i}_z) = v_y H_0 \mu_0 \hat{i}_x - v_x H_0 \mu_0 \hat{i}_y$$

$$m \frac{dv_x}{dt} = \frac{V_0 e}{s} - e\mu_0 H_0 v_y$$

$$m \frac{dv_y}{dt} = e\mu_0 H_0 v_x$$

$$\rightarrow m \frac{d^2 v_y}{dt^2} = \frac{e\mu_0 H_0}{m} \left(\frac{eV_0}{s} - e\mu_0 H_0 v_y \right)$$

$$\frac{d^2 v_y}{dt^2} + \left(\frac{e\mu_0 H_0}{m} \right)^2 v_y = \frac{e^2 \mu_0 H_0}{m^2 s} V_0 \quad (1)$$

$$m \frac{d^2 v_x}{dt^2} = -\frac{e^2 \mu_0^2 H_0^2}{m} v_x$$

$$\frac{d^2 v_x}{dt^2} + \left(\frac{e\mu_0 H_0}{m} \right)^2 v_x = 0 \quad (2)$$

Solution to equations 1 and 2 (homogenous + particular):

$$v_y = \frac{V_0}{\mu_0 H_0 s} \left(1 + c_1 \sin \left(\frac{e\mu_0 H_0}{m} t \right) + c_2 \cos \left(\frac{e\mu_0 H_0}{m} t \right) \right)$$

$$v_x = \frac{V_0}{\mu_0 H_0 s} \left(c_1 \cos \left(\frac{e\mu_0 H_0}{m} t \right) - c_2 \sin \left(\frac{e\mu_0 H_0}{m} t \right) \right)$$

(v_x from $e\mu_0 H_0 v_x = m \frac{dv_y}{dt}$)

$$v_x(t=0) = v_y(t=0) = 0 \Rightarrow c_1 = 0, c_2 = -1$$

$$\vec{v}(t) = \frac{V_0}{\mu_0 H_0 s} \left(\sin \left[\frac{e\mu_0 H_0}{m} t \right] \hat{i}_x + \left\{ 1 - \cos \left[\frac{e\mu_0 H_0}{m} t \right] \right\} \hat{i}_y \right)$$

$$\int \vec{v}(t) dt = \vec{d}(t) = \frac{mV_0}{\mu_0^2 H_0^2 s e} \left(1 - \cos \left[\frac{e\mu_0 H_0}{m} t \right] \right) \hat{i}_x + \left(\frac{V_0}{\mu_0 H_0 s} t - \frac{mV_0}{es\mu_0^2 H_0^2} \sin \left[\frac{e\mu_0 H_0}{m} t \right] \right) \hat{i}_y$$

ii

$$|d_x(t)|_{\max} < s \Rightarrow \frac{2mV_0}{\mu_0^2 H_0^2 s e} < s$$

$$H_0 > \sqrt{\frac{2mV_0}{\mu_0^2 s^2 e}}$$

Problem 5.2

A

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{A}$$

For contour through space to left of block

$$H_y^L s = NI = NI_0 \cos \omega t$$

$$H_y^L = \frac{NI_0}{s} \sin \omega t$$

For contour through space to right of block

$$H_y^R s = NI_0 \cos \omega t + \int_{\text{Block}} \vec{J}_z da \quad (\text{all current in } +z \text{ direction must return in } -z \text{ direction})$$

$$H_y^R = \frac{NI_0}{s} \sin \omega t$$

B

$$\nabla \times \vec{H} = \vec{J} \tag{3}$$

For Block A, $\vec{J} = \sigma \vec{E}$.

$$\nabla \times \vec{H} = \sigma \vec{E}$$

For block B

$$\frac{\partial \vec{J}}{\partial t} = \omega_p^2 \epsilon \vec{E} \tag{4}$$

$$\frac{\partial(\nabla \times \vec{H})}{\partial t} = \frac{\partial}{\partial t}(\vec{J})$$

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \frac{\partial \vec{J}}{\partial t} \tag{5}$$

4→5:

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \omega_p^2 \epsilon \vec{E}$$

C

Block A

$$\nabla \times \vec{H} = \sigma \vec{E}$$

$$\nabla \times \nabla \times \vec{H} = \nabla \times \sigma \vec{E}$$

Assume uniform σ , and use $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.

$$\nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H} = \sigma(\nabla \times \vec{E})$$

Faraday's Law

$$\nabla \times \vec{E} = -\frac{\partial \mu \vec{H}}{\partial t}$$

Flux continuity: $\nabla \cdot \mu \vec{H} = 0$; for uniform $\mu \Rightarrow \nabla \cdot \vec{H} = 0$.

$$\nabla^2 \vec{H} = \sigma \mu \frac{\partial \vec{H}}{\partial t}$$

Block B

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \omega_p^2 \varepsilon \vec{E}$$

$$\nabla \times \nabla \times \frac{\partial \vec{H}}{\partial t} = \nabla \times \omega_p^2 \varepsilon \vec{E}$$

Using $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ and assuming uniform properties

$$\nabla(\nabla \cdot \frac{\partial \vec{H}}{\partial t}) - \nabla^2 \frac{\partial \vec{H}}{\partial t} = \omega_p^2 \varepsilon (\nabla \times \vec{E})$$

Flux continuity for uniform $\mu \rightarrow \nabla \cdot \vec{H} = 0$ and using Faraday's Law

$$\nabla^2 \frac{\partial \vec{H}}{\partial t} = \omega_p^2 \varepsilon \mu \frac{\partial \vec{H}}{\partial t}$$

Integrate and assume that integration constant is zero

$$\nabla^2 \vec{H} = \omega_p^2 \varepsilon \mu \vec{H} = \frac{\omega_p^2}{c^2} \vec{H}$$

$$c = \frac{1}{\sqrt{\varepsilon \mu}} = \text{speed of light in material}$$

D

Assume $\vec{H} = \text{Im} \left\{ \hat{H}_y(x) e^{j\omega t} \hat{i}_y \right\}$

$$\nabla^2 \vec{H} = \sigma \mu \frac{\partial \vec{H}}{\partial t} \text{ for Block A.}$$

$$\frac{\partial^2}{\partial x^2} \hat{H}_y(x) e^{j\omega t} = \sigma \mu j \omega \hat{H}_y(x) e^{j\omega t}$$

$$\frac{\partial^2}{\partial x^2} \hat{H}_y(x) = \sigma \mu j \omega \hat{H}_y(x)$$

Assume $\hat{H}_y(x) = \hat{H}_0 e^{jkx}$.

$$-k^2 \hat{H}_0 e^{jkx} = \sigma \mu \hat{H}_0 e^{jkx} j \omega$$

$$-k^2 = \sigma \mu j \omega, k = \pm \sqrt{-\omega \mu j \sigma}$$

$$k = \pm \sqrt{\frac{\sigma \mu \omega}{2}} (1 - j)$$

$$\alpha \equiv \sqrt{\frac{\sigma \mu \omega}{2}}$$

$$\vec{H}_y = \text{Im} \left\{ \hat{H}_0^1 e^{\alpha x} e^{j\alpha x} e^{j\omega t} + \hat{H}_0^2 e^{-\alpha x} e^{-j\alpha x} e^{j\omega t} \right\}$$

Apply B.C. to complex H . At $x = 0$, $K = 0 \Rightarrow H_y(x = 0, t) = \frac{NI_0}{s} \sin \omega t$.

$$= \text{Im} \left\{ \frac{NI_0}{s} e^{j\omega t} \right\}$$

$$\hat{H}_0^1 e^0 e^0 e^{j\omega t} + \hat{H}_0^2 e^0 e^0 e^{j\omega t}$$

$$= \frac{NI_0}{s} e^{j\omega t}$$

$$\hat{H}_0^1 + \hat{H}_0^2 = \frac{NI_0}{s}$$

At $x = d$, $K = 0 \Rightarrow H_y(x = d, t) = \frac{NI_0}{s} \sin \omega t$

$$\hat{H}_0^1 e^{\alpha d} e^{j\alpha d} e^{j\omega t} + \hat{H}_0^2 e^{-\alpha d} e^{-j\alpha d} e^{j\omega t} = \frac{NI_0}{s} e^{j\omega t}$$

$$\Rightarrow \hat{H}_0^1 e^{\alpha d} e^{j\alpha d} + \hat{H}_0^2 e^{-\alpha d} e^{-j\alpha d} = \frac{NI_0}{s}$$

From B.C. at $x = 0$

$$\hat{H}_0^1 = \frac{NI_0}{s} - \hat{H}_0^2$$

$$\frac{NI_0}{s} e^{\alpha d} e^{j\alpha d} - \hat{H}_0^2 [e^{\alpha d} e^{j\alpha d} - e^{-\alpha d} e^{-j\alpha d}] = \frac{NI_0}{s}$$

$$\hat{H}_0^2 = \frac{\frac{NI_0}{s} (1 - e^{\alpha d} e^{j\alpha d})}{(e^{-\alpha d} e^{-j\alpha d} - e^{\alpha d} e^{j\alpha d})}$$

$$\hat{H}_0^1 = \frac{NI_0}{s} - \frac{NI_0}{s} \left[\frac{1 - e^{\alpha d} e^{j\alpha d}}{e^{-\alpha d} e^{-j\alpha d} - e^{\alpha d} e^{j\alpha d}} \right]$$

$$= \frac{NI_0}{s} \left[\frac{e^{-\alpha d} e^{-j\alpha d} - 1}{e^{-\alpha d} e^{-j\alpha d} - e^{\alpha d} e^{j\alpha d}} \right]$$

$$\vec{H} = \text{Re} \left\{ \hat{H}_0^1 e^{\alpha x} e^{j\alpha x} e^{j\omega t} + \hat{H}_0^2 e^{-\alpha x} e^{-j\alpha x} e^{j\omega t} \right\} \hat{i}_y$$

Or

$$\vec{H} = \text{Im} \left[\frac{NI_0}{s} \frac{\cosh \gamma (x - \frac{d}{2})}{\cosh \frac{\gamma d}{2}} e^{j\omega t} \right] \hat{i}_y$$

with $\gamma = \frac{1+j}{\delta}$, $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$, $\alpha = \sqrt{\frac{\sigma\mu\omega}{2}} = \frac{1}{\delta}$.

For Block B: Assume $H_y(x, t) = \text{Im} \left\{ \hat{H}_0 e^{jkx} e^{j\omega t} \right\}$.

$$\nabla^2 \vec{H} = \omega_p^2 \varepsilon \mu \vec{H}$$

$$\frac{\partial^2}{\partial x^2} \hat{H}_0 e^{jkx} e^{j\omega t} = \omega_p^2 \varepsilon \mu \hat{H}_0 e^{jkx} e^{j\omega t}$$

$$-k^2 = \omega_p^2 \varepsilon \mu$$

$$k = \pm \sqrt{\omega_p^2 \varepsilon \mu} = \pm \omega_p \sqrt{\varepsilon \mu} j$$

$$\vec{H}_y(x) = \text{Im} \left\{ \hat{H}_0^1 e^{\alpha x} e^{j\omega t} + \hat{H}_0^2 e^{-\alpha x} e^{j\omega t} \right\}$$

$$\alpha = \omega_p \sqrt{\varepsilon \mu}$$

Apply B.C. at $x = 0$, $K = 0 \Rightarrow H_y^{0+} = \frac{I_0 N}{s} \sin \omega t$.

$$\hat{H}_0^1 e^0 e^{j\omega t} + \hat{H}_0^2 e^0 e^{j\omega t} = \frac{I_0 N}{s} e^{j\omega t}$$

$$\rightarrow \hat{H}_0^1 + \hat{H}_0^2 = \frac{NI_0}{s}$$

Apply B.C. at $x = d$, $K = 0 \Rightarrow H_y^d = \frac{I_0 N}{s} \sin \omega t$.

$$\hat{H}_0^1 e^{\alpha d} e^{j\omega t} + \hat{H}_0^2 e^{-\alpha d} e^{j\omega t} = \frac{I_0 N}{s} e^{j\omega t}$$

$$\hat{H}_0^1 e^{\alpha d} + \hat{H}_0^2 e^{-\alpha d} = \frac{NI_0}{s}$$

$$\left[\frac{NI_0}{s} - \hat{H}_0^2 \right] e^{\alpha d} + \hat{H}_0^2 e^{-\alpha d} = \frac{NI_0}{s}$$

$$\hat{H}_0^2 [e^{-\alpha d} - e^{\alpha d}] = \frac{NI_0}{s} [1 - e^{\alpha d}]$$

$$\hat{H}_0^2 = \frac{NI_0}{s} \left[\frac{1 - e^{\alpha d}}{e^{-\alpha d} - e^{\alpha d}} \right]$$

$$\hat{H}_0^1 = \frac{NI_0}{s} \left[\frac{e^{-\alpha d} - 1}{e^{-\alpha d} - e^{\alpha d}} \right]$$

$$\vec{H}_y(x) = \text{Im} \left\{ \hat{H}_0^1 e^{\alpha x} e^{j\omega t} + \hat{H}_0^2 e^{-\alpha x} e^{j\omega t} \right\}$$

$$\alpha = \omega_p \sqrt{\varepsilon \mu}$$

Or

$$H_0^2 = \frac{NI_0}{s} \left[\frac{1 - e^{\alpha d}}{e^{-\alpha d} - e^{\alpha d}} \right]$$

$$H_0^1 = \frac{NI_0}{s} \left[\frac{e^{-\alpha d} - 1}{e^{-\alpha d} - e^{\alpha d}} \right]$$

$$H_y(x, t) = [H_0^1 e^{\alpha x} + H_0^2 e^{-\alpha x}] \sin \omega t$$

$$H_y(x, t) = \frac{NI_0}{s} \frac{\cosh \alpha \left(x - \frac{d}{2} \right)}{\cosh \frac{\alpha d}{2}} \sin \omega t, \quad \text{with } \alpha = \frac{\omega_p}{c} = \omega_p \sqrt{\varepsilon \mu}$$

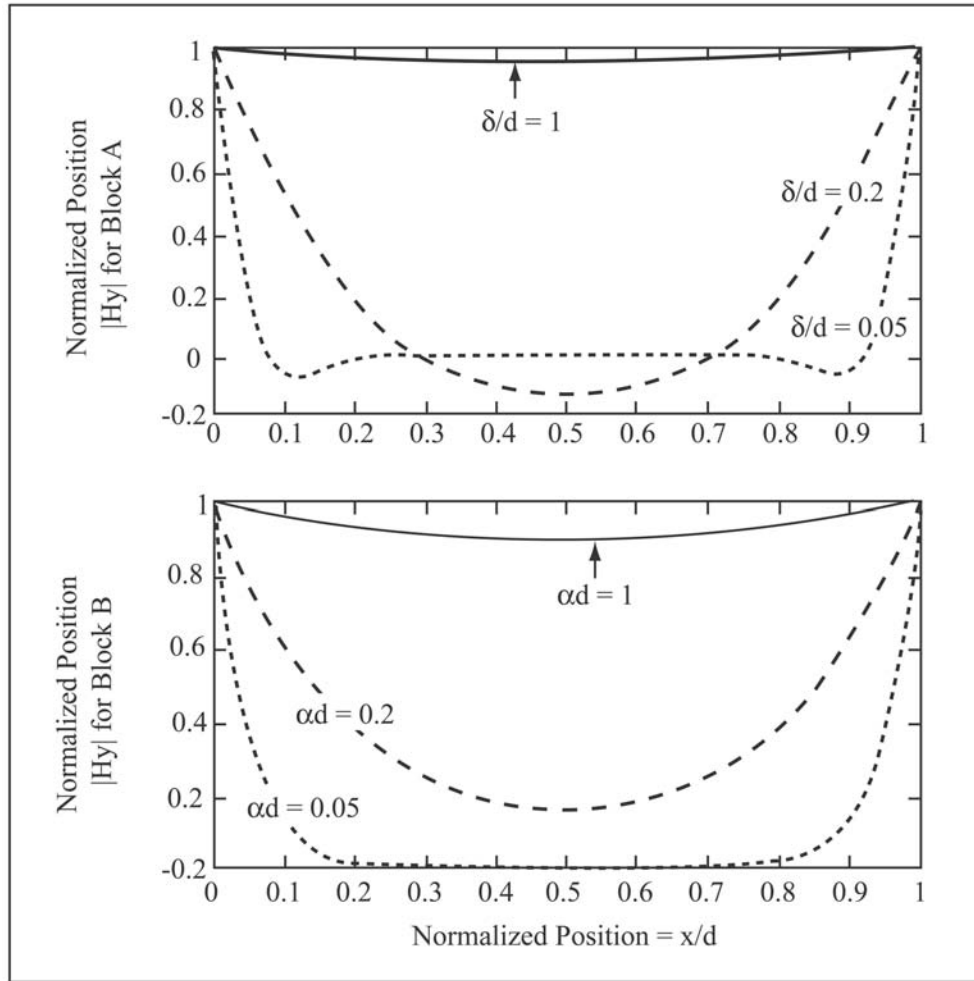


Figure 1: $|H_y|$ for blocks A and B for Problem 5.2 (Image by MIT OpenCourseWare.)

E

For Block A

$$\nabla \times \vec{H} = \vec{J}$$

Since $\vec{H} = \vec{H}_y(x)$

$$\vec{J} = \hat{z} \frac{\partial H_y(x)}{\partial x}$$

$$\vec{J} = \text{Im} \left\{ \hat{H}_0^1 (\alpha + j\alpha) e^{\alpha x} e^{j\alpha x} e^{j\omega t} + \hat{H}_0^2 (-\alpha - j\alpha) e^{-\alpha x} e^{-j\alpha x} e^{j\omega t} \right\} \hat{z} = \text{Im} \left[\frac{NI_0}{s} \frac{\gamma \sinh \gamma (x - \frac{d}{2})}{\cosh(\frac{\gamma d}{2})} e^{j\omega t} \right] \hat{i}_z$$

$$\gamma = \frac{1+j}{\delta}, \quad \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

For Block B

$$\vec{J} = \hat{z} \frac{\partial H_y(x)}{\partial x}$$

$$\vec{J} = \text{Im} \left\{ \hat{H}_0^1 \alpha e^{\alpha x} e^{j\omega t} - \hat{H}_0^2 \alpha e^{-\alpha x} e^{j\omega t} \right\} \hat{z}$$

or

$$\begin{aligned} \vec{J}(x,t) &= [H_0^1 \alpha e^{\alpha x} - H_0^2 \alpha e^{-\alpha x}] \sin \omega t \hat{z} \\ &= \frac{NI_0}{s} \frac{k \sinh k(x - \frac{d}{2})}{\cosh \frac{kd}{2}} \sin \omega t \hat{z}, \quad k = \frac{\omega_p}{c} \end{aligned}$$

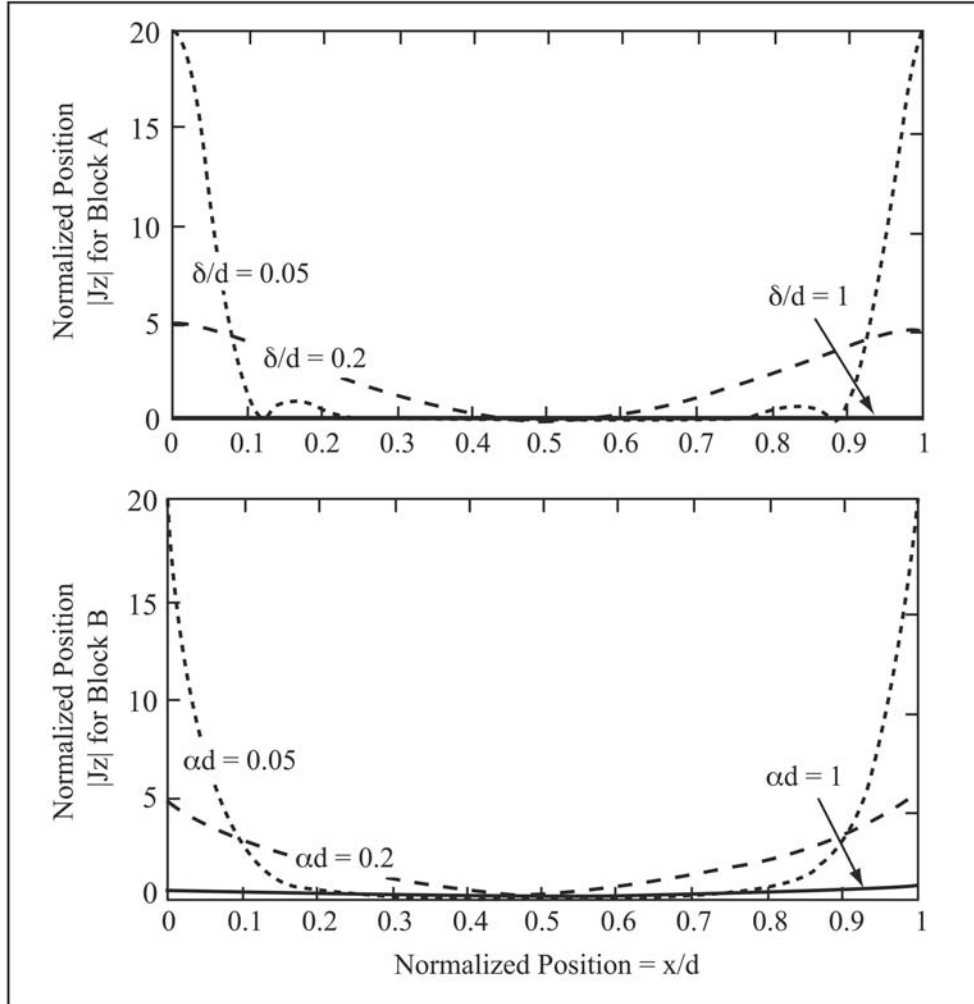


Figure 2: $|J_z|$ for blocks A and B for Problem 5.2 (Image by MIT OpenCourseWare.)

Problem 5.3

A

From the Boundary condition $\vec{i}_x \cdot [\varepsilon_0 \vec{E}(x = 0^+) - \varepsilon_0 \vec{E}(x = 0^-)] = \sigma_s$, we know

$$E_x(x = 0^+) - E_x(x = 0^-) = \frac{\sigma_s}{\varepsilon_0}$$

and from the symmetry, we know $E_x(x = 0^+) = -E_x(x = 0^-)$. So $E_x(x = 0^+) = -E_x(x = 0^-) = \frac{\sigma_s}{2\epsilon_0} = \frac{\sigma_0 \cos(ay)}{2\epsilon_0}$. We can build the boundary condition for the scalar potential:

$$BC1 : \Phi(x \rightarrow \infty) = 0, \Phi(x \rightarrow -\infty) = 0$$

$$BC2 : -\frac{\partial\Phi(x, y)}{\partial x}(x = 0^+) = +\frac{\partial\Phi(x, y)}{\partial x}(x = 0^-) = \frac{\sigma_0 \cos(ay)}{2\epsilon_0}$$

Here we assume the general solution for the scalar potential is given: $\Phi(x, y) = AX(x)\Psi(y)$. As BC1 implied, $X(x) = e^{-ax}, x > 0$; and $e^{ax}, x < 0$. With BC2, we find the scalar potential as:

$$\Phi(x, y) = \frac{\sigma_0 e^{-ax} \cos(ay)}{2a\epsilon_0}, x > 0$$

$$\Phi(x, y) = \frac{\sigma_0 e^{ax} \cos(ay)}{2a\epsilon_0}, x < 0$$

With $\bar{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial x}\bar{i}_x - \frac{\partial\Phi}{\partial y}\bar{i}_y$.

$$\bar{E} = \frac{\sigma_0 e^{-ax}}{2\epsilon_0} (\cos(ay)\bar{i}_x + \sin(ay)\bar{i}_y), x > 0$$

$$\bar{E} = -\frac{\sigma_0 e^{ax}}{2\epsilon_0} (\cos(ay)\bar{i}_x - \sin(ay)\bar{i}_y), x < 0$$

B

For the electric field line, we have $\frac{dy}{dx} = \frac{E_y}{E_x}$. At $x > 0$,

$$dx = \frac{\cos(ay)}{\sin(ay)} dy \Rightarrow e^{-ax} \sin(ay) = \text{constant}$$

At $x < 0$,

$$dx = -\frac{\cos(ay)}{\sin(ay)} dy \Rightarrow e^{ax} \sin(ay) = \text{constant}$$

C

See Figure 3

Problem 5.4

A

At region $x > 0$, we have the general solution $\Psi_I = e^{-ax} A \sin(ay)$. At region $x < 0$, we have solution $\Psi_{II} = e^{ax} B \sin(ay)$. We chose exponentials in x for 0 potential at $x = \pm\infty$. We chose sines as $\bar{H} = -\nabla\Psi$ will have to satisfy $\cos(ay)$ and the boundary condition for $x = 0$.

B

Boundary Condition 1: $\bar{n} \cdot [\bar{B}_I - \bar{B}_{II}] = 0 \Rightarrow \mu_0 H_{Ix}|_{x=0^+} = \mu H_{IIx}|_{x=0^-}$.

Boundary Condition 2: $\bar{n} \times [\bar{H}_I - \bar{H}_{II}] = K_0 \cos(ay)\bar{i}_x$.

$$H_{Iy}|_{x=0^+} - H_{IIy}|_{x=0^-} = K_0 \cos(ay)$$

Here $\bar{H} = -\nabla\Psi, \bar{H}_I = ae^{-ax} A \sin(ay)\bar{i}_x - ae^{-ax} A \cos(ay)\bar{i}_y$

$$\bar{H}_{II} = -ae^{ax} B \sin(ay)\bar{i}_x - ae^{ax} B \cos(ay)\bar{i}_y$$

By BC1: $\mu_0 A = -\mu B$; By BC2: $B - A = \frac{K_0}{a}$. We can solve: $A = -\frac{\mu K_0}{a(\mu_0 + \mu)}$ and $B = \frac{\mu_0 K_0}{a(\mu_0 + \mu)}$.

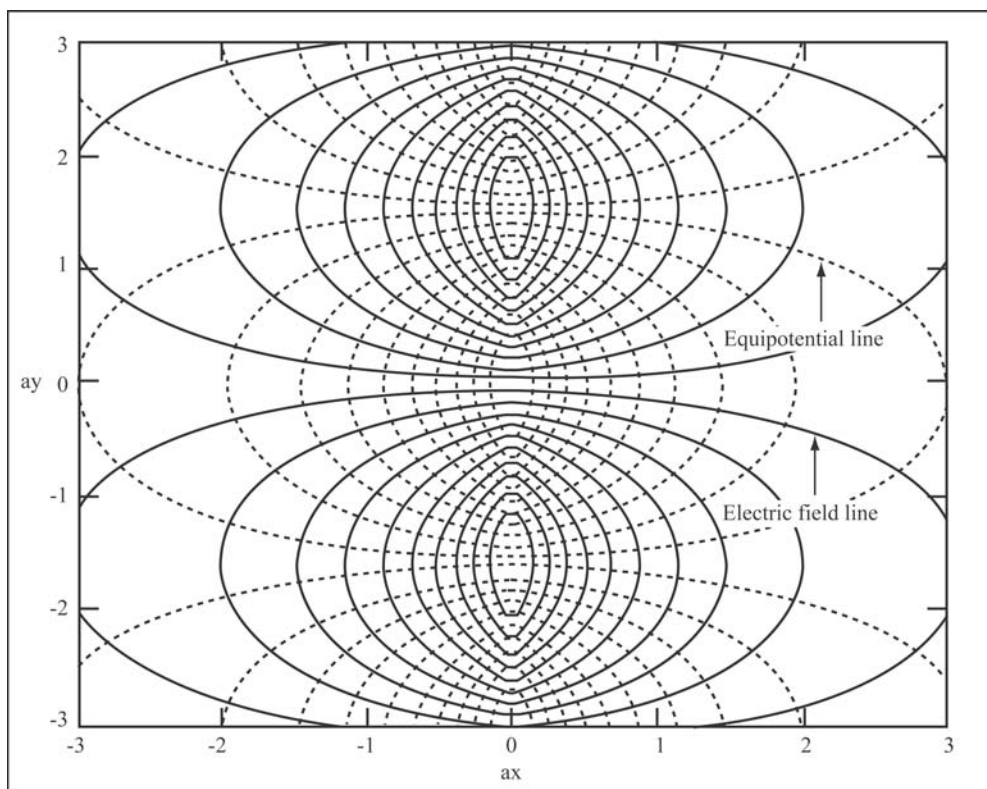


Figure 3: Electric Field Lines and Equipotential Lines for Problem 5.3C (Image by MIT OpenCourseWare.)

C

The solution for H field:

$$\vec{H}_I = \frac{\mu K_0}{(\mu_0 + \mu)} [-e^{-ax} \sin(ay) \vec{i}_x + e^{-ax} \cos(ay) \vec{i}_y] \quad x > 0$$

$$\vec{H}_{II} = \frac{\mu_0 K_0}{(\mu_0 + \mu)} [-e^{ax} \sin(ay) \vec{i}_x - e^{ax} \cos(ay) \vec{i}_y] \quad x < 0$$

Problem 5.5

Line current I of infinite extent above a plane of material of infinite permeability, $\mu \rightarrow 0$.

A

$\vec{B} = \mu \vec{H} \Rightarrow$ for $\mu \rightarrow \infty$, in order to have B finite, we need H zero \Rightarrow continuity of normal \vec{B} and tangential \vec{H} at the surface.

B

Using method of images to satisfy because at $y = 0$ for medium where $\mu \rightarrow \infty$ because $\Rightarrow H_x = H_z = 0$ at $y = 0$

For a line current at origin (see Figure 6)

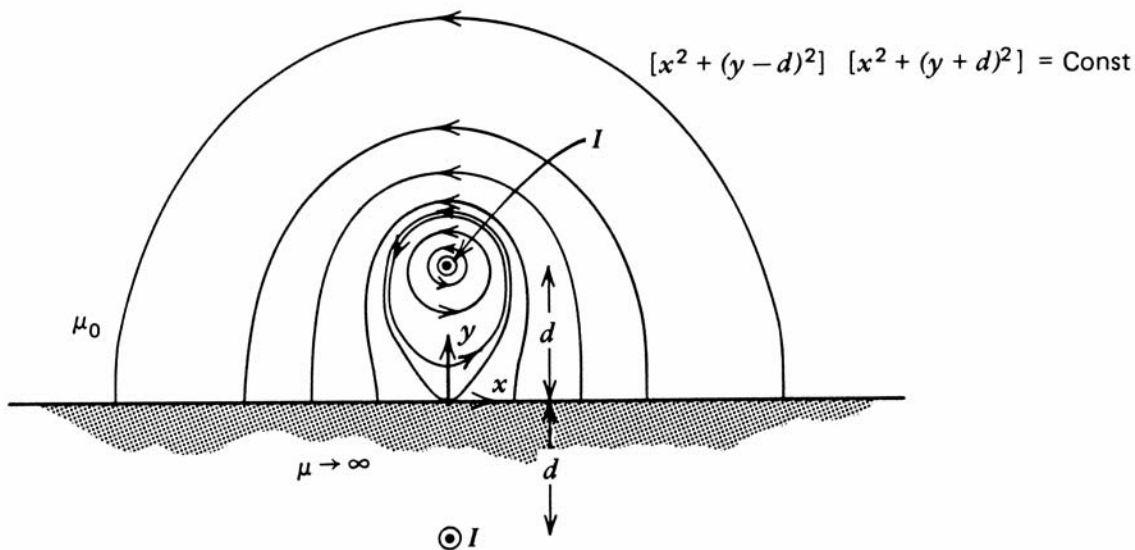


Figure 4: A diagram showing a line current I of infinite extent above a plane of material of infinite permeability with field lines.

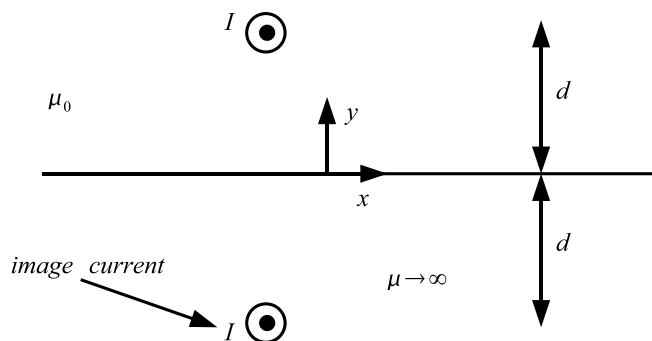


Figure 5: A diagram showing how to apply the method of images with the image current in the same direction as the source current. (Image by MIT OpenCourseWare).

$$\oint_C \vec{H} \cdot d\vec{l} = I \Rightarrow H_\phi = \frac{I}{2\pi r}$$

$$\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r} \hat{i}_\phi, \text{ since } \vec{B} = \nabla \times \vec{A} \Rightarrow -\frac{\partial A_z}{\partial r} = \frac{I\mu_0}{2\pi r}$$

$$\Rightarrow A_z = -\frac{I\mu_0}{2\pi} \ln r \rightarrow \text{constant}$$

$$\Rightarrow \text{for line currents } I \text{ at } z = d \text{ and } I \text{ at } z = -d$$

$$A_z = -\frac{I\mu_0}{2\pi} \left\{ \ln \left[\sqrt{x^2 + (y - d)^2} \right] + \ln \left[\sqrt{x^2 + (y + d)^2} \right] \right\}$$

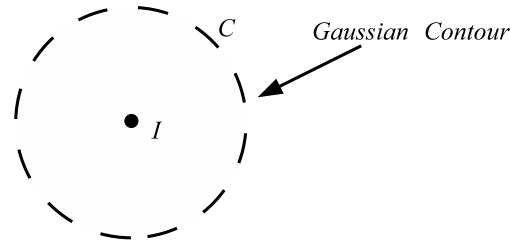


Figure 6: A diagram depicting a Gaussian Contour surrounding a line current (Image by MIT OpenCourseWare).

$$\Rightarrow A_z = -\frac{I\mu_0}{4\pi} \left\{ \ln \left[x^2 + (y-d)^2 \right] \left[x^2 + (y+d)^2 \right] \right\}$$

C

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} = -\frac{\partial}{\partial x} A_z \vec{i}_y + \frac{\partial}{\partial y} A_z \vec{i}_x \\ &= \frac{I\mu_0}{4\pi} \frac{2x \left[x^2 + (y+d)^2 \right] + 2x \left[x^2 + (y-d)^2 \right]}{\left[x^2 + (y-d)^2 \right] \left[x^2 + (y+d)^2 \right]} \vec{i}_y - \frac{I\mu_0}{4\pi} \frac{2(y-d) \left[x^2 + (y+d)^2 \right] + 2(y+d) \left[x^2 + (y-d)^2 \right]}{\left[x^2 + (y-d)^2 \right] \left[x^2 + (y+d)^2 \right]} \vec{i}_x \\ &= \frac{I\mu_0}{2\pi} \left[\frac{(y+d)\vec{i}_x - x\vec{i}_y}{x^2 + (y+d)^2} + \frac{(y-d)\vec{i}_x - x\vec{i}_y}{x^2 + (y-d)^2} \right] \end{aligned}$$

D

Force is applied on the line current due to the image line current

Force per unit length:

$$\begin{aligned} \vec{F} &= \vec{I} \times \vec{B} \leftarrow \text{field due to image charge at } (x=0, y=-d) \\ &= \vec{I} \vec{i}_z \times \left(\frac{\mu_0 I}{2\pi} \right) \frac{1}{2d} \vec{i}_x \\ &= \left[-\frac{\mu_0 I^2}{4\pi d} \vec{i}_y \right] \text{ so line current is attracted to the surface} \end{aligned}$$